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LITTLEWOOD'S INEQUALITY FOR $p-$BIMEASURES

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Abstract

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Abstract

In this paper we extend an inequality of Littlewood concerning the higher variations of functions of bounded Fréchet variations of two variables (bimeasures) to a class of functions that are p-bimeasures, by using the machinery of vector measures. Using random estimates of Kahane-Salem-Zygmund, we show that the inequality is sharp.

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1. Introduction

Let μ be a set function defined on the product $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$ of 2 σ-fields, such that it is a finite complex measure in each coordinate. More precisely, for each fixed $A \in \sigma(\mathcal{B}_1)$ the set function $\mu(A, \cdot)$ is a complex measure defined on $\sigma(\mathcal{B}_2)$. Similarly for each $B \in \sigma(\mathcal{B}_2)$, the set function μ gives rise to a measure in the first coordinate. Such set functions dubbed *bimeasures* by Morse and Transue were studied extensively by these and other authors (see [1, 2, 3, 5, 6, 7, 10, 11, 12]). It is well known that such set functions need not be extendible to a measure on the $\sigma-$Algebra generated by $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$. Now suppose that μ is a set function defined on $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$, such that it has finite *semi-variation*; that is,

$$\|\mu\|_F = \sup \left\{ \left\| \sum_{j,k} \mu(A_j \times B_k) r_j \otimes r_k \right\|_\infty \right\} < \infty, \tag{1.1}$$

where sup is taken over all measurable partitions $\{A_j\}$, $\{B_k\}$ of Ω_1 and Ω_2, respectively. Here $\{r_j\}$ is the usual system of Rademachers, realized as functions on the interval $[0, 1]$. By a partition of Ω, we mean a finite collection of mutually disjoint measurable sets whose union is Ω. F in $||\cdot||_F$ is for Fréchet. It is clear that a set function μ with finite semi-variation is also a bimeasure. It is interesting that the converse also holds. That is, a bimeasure has finite semi-variation. This follows easily from the machinery of vector measure theory. On the other hand, it is well known that a set function which has finite semi-variation need not have finite total variation (in the sense of Vitali), hence it may not be extendible to a measure [2, 9]. However, all is not lost, in his 1930 paper, Littlewood showed that a bimeasure has finite 4/3-variation. To make this precise we first introduce the notion of mixed variation of μ. Let $p, q > 0$, and define the mixed (p, q)-variation of μ to be

$$\|\mu\|_{p,q} = \sup\left\{\left(\sum_k\left(\sum_j|\mu(A_j\times B_k)|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\right\}, \tag{1.2}$$

where the sup is taken over all finite measurable partitions $\{A_j\}$ and $\{B_k\}$ of Ω_1 and Ω_2 respectively. In the case that $p = q$, we simply write $\|\mu\|_p$, that is $\|\mu\|_p = \|\mu\|_{p,p}$. We now state Littlewood's $4/3$ inequalities.

1.1. Littlewood's Inequalities

$$\|\mu\|_{2,1} + \|\mu\|_{1,2} + \|\mu\|_{4/3} \le c\,\|\mu\|_F\,, \tag{1.3}$$

where c is a fixed universal constant. The result is sharp in the sense that, there exists $\mu \in$ such that $\|\mu\|_p$ and $\|\mu\|_{q,1/q}$ are infinite for all $p < 4/3$ and for all $q < 2$. Extension of Littlewood's inequality to a larger class of functions of two variables is the main result of this paper.

Definition 1.1. *A set function μ defined on product of two algebras $\mathcal{B}_1\times\mathcal{B}_2$ is called a pre-p-bimeasure, if it is finitely additive in each coordinate, and for each fixed $A \in\mathcal{B}_1$, the quantity*

$$BV_p(\mu(A,\cdot)) := \sup\left\{\sum_k|\mu(A\times B_k|^p\right\}$$

is finite, and for each fixed $B \in \mathcal{B}_2$, $BV_p(\mu(\cdot, B))$ is finite. Here $\sup$ is taken over all finite measurable partitions of Ω_2.

If the set function is defined on the product of two σ-algebras with above properties, then it is called a p-bimeasure.

Definition 1.2. *A pre-p-bimeasure μ defined on product of two algebras $\mathcal{B}_1 \times \mathcal{B}_2$, is said to be bounded, if there exists a positive constant M such that $BV_p(\mu(A, \cdot)) + BV_p(\mu(\cdot, B)) \leq M$, for all $A \in \mathcal{B}_1$ and for all $B \in \mathcal{B}_2$.*

We prove the following result.

Theorem 1.1. *Suppose that either μ is a p-bimeasure defined on $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$, or that it is a bounded pre-p-bimeasure defined on $\mathcal{B}_1 \times \mathcal{B}_2$. If $1 \leq p \leq 2$ then*

$$\|\mu\|_{2,p} + \|\mu\|_{p,2} + \|\mu\|_{\frac{4p}{2+p}} < \infty. \tag{1.4}$$

In the case that $p \geq 2$, then

$$\|\mu\|_p < \infty. \tag{1.5}$$

Furthermore, the result is sharp, in the sense that, there exists a p-bimeasure such that $\|\mu\|_q = \infty$, for all $q < \frac{4p}{2+p}$.

To prove Theorem 1.1 we collect some definitions and results about vector measures. Much of the following can be found in Chapter 1 of [4].

Definition 1.3. *A function μ from a field $\mathcal{B}$ of a set Ω to a Banach space is called a finitely additive vector measure, or simply a vector measure, if whenever A_1*

and A_2 *are disjoint members of* $\mathcal{B}$ *then* $\mu(A_1 \bigcup A_2) = \mu(A_1) + \mu(A_2)$. *The variation of a vector measure* μ *is the extended nonnegative function* $|\mu|$ *whose value on the set* E *is given by*

$$|\mu|(A) = \sup_{\pi} \sum_{A \in \pi} ||\mu(A)||,$$

where the sup is taken over all partitions π *of* A *into a finite number of disjoint members of* $\mathcal{B}$. *If* $|\mu|(\Omega)$ *is finite, then* μ *will be called a measure of bounded variation.*

A different type of variation related to a vector measure μ is the so called *semi-variation* of μ. More precisely, the semi-variation of μ is the extended nonnegative function $\|\mu\|_F$ whose value on a measurable set A is given by

$$\|\mu\|_F(A) = \sup\{|x^*(\mu)|(A) : x^* \in X^*, \|x^*\| \leq 1\},$$

where $|x^*(\mu)|$ is the variation of the real-valued measure (finitely additive measure) $x^*(\mu)$. If $\|\mu\|_F(\Omega)$ is finite, then μ will be called a *measure of bounded semi-variation.*

2. Proof of Theorem 1.1

We now prove Theorem 1.1. Suppose that $1 \leq p < 2$. Let X_1 be the space of finitely additive set functions defined on $\sigma(\mathcal{B}_1)$, which have finite p-variations. Similarly let X_2 be the set finitely additive functions defined on $\sigma(\mathcal{B}_2)$ which have finite p-variations. It can be shown that equipped with p-variation norm, X_1 and X_2 are Banach spaces. Let L be the X_1-valued function defined on $\sigma(\mathcal{B}_2)$ as follows: $L(A) = \mu(\cdot, A)$, where $A \in \sigma(\mathcal{B}_2)$. Let R be the X_2-valued function defined on $\sigma(\mathcal{B}_1)$ as follows: $R(A) = \mu(A, \cdot)$, where $A \in \sigma(\mathcal{B}_1)$. If μ is a p-bimeasure then by the Nikodym Boundedness Theorem (see [4, Theorem 1, page 14]), L and R have finite semi-variations. If μ is a bounded pre-p-bimeasure then by general properties of vector measures (see e.g., [4, Proposition 11, page 4]), L and R have finite semi-variations. Let $\{A_n\}$ be a finite measurable partition of Ω_2 and $\{B_k\}$ be a finite measurable partition of Ω_1, then

$$\begin{aligned}
\infty &> ||L||_F(\Omega_2) \\
&\geq \left\| BV_p\left(\sum_n r_n \mu(A_n, \cdot)\right)\right\|_\infty \\
&\geq \left\| \left(\sum_k \left|\sum_n r_n \mu(A_n, B_k)\right|^p\right)^{\frac{1}{p}}\right\|_\infty \\
&\geq \left(\int_0^1 \sum_k \left|\sum_n r_n(x)\mu(A_n, B_k)\right|^p dx\right)^{\frac{1}{p}}
\end{aligned} \tag{2.1}$$

$$\text{(Khinchin's inequality)} \Rightarrow \geq c \left(\sum_k \left(\sum_n |\mu(A_n, B_k)|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

Similarly,

$$\infty > ||R||_F(\Omega_1) \geq c \left(\sum_n \left(\sum_k |\mu(A_n, B_k)|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}. \tag{2.2}$$

(2.2) and (2.3) imply that, $\|\mu\|_{2,p}$ is finite. Applying Minkowski's inequality we obtain $\|\mu\|_{p,2} \leq \|\mu\|_{2,p} < \infty$. We now show that $\|\mu\|_{\frac{4p}{2+p}}$ is finite. Let $a_{n,k} = \mu(A_n, B_k)$. Applying Hölder's inequality with exponents $\frac{2+p}{p}$ and $\frac{2+p}{2}$, we obtain

$$\begin{aligned} \sum_{n,k} |a_{n,k}|^{\frac{4p}{2+p}} &= \sum_{n,k} |a_{n,k}|^{\frac{2p}{2+p}} |a_{n,k}|^{\frac{2p}{2+p}} \\ &\leq \sum_n \left[\sum_k |a_{n,k}|^2 \right]^{\frac{p}{2+p}} \left[\sum_k |a_{n,k}|^p \right]^{\frac{2}{p+2}} \\ &\leq \left[\sum_n (\sum_k |a_{n,k}|^2)^{\frac{p}{2}} \right]^{\frac{2}{2+p}} \left[\sum_n \left(\sum_k |a_{n,k}|^p \right)^{\frac{2}{p}} \right]^{\frac{p}{2+p}} \\ &\leq \left(\|\mu\|_{2,p} \|\mu\|_{p,2} \right)^{\frac{2p}{p+2}} < \infty. \end{aligned} \tag{2.3}$$

This proves inequality (1.5). If $p \geq 2$ then $p/2 \geq 1$, consequently

$$||R||_F(\Omega_1) \geq c \left(\sum_n \left(\sum_k |\mu(A_n, B_k)|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \tag{2.4}$$
$$\geq c \left(\sum_k \left(\sum_n |\mu(A_n, B_k)|^p \right) \right)^{\frac{1}{p}}.$$

Similarly

$$||L||_F(\Omega_2) \geq c \left(\sum_k \left(\sum_n |\mu(A_n, B_k)|^p \right) \right)^{\frac{1}{p}}.$$

This proves inequality (2.1).

We now show that the exponent $\frac{4p}{p+2}$ is sharp. We only consider the case $1 < p < 2$. Sharpness of Theorem 1.1 for the case $p = 1$ is known [9]. Sharpness of Theorem 1.1 for $p \geq 2$ is trivial.

We need the following result, which is a consequence of Kahane-Salem-Zygmund estimates (see [8, Theorem 3, p. 70]).

Lemma 2.1. *Let $X_{n_1,n_2,\ldots n_s}$ be a subnormal collection of independent random variables. Given complex numbers $c_{n_1,n_2,\ldots,n_s}$, where the multi-index $(n_1, n_2, \ldots, n_s)$ satisfies $|n_1| + |n_2| + \cdots + |n_s| \leq N$, then*

$$\Pr\left\{ \sup_{t_1,\ldots,t_s} \left| \sum X_{n_1,n_2,\ldots n_s} c_{n_1,n_2,\ldots,n_s} e^{i(n_1 t_1 + \cdots n_s t_s)} \right| \right. \tag{2.5}$$

$$\geq C\left[s\sum |c_{n_1,n_2,\ldots,n_s}|^2\log N\right]^{\frac{1}{2}}\Bigg\}\leq N^{-2}e^{-s},$$

where C is an independent constant.

To apply Lemma 2.1, we will need to construct an appropriate sequence of independent subnormal random variables. We will construct a Radamacher type of system, which we will call the 4-level Radamacher system.

2.1. 4-level Radamacher System

4-level Radamacher system is the sequence of independent random variables, $\{w_j(x)\}_{j=1}^{\infty}$, defined on the unit interval $[0,1]$, such that each w_j takes on 4 discrete values $\{2,-2,1,-1\}$, each with probability $\frac{1}{4}$. Such a system can be constructed similar to the usual Radamacher system. Observe that, M 4-level Radamacher system generate 4^M distinct vectors of length M. On the other hand the set $\{1,2,\ldots,M\}$ has 2^M distinct subsets.

By Lemma 2.1, for $j,k=1,\ldots,M$, there exists a vector $\vec{t}=(t_1,t_2)$ and choice of scalers $\{b_{jk}\}_{j,k=1}^{M}$ (approximately as many as $\left(1-\frac{1}{M^2}\right)4^{M^2}-2^{M^2}$), such that $b_{jk}\in\{2,-2,1,-1\}$, and for any subset A of $\{1,2,\ldots,M\}$,

$$\left|\sum_{j\in A} b_{jk}e^{i(kt_1+jt_2)}\right|\leq C[4M\log(2M)]^{\frac{1}{2}}, \tag{2.6}$$

and

$$\left|\sum_{k\in A} b_{jk}e^{i(kt_1+jt_2)}\right|\leq C[4M\log(2M)]^{\frac{1}{2}}. \tag{2.7}$$

Let

$$(a) = \{a_{jk}\}_{j,k} = \left\{b_{jk}e^{i(jt_1+kt_2)}\right\}_{j,k=1}^{M}. \tag{2.8}$$

Let $A, B \subset \{1, 2, ..., M\}$ and define

$$a(A, B) = \sum_{j\in A}\sum_{k\in B} a_{jk}, \tag{2.9}$$

then by virtue of inequalities (2.7) and (2.8),

$$||a||_F \leq C_p M^{\frac{1}{2}+\frac{1}{p}}\sqrt{\log(2M)}. \tag{2.10}$$

On the other hand for any $r > 0$,

$$||a||_r = \left[\sum_{j=1}^{M}\sum_{k=1}^{M}|a_{jk}|^r\right]^{\frac{1}{r}} \geq M^{\frac{2}{r}}. \tag{2.11}$$

We see that if $r < \frac{4p}{p+2}$,

$$\lim_{M\to\infty}\frac{||a||_r}{||a||_F} = \infty. \tag{2.12}$$

This shows that $\frac{4p}{p+2}$ is sharp. The proof for the case that μ is a bounded-pre-bimeasure is similar.

3. Functions of Bounded p-Variations and Related Function Spaces

Let $p \geq 1$ and f be a function defined on $[0,1]^2$. Let

$$V_p^{(2)}(f,[0,1]^2) = \left(\sup_{\pi_1,\pi_2} \sum_{i,j} |\Delta_{i,j}^{\pi_1,\pi_2} f|^p \right)^{1/p}.$$

Here $\pi_1 = \{0 = x_0 < x_1, < \cdots < x_m = 1\}$, and $\pi_2 = \{0 = y_0 < y_1, < \cdots < y_n = 1\}$, are partitions of $[0,1]$ and

$$\Delta_{i,j}^{\pi_1,\pi_2}(f) = f(x_i,y_j) - f(x_i,y_{j-1}) - f(x_{i-1},y_j) + f(x_{i-1},y_{j-1}).$$

Let $W_p^{(2)}([0,1]^2) = W_p^{(2)}$ denote the class of functions f on $[0,1]^2$ such that,

$$\begin{aligned} ||f||_{W_p^2} &= V_p^{(2)}(f,[0,1]^2) + V_p^{(2)}(f(0,\cdot),[0,1]) \\ &\quad + V_p^{(1)}(f(\cdot,0),[0,1]) + |f(0,0,0)| \\ &< \infty. \end{aligned}$$

Let $\vec{x} = (x_1,x_2)$, $\vec{y} = (y_1,y_2) \in [0,1]^2$, and f be a function defined on $[0,1]^2$. Let

$$f_{\vec{y}}(\vec{x}) = f(x_1,x_2) - f(x_1,y_2) - f(y_1,x_2) + f(y_1,y_2).$$

We say that f is a *Lipschitz function of order α of first type*, if there exists a constant C such that for all $\vec{x}$ and $\vec{y}$ in $[0,1]^2$,

$$|f(\vec{x}) - f(\vec{y})| \leq C||\vec{x} - \vec{y}||_2^{\alpha}. \tag{3.1}$$

Here $||\cdot||_2$ refers to the usual l_2-norm. The class of Lipschitz functions of order α of first type is denoted by $\Lambda^1_\alpha(2)$. We say that f is a *Lipschitz function of order α of second type*, if there exists a constant C such that for all $\vec{x}$ and $\vec{y}$ in $[0,1]^2$,

$$|f_{\vec{y}}(\vec{x})| \leq C||\vec{x}-\vec{y}||_2^\alpha. \tag{3.2}$$

The class of Lipschitz functions of order α of second type is denoted by $\Lambda^2_\alpha(2)$. If $f \in \Lambda^1_\alpha(2)$ then

$$|f_{\vec{y}}(\vec{x})| \leq 4C \min\{|x_j - y_j|^\alpha : 1 \leq j \leq 2\} \leq C_2||\vec{x}-\vec{y}||_2^\alpha\}.$$

Therefore, $\Lambda^1_\alpha(2) \subset \Lambda^2_\alpha(2)$. Using Theorem 1.1 we obtain

Corollary 3.1. *Let f be a function defined on $[0,1]^2$. Suppose that for any $1 \leq j \leq n$ and for any fixed partitions π_1 and π_2 of the interval $[0,1]$, we have*

$$\sup_\pi \left[\sum_{i,j} |\Delta^{\pi_1,\pi}_{i,j} f|^p\right]^{1/p} + \sup_\pi \left[\sum_{i,j} |\Delta^{\pi,\pi_2}_{i,j} f|^p\right]^{1/p} \leq M < \infty, \tag{3.3}$$

then $f \in W^{(2)}_{\frac{4p}{2+p}}$.

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