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## LITTLEWOOD'S INEQUALITY FOR $p$ -BIMEASURES

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Abstract

Contents



Home Page

Go Back

Close

Quit

## Abstract

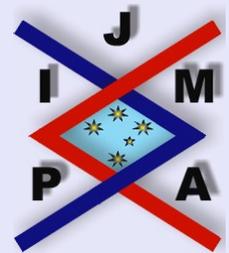
In this paper we extend an inequality of Littlewood concerning the higher variations of functions of bounded Fréchet variations of two variables (bimeasures) to a class of functions that are  $p$ -bimeasures, by using the machinery of vector measures. Using random estimates of Kahane-Salem-Zygmund, we show that the inequality is sharp.

*2000 Mathematics Subject Classification:* Primary 26B15, 26A42, Secondary 28A35, 28A25.

*Key words:* Inequalities, Bimeasures, Fréchet variation,  $p$ -variations, Bounded variations.

## Contents

1	Introduction .....	3
1.1	Littlewood's Inequalities .....	4
2	Proof of Theorem 1.1 .....	7
2.1	4-level Radamacher System .....	10
3	Functions of Bounded $p$ -Variations and Related Function Spaces .....	12
	References .....	



### Littlewood's Inequality for $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

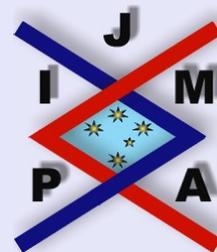
Page 2 of 15

# 1. Introduction

Let  $\mu$  be a set function defined on the product  $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$  of 2  $\sigma$ -fields, such that it is a finite complex measure in each coordinate. More precisely, for each fixed  $A \in \sigma(\mathcal{B}_1)$  the set function  $\mu(A, \cdot)$  is a complex measure defined on  $\sigma(\mathcal{B}_2)$ . Similarly for each  $B \in \sigma(\mathcal{B}_2)$ , the set function  $\mu$  gives rise to a measure in the first coordinate. Such set functions dubbed *bimeasures* by Morse and Transue were studied extensively by these and other authors (see [1, 2, 3, 5, 6, 7, 10, 11, 12]). It is well known that such set functions need not be extendible to a measure on the  $\sigma$ -Algebra generated by  $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$ . Now suppose that  $\mu$  is a set function defined on  $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$ , such that it has finite *semi-variation*; that is,

$$(1.1) \quad \|\mu\|_F = \sup \left\{ \left\| \sum_{j,k} \mu(A_j \times B_k) r_j \otimes r_k \right\|_\infty \right\} < \infty,$$

where sup is taken over all measurable partitions  $\{A_j\}$ ,  $\{B_k\}$  of  $\Omega_1$  and  $\Omega_2$ , respectively. Here  $\{r_j\}$  is the usual system of Rademachers, realized as functions on the interval  $[0, 1]$ . By a partition of  $\Omega$ , we mean a finite collection of mutually disjoint measurable sets whose union is  $\Omega$ .  $F$  in  $\|\cdot\|_F$  is for Fréchet. It is clear that a set function  $\mu$  with finite semi-variation is also a bimeasure. It is interesting that the converse also holds. That is, a bimeasure has finite semi-variation. This follows easily from the machinery of vector measure theory. On the other hand, it is well known that a set function which has finite semi-variation need not have finite total variation (in the sense of Vitali), hence it may not be extendible to a measure [2, 9]. However, all is not lost, in his 1930



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Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

---

Title Page

Contents



Go Back

Close

Quit

Page 3 of 15

paper, Littlewood showed that a bimeasure has finite  $4/3$ -variation. To make this precise we first introduce the notion of mixed variation of  $\mu$ . Let  $p, q > 0$ , and define the mixed  $(p, q)$ -variation of  $\mu$  to be

$$(1.2) \quad \|\mu\|_{p,q} = \sup \left\{ \left( \sum_k \left( \sum_j |\mu(A_j \times B_k)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \right\},$$

where the sup is taken over all finite measurable partitions  $\{A_j\}$  and  $\{B_k\}$  of  $\Omega_1$  and  $\Omega_2$  respectively. In the case that  $p = q$ , we simply write  $\|\mu\|_p$ , that is  $\|\mu\|_p = \|\mu\|_{p,p}$ . We now state Littlewood's  $4/3$  inequalities.

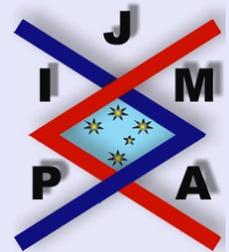
## 1.1. Littlewood's Inequalities

$$(1.3) \quad \|\mu\|_{2,1} + \|\mu\|_{1,2} + \|\mu\|_{4/3} \leq c \|\mu\|_F,$$

where  $c$  is a fixed universal constant. The result is sharp in the sense that, there exists  $\mu \in \mathcal{M}$  such that  $\|\mu\|_p$  and  $\|\mu\|_{q,1/q}$  are infinite for all  $p < 4/3$  and for all  $q < 2$ . Extension of Littlewood's inequality to a larger class of functions of two variables is the main result of this paper.

**Definition 1.1.** A set function  $\mu$  defined on product of two algebras  $\mathcal{B}_1 \times \mathcal{B}_2$  is called a *pre- $p$ -bimeasure*, if it is finitely additive in each coordinate, and for each fixed  $A \in \mathcal{B}_1$ , the quantity

$$BV_p(\mu(A, \cdot)) := \sup \left\{ \sum_k |\mu(A \times B_k)|^p \right\}$$



Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

Page 4 of 15

is finite, and for each fixed  $B \in \mathcal{B}_2$ ,  $BV_p(\mu(\cdot, B))$  is finite. Here sup is taken over all finite measurable partitions of  $\Omega_2$ .

If the set function is defined on the product of two  $\sigma$ -algebras with above properties, then it is called a  $p$ -bimeasure.

**Definition 1.2.** A pre- $p$ -bimeasure  $\mu$  defined on product of two algebras  $\mathcal{B}_1 \times \mathcal{B}_2$ , is said to be bounded, if there exists a positive constant  $M$  such that  $BV_p(\mu(A, \cdot)) + BV_p(\mu(\cdot, B)) \leq M$ , for all  $A \in \mathcal{B}_1$  and for all  $B \in \mathcal{B}_2$ .

We prove the following result.

**Theorem 1.1.** Suppose that either  $\mu$  is a  $p$ -bimeasure defined on  $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$ , or that it is a bounded pre- $p$ -bimeasure defined on  $\mathcal{B}_1 \times \mathcal{B}_2$ . If  $1 \leq p \leq 2$  then

$$(1.4) \quad \|\mu\|_{2,p} + \|\mu\|_{p,2} + \|\mu\|_{\frac{4p}{2+p}} < \infty.$$

In the case that  $p \geq 2$ , then

$$(1.5) \quad \|\mu\|_p < \infty.$$

Furthermore, the result is sharp, in the sense that, there exists a  $p$ -bimeasure such that  $\|\mu\|_q = \infty$ , for all  $q < \frac{4p}{2+p}$ .

To prove Theorem 1.1 we collect some definitions and results about vector measures. Much of the following can be found in Chapter 1 of [4].

**Definition 1.3.** A function  $\mu$  from a field  $\mathcal{B}$  of a set  $\Omega$  to a Banach space is called a finitely additive vector measure, or simply a vector measure, if whenever  $A_1$



### Littlewood's Inequality for $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

Page 5 of 15

and  $A_2$  are disjoint members of  $\mathcal{B}$  then  $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ . The variation of a vector measure  $\mu$  is the extended nonnegative function  $|\mu|$  whose value on the set  $E$  is given by

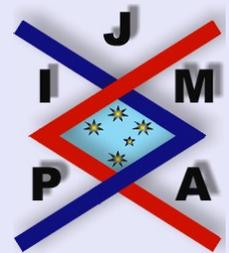
$$|\mu|(A) = \sup_{\pi} \sum_{A \in \pi} \|\mu(A)\|,$$

where the sup is taken over all partitions  $\pi$  of  $A$  into a finite number of disjoint members of  $\mathcal{B}$ . If  $|\mu|(\Omega)$  is finite, then  $\mu$  will be called a measure of bounded variation.

A different type of variation related to a vector measure  $\mu$  is the so called *semi-variation* of  $\mu$ . More precisely, the semi-variation of  $\mu$  is the extended nonnegative function  $\|\mu\|_F$  whose value on a measurable set  $A$  is given by

$$\|\mu\|_F(A) = \sup \{ |x^*(\mu)|(A) : x^* \in X^*, \|x^*\| \leq 1 \},$$

where  $|x^*(\mu)|$  is the variation of the real-valued measure (finitely additive measure)  $x^*(\mu)$ . If  $\|\mu\|_F(\Omega)$  is finite, then  $\mu$  will be called a *measure of bounded semi-variation*.




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### Littlewood's Inequality for $p$ -Bimeasures

Nasser Towghi

---

Title Page

Contents



Go Back

Close

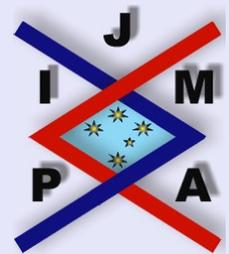
Quit

Page 6 of 15

## 2. Proof of Theorem 1.1

We now prove Theorem 1.1. Suppose that  $1 \leq p < 2$ . Let  $X_1$  be the space of finitely additive set functions defined on  $\sigma(\mathcal{B}_1)$ , which have finite  $p$ -variations. Similarly let  $X_2$  be the set finitely additive functions defined on  $\sigma(\mathcal{B}_2)$  which have finite  $p$ -variations. It can be shown that equipped with  $p$ -variation norm,  $X_1$  and  $X_2$  are Banach spaces. Let  $L$  be the  $X_1$ -valued function defined on  $\sigma(\mathcal{B}_2)$  as follows:  $L(A) = \mu(\cdot, A)$ , where  $A \in \sigma(\mathcal{B}_2)$ . Let  $R$  be the  $X_2$ -valued function defined on  $\sigma(\mathcal{B}_1)$  as follows:  $R(A) = \mu(A, \cdot)$ , where  $A \in \sigma(\mathcal{B}_1)$ . If  $\mu$  is a  $p$ -bimeasure then by the Nikodym Boundedness Theorem (see [4, Theorem 1, page 14]),  $L$  and  $R$  have finite semi-variations. If  $\mu$  is a bounded pre- $p$ -bimeasure then by general properties of vector measures (see e.g., [4, Proposition 11, page 4]),  $L$  and  $R$  have finite semi-variations. Let  $\{A_n\}$  be a finite measurable partition of  $\Omega_2$  and  $\{B_k\}$  be a finite measurable partition of  $\Omega_1$ , then

$$\begin{aligned}
 (2.1) \quad & \infty > \|L\|_F(\Omega_2) \\
 & \geq \left\| BV_p \left( \sum_n r_n \mu(A_n, \cdot) \right) \right\|_{\infty} \\
 & \geq \left\| \left( \sum_k \left| \sum_n r_n \mu(A_n, B_k) \right|^p \right)^{\frac{1}{p}} \right\|_{\infty} \\
 & \geq \left( \int_0^1 \sum_k \left| \sum_n r_n(x) \mu(A_n, B_k) \right|^p dx \right)^{\frac{1}{p}}
 \end{aligned}$$



Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

Page 7 of 15

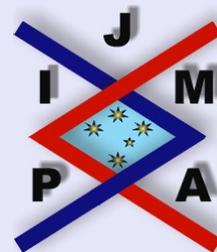
$$\text{(Khinchin's inequality)} \Rightarrow \geq c \left( \sum_k \left( \sum_n |\mu(A_n, B_k)|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

Similarly,

$$(2.2) \quad \infty > \|R\|_F(\Omega_1) \geq c \left( \sum_n \left( \sum_k |\mu(A_n, B_k)|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

(2.2) and (2.3) imply that,  $\|\mu\|_{2,p}$  is finite. Applying Minkowski's inequality we obtain  $\|\mu\|_{p,2} \leq \|\mu\|_{2,p} < \infty$ . We now show that  $\|\mu\|_{\frac{4p}{2+p}}$  is finite. Let  $a_{n,k} = \mu(A_n, B_k)$ . Applying Hölder's inequality with exponents  $\frac{2+p}{p}$  and  $\frac{2+p}{2}$ , we obtain

$$\begin{aligned} (2.3) \quad \sum_{n,k} |a_{n,k}|^{\frac{4p}{2+p}} &= \sum_{n,k} |a_{n,k}|^{\frac{2p}{2+p}} |a_{n,k}|^{\frac{2p}{2+p}} \\ &\leq \sum_n \left[ \sum_k |a_{n,k}|^2 \right]^{\frac{p}{2+p}} \left[ \sum_k |a_{n,k}|^p \right]^{\frac{2}{p+2}} \\ &\leq \left[ \sum_n \left( \sum_k |a_{n,k}|^2 \right)^{\frac{p}{2}} \right]^{\frac{2}{2+p}} \left[ \sum_n \left( \sum_k |a_{n,k}|^p \right)^{\frac{2}{p}} \right]^{\frac{p}{2+p}} \\ &\leq \left( \|\mu\|_{2,p} \|\mu\|_{p,2} \right)^{\frac{2p}{p+2}} < \infty. \end{aligned}$$



Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

Page 8 of 15

This proves inequality (1.5). If  $p \geq 2$  then  $p/2 \geq 1$ , consequently

$$(2.4) \quad \begin{aligned} \|R\|_F(\Omega_1) &\geq c \left( \sum_n \left( \sum_k |\mu(A_n, B_k)|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &\geq c \left( \sum_k \left( \sum_n |\mu(A_n, B_k)|^p \right) \right)^{\frac{1}{p}}. \end{aligned}$$

Similarly

$$\|L\|_F(\Omega_2) \geq c \left( \sum_k \left( \sum_n |\mu(A_n, B_k)|^p \right) \right)^{\frac{1}{p}}.$$

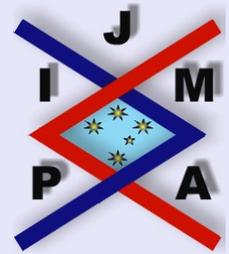
This proves inequality (2.1).

We now show that the exponent  $\frac{4p}{p+2}$  is sharp. We only consider the case  $1 < p < 2$ . Sharpness of Theorem 1.1 for the case  $p = 1$  is known [9]. Sharpness of Theorem 1.1 for  $p \geq 2$  is trivial.

We need the following result, which is a consequence of Kahane-Salem-Zygmund estimates (see [8, Theorem 3, p. 70]).

**Lemma 2.1.** *Let  $X_{n_1, n_2, \dots, n_s}$  be a subnormal collection of independent random variables. Given complex numbers  $c_{n_1, n_2, \dots, n_s}$ , where the multi-index  $(n_1, n_2, \dots, n_s)$  satisfies  $|n_1| + |n_2| + \dots + |n_s| \leq N$ , then*

$$(2.5) \quad \Pr \left\{ \sup_{t_1, \dots, t_s} \left| \sum X_{n_1, n_2, \dots, n_s} c_{n_1, n_2, \dots, n_s} e^{i(n_1 t_1 + \dots + n_s t_s)} \right| \right\}$$



Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

Page 9 of 15

$$\geq C \left[ s \sum |c_{n_1, n_2, \dots, n_s}|^2 \log N \right]^{\frac{1}{2}} \} \leq N^{-2} e^{-s},$$

where  $C$  is an independent constant.

To apply Lemma 2.1, we will need to construct an appropriate sequence of independent subnormal random variables. We will construct a Radamacher type of system, which we will call the 4-level Radamacher system.

## 2.1. 4-level Radamacher System

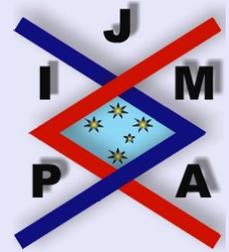
4-level Radamacher system is the sequence of independent random variables,  $\{w_j(x)\}_{j=1}^{\infty}$ , defined on the unit interval  $[0, 1]$ , such that each  $w_j$  takes on 4 discrete values  $\{2, -2, 1, -1\}$ , each with probability  $\frac{1}{4}$ . Such a system can be constructed similar to the usual Radamacher system. Observe that,  $M$  4-level Radamacher system generate  $4^M$  distinct vectors of length  $M$ . On the other hand the set  $\{1, 2, \dots, M\}$  has  $2^M$  distinct subsets.

By Lemma 2.1, for  $j, k = 1, \dots, M$ , there exists a vector  $\vec{t} = (t_1, t_2)$  and choice of scalars  $\{b_{jk}\}_{j,k=1}^M$  (approximately as many as  $(1 - \frac{1}{M^2}) 4^{M^2} - 2^{M^2}$ ), such that  $b_{jk} \in \{2, -2, 1, -1\}$ , and for any subset  $A$  of  $\{1, 2, \dots, M\}$ ,

$$(2.6) \quad \left| \sum_{j \in A} b_{jk} e^{i(kt_1 + jt_2)} \right| \leq C[4M \log(2M)]^{\frac{1}{2}},$$

and

$$(2.7) \quad \left| \sum_{k \in A} b_{jk} e^{i(kt_1 + jt_2)} \right| \leq C[4M \log(2M)]^{\frac{1}{2}}.$$



Littlewood's Inequality for  $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

Page 10 of 15

Let

$$(2.8) \quad (a) = \{a_{jk}\}_{j,k} = \{b_{jk} e^{i(jt_1 + kt_2)}\}_{j,k=1}^M.$$

Let  $A, B \subset \{1, 2, \dots, M\}$  and define

$$(2.9) \quad a(A, B) = \sum_{j \in A} \sum_{k \in B} a_{jk},$$

then by virtue of inequalities (2.7) and (2.8),

$$(2.10) \quad \|a\|_F \leq C_p M^{\frac{1}{2} + \frac{1}{p}} \sqrt{\log(2M)}.$$

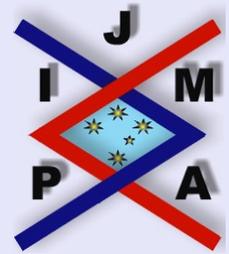
On the other hand for any  $r > 0$ ,

$$(2.11) \quad \|a\|_r = \left[ \sum_{j=1}^M \sum_{k=1}^M |a_{jk}|^r \right]^{\frac{1}{r}} \geq M^{\frac{2}{r}}.$$

We see that if  $r < \frac{4p}{p+2}$ ,

$$(2.12) \quad \lim_{M \rightarrow \infty} \frac{\|a\|_r}{\|a\|_F} = \infty.$$

This shows that  $\frac{4p}{p+2}$  is sharp. The proof for the case that  $\mu$  is a bounded-pre-bimeasure is similar.



Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

Page 11 of 15

### 3. Functions of Bounded $p$ -Variations and Related Function Spaces

Let  $p \geq 1$  and  $f$  be a function defined on  $[0, 1]^2$ . Let

$$V_p^{(2)}(f, [0, 1]^2) = \left( \sup_{\pi_1, \pi_2} \sum_{i,j} |\Delta_{i,j}^{\pi_1, \pi_2} f|^p \right)^{1/p}.$$

Here  $\pi_1 = \{0 = x_0 < x_1, < \dots < x_m = 1\}$ , and  $\pi_2 = \{0 = y_0 < y_1, < \dots < y_n = 1\}$ , are partitions of  $[0, 1]$  and

$$\Delta_{i,j}^{\pi_1, \pi_2}(f) = f(x_i, y_j) - f(x_i, y_{j-1}) - f(x_{i-1}, y_j) + f(x_{i-1}, y_{j-1}).$$

Let  $W_p^{(2)}([0, 1]^2) = W_p^{(2)}$  denote the class of functions  $f$  on  $[0, 1]^2$  such that,

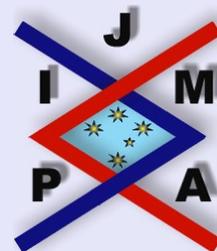
$$\begin{aligned} \|f\|_{W_p^{(2)}} &= V_p^{(2)}(f, [0, 1]^2) + V_p^{(2)}(f(0, \cdot), [0, 1]) \\ &\quad + V_p^{(1)}(f(\cdot, 0), [0, 1]) + |f(0, 0, 0)| \\ &< \infty. \end{aligned}$$

Let  $\vec{x} = (x_1, x_2)$ ,  $\vec{y} = (y_1, y_2) \in [0, 1]^2$ , and  $f$  be a function defined on  $[0, 1]^2$ . Let

$$f_{\vec{y}}(\vec{x}) = f(x_1, x_2) - f(x_1, y_2) - f(y_1, x_2) + f(y_1, y_2).$$

We say that  $f$  is a *Lipschitz function of order  $\alpha$  of first type*, if there exists a constant  $C$  such that for all  $\vec{x}$  and  $\vec{y}$  in  $[0, 1]^2$ ,

$$(3.1) \quad |f(\vec{x}) - f(\vec{y})| \leq C \|\vec{x} - \vec{y}\|_2^\alpha.$$



Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

Close

Quit

Page 12 of 15

Here  $\|\cdot\|_2$  refers to the usual  $l_2$ -norm. The class of Lipschitz functions of order  $\alpha$  of first type is denoted by  $\Lambda_\alpha^1(2)$ . We say that  $f$  is a *Lipschitz function of order  $\alpha$  of second type*, if there exists a constant  $C$  such that for all  $\vec{x}$  and  $\vec{y}$  in  $[0, 1]^2$ ,

$$(3.2) \quad |f_{\vec{y}}(\vec{x})| \leq C \|\vec{x} - \vec{y}\|_2^\alpha.$$

The class of Lipschitz functions of order  $\alpha$  of second type is denoted by  $\Lambda_\alpha^2(2)$ . If  $f \in \Lambda_\alpha^1(2)$  then

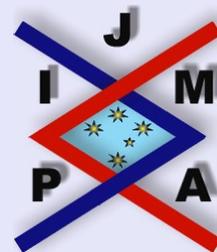
$$|f_{\vec{y}}(\vec{x})| \leq 4C \min\{|x_j - y_j|^\alpha : 1 \leq j \leq 2\} \leq C_2 \|\vec{x} - \vec{y}\|_2^\alpha.$$

Therefore,  $\Lambda_\alpha^1(2) \subset \Lambda_\alpha^2(2)$ . Using Theorem 1.1 we obtain

**Corollary 3.1.** *Let  $f$  be a function defined on  $[0, 1]^2$ . Suppose that for any  $1 \leq j \leq n$  and for any fixed partitions  $\pi_1$  and  $\pi_2$  of the interval  $[0, 1]$ , we have*

$$(3.3) \quad \sup_{\pi} \left[ \sum_{i,j} |\Delta_{i,j}^{\pi_1, \pi} f|^p \right]^{1/p} + \sup_{\pi} \left[ \sum_{i,j} |\Delta_{i,j}^{\pi, \pi_2} f|^p \right]^{1/p} \leq M < \infty,$$

then  $f \in W_{\frac{4p}{2+p}}^{(2)}$ .



Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

Title Page

Contents



Go Back

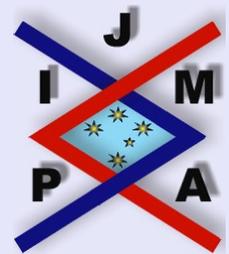
Close

Quit

Page 13 of 15

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Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

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Title Page

Contents



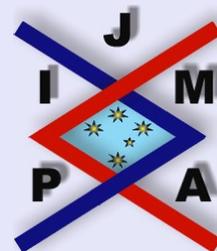
Go Back

Close

Quit

Page 14 of 15

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Littlewood's Inequality for  
 $p$ -Bimeasures

Nasser Towghi

---

Title Page

Contents



Go Back

Close

Quit

Page 15 of 15