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TWO REMARKS ON THE STABILITY OF GENERALIZED HEMIVARIATIONAL INEQUALITIES

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ABSTRACT. The present paper is devoted to the stability analysis of a general class of hemivariational inequalities. Essentially, we present two approaches for this class of problems. First, using a general version of Minty's Lemma and the convergence result of generalized gradients due to T. Zolezzi [23], we prove a stability result in the spirit of Mosco's results on the variational inequalities [14]. Second, we provide a quite different stability result with an estimate for the rate of convergence of solutions when the given perturbed data are converging with respect to an appropriate distance. Illustration is given with respect to a hemivariational inequality modelling the buckling of adhesively connected von kármán plates.

Key words and phrases: Generalized hemivariational inequalities, Clarke's gradient, Perturbation, Epi-convergence, Stability, Rate of convergence, Equilibrium problems, von kármán plates.

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1. INTRODUCTION

The theory of inequalities has received remarkable developments in both pure and applied mathematics as well as in mechanics, engineering sciences and economics. This theory has been a key feature in the understanding and solution of many practical problems such as market price equilibria, traffic assignments, monetary policy setting and so on. In this context, variational inequalities have been the appropriate framework for studying some of these problems during the last forty years. More recently, new and efficient mathematical inequalities, called hemivariational inequalities, have facilitated the solution to many challenging open questions in mechanics and engineering. This class of problems has been pioneered by the work of Panagiotopoulos [18] who introduced a variational formulation involving nonconvex and nonsmooth

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energy functions. Subsequently, it has been developed from the point of view of existence results by many authors, we refer to [5, 6], [10], [17], [16], [20] and references therein.

In this paper, we attempt to investigate stability results for the following generalized hemivariational inequalities: for any $n \in \mathbb{N}$, find $u_n \in X$ such that for all $v \in X$

$$(GHI_n) \qquad \Phi_n(u_n, v) + \Psi_n(u_n, v) + J_n^0(u_n; v - u_n) + \varphi_n(v) - \varphi_n(u_n) \geq 0.$$

holds.

Here, X is a Banach space, $(\Phi_n)_{n\geq 0}, (\Psi_n)_{n\geq 0}$ are sequences of real valued bifunctions defined on $X \times X$, $(\varphi_n)_{n\geq 0}$ a sequence of extended real valued functions and $(J_n)_n$ a sequence of real locally Lipschitz functions; J_n^0 is the Clarke's derivative of J_n. The main question is then the following : under what conditions do the solutions u_n to (GHI_n) converge to a solution of the initial problem (GHI_0)?

The remainder of the paper is organized as follows. In Section 2, we discuss a concrete mechanical example that has motivated our study. Section 3 is devoted to our main stability results. We present two approaches. Namely, we first propose a general version of Minty's Lemma and proceed by the epi-convergence method, Theorem 3.2. Further, we define a "distance" between two bifunctions and present a stability result with an estimate for the rate of convergence of solutions in terms of the given data rate of convergence: Theorem 3.20 is first stated in equilibrium problems formulation and Corollary 3.21 is then derived for (GHI_0). In Section 4, we illustrate the abstract results by an application to a hemivariational inequality that models the buckling of adhesively connected von kármán plates allowing for delamination. Finally, we conclude with some comments.

2. MECHANICAL EXAMPLE

To illustrate the idea of hemivariational inequalities and explain how important this class of inequalities is, we suggest the following model[1] summarized from [15], further details and similar models can be found in [16, 17, 18, 20]. The model is concerned with the buckling of adhesively connected von kármán plates allowing for delamination. Roughly speaking, it consists of characterizing the position on equilibrium of the plates and lead to research of solution to special problem formulated as a hemivariational inequality. Let us now formulate the problem. Consider a plate Ω and the binding material on Ω'. In the undeformed state, the middle of the plate occupies an open, bounded and connected subset Ω of $\mathbb{R}^2$, referred to a fixed right-handed Cartesian coordinate system $Ox_1x_2x_3$. Let Γ be the boundary of the plate: Γ is assumed to be appropriately regular. Let also the binding material occupy a subset Ω' such that $\Omega' \subset \Omega$ and $\bar{\Omega}' \bigcap \Gamma = \emptyset$. We denote by $\zeta(x)$ the vertical deflection of the point $x \in \Omega$ of the plate, and by $f = (0, 0, f_3(x))$ the distributed vertical load. Further, let $u = \{u_1, u_2\}$ be the in-plane displacement of the plate. We assume that the plate has constant thickness h. Moreover, we assume that the plate obeys the Von kármán theory, i.e. it is a thin plate having large deflections. The von kármán plates verify the following system of differential equations:

$$(2.1) \qquad K\triangle\triangle\zeta - h(\sigma_{\alpha\beta}\zeta_{,\beta})_{,\alpha} = f \text{ in } \Omega_j,$$

$$(2.2) \qquad \sigma_{\alpha\beta,\beta} = 0 \text{ in } \Omega_j,$$

$$(2.3) \qquad \sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}(\epsilon_{\gamma\delta}(u) + \frac{1}{2}\zeta_{,\gamma}\zeta_{,\delta}) \text{ in } \Omega_j.$$

Here the subscripts, $\alpha, \beta, \gamma, \delta = 1, 2$ correspond to the coordinate directions: $\{\sigma_{\alpha\beta}\}$, $\{\varepsilon_{\alpha\beta}\}$ and $C_{\alpha\beta\gamma\delta}$ denotes the stress, strain and elasticity tensors in the plane of the plate. The components

[1] We have recalled in details this model as it was stated in [15] in concern with existence of solutions, here we deal with stability issue under data perturbation for hemivariational inequalities modelling such problems.

of C are elements of $L^\infty(\Omega)$ and have the usual symmetry and ellipticity properties (further explanations and figures can be found in [15]). Moreover, $K = \frac{Eh^3}{12(1-\nu^2)}$ is the bending rigidity of the plate with E the modulus of elasticity and ν the Poisson ratio. For the sake of simplicity, we consider here isotropic homogeneous plates of constant thickness. In laminated and layered plates, the interlaminar normal stress σ_{33} is one of the main cause for delamination effects. Note that this is a simplification of the problem. In order to model the action of σ_{33}, f is split into a vector $\bar{f}$, which describes the action of the adhesive and $\overline{\overline{f}} \in L^2(\Omega)$, which represents the external loading applied on the plate:

$$f = \overline{f} + \overline{\overline{f}} \text{ in } \Omega.$$

We introduce now a phenomenological law connecting $\bar{f}$ with the corresponding deflection of the plate describing the action of adhesive material. We assume that:

$$-\overline{f} \in \overline{\beta}(\zeta) \text{ in } \Omega', \tag{2.4}$$

where $\overline{\beta}$ is a multivalued function defined as in [19] (by filling in the jumps in the graph of a function $\beta \in L^\infty_{loc}(\mathbb{R})$). We note here that cracking as well as crushing effects of either a brittle or semi-brittle nature can be accounted for by means of this law. The following relation completes in a natural way the definition of $\overline{f}$:

$$\overline{f} = 0 \text{ in } \Omega - \Omega'.$$

In order to obtain a variational formulation of the problem, we express relation (2.4) in a superpotential form. If $\beta(\xi \pm 0)$ exists for every $\xi \in \mathbb{R}$ then, from [7] and [19] a locally Lipschitz (nonconvex) function $J : \mathbb{R} \to \mathbb{R}$ can be determined up to an additive constant such that

$$\overline{\beta}(\xi) = \partial J(\xi),$$

where ∂ is the generalized gradient of Clarke[1]. Moreover, we suppose the following boundary condition on the plate boundary:

$$\zeta = 0 \text{ on } \Gamma.$$

Now, let us denote by n the outward normal unit vector to Γ and by g_α ($\alpha = 1, 2$) the self-equilibrating forces and assume for the in-plane action the boundary conditions

$$\sigma_{\alpha\beta} n_\beta = g_\alpha \text{ on } \Gamma \;\; \alpha = 1, 2. \tag{2.5}$$

Notice that in [15], (2.5) involves an eigenvalue λ such that $\sigma_{\alpha\beta} n_\beta = \lambda g_\alpha$. Here we take $\lambda = 1$. For the moment we assume that $g_\alpha = 0 \;\; \alpha = 1, 2$. We can now derive the variational formulation of the problem. From (2.1), by assuming sufficiently regular functions, multiplying by $z^{(j)} - \zeta^{(j)}$, integrating and applying the Green-Gauss theorem, we obtain the expression (E):

$$\begin{aligned} \alpha(\zeta, z-\zeta) + \int_\Omega h\sigma_{\alpha\beta}\zeta,\alpha(z-\zeta)_{,\beta} d\Omega \;&=\; \int_\Gamma h\sigma_{\alpha\beta}\zeta_{,\beta} n_\alpha (z-\zeta) d\Gamma + \int_\Omega \overline{\overline{f}}(z-\zeta) d\Omega \\ &\quad + \int_\Gamma K_n(\zeta)(z-\zeta) d\Gamma - \int_\Gamma M_n(\zeta) \frac{\partial(z-\zeta)}{\partial n} d\Gamma. \end{aligned}$$

Here, $\alpha, \beta = 1$, n denotes the outward normal unit vector to Γ,

$$\alpha(\zeta, z) = K \int_\Omega [(1-\nu)\zeta_{,\alpha\beta} z_{,\alpha\beta} + \nu \triangle\zeta \triangle z] d\Omega, \;\; 0 < \nu < 0.5, \tag{2.6}$$

$$M_n(\zeta) = -K\left[\nu\triangle\zeta + (1-\nu)\left(2n_1 n_2 \zeta_{,12} + n_1^2 \zeta_{,11} + n_2^2 \zeta_{,22}\right)\right] \tag{2.7}$$

[1]For the convenience of the reader we recall (see [8]) that ∂j is defined by $\partial j(x) = \{\zeta \in Z^* : \langle \zeta, v\rangle \leq j^0(x; v) \text{ for all } v \text{ in } Z\}$ and $j^0(x; v) := \limsup_{\substack{y\to x \\ t\searrow 0}} \frac{1}{t}\left(j\left(y+tv\right) - j\left(y\right)\right)$.

and

$$K_n(\zeta) = -K\left[\frac{\partial \triangle \zeta}{\partial n} + (1-\nu)\frac{\partial}{\partial \tau}\left[n_1 n_2(\zeta_{,22} - \zeta_{,11}) + (n_1^2 - n_2^2)\zeta_{,12}\right]\right], \tag{2.8}$$

where τ is the unit vector tangential to Γ such that $\nu.\tau$ and the Ox_3-form a right-handed system. A similar argument applied to (2.2) leads to the following expression

$$\int_\Omega \sigma_{\alpha\beta}\varepsilon_{\alpha\beta}(v-u)d\Omega = \int_\Gamma \sigma_{\alpha\beta}n_\beta(v_\alpha - u_\alpha)d\Gamma.\ \alpha,\beta = 1,2. \tag{2.9}$$

Further, the following notations are introduced:

$$R(m,k) = \int_\Omega C_{\alpha\beta\gamma\delta}m_{\alpha\beta}k_{\alpha\beta}d\Omega.\ \alpha,\beta,\gamma,\delta = 1,2. \tag{2.10}$$

and

$$P(\zeta, z) = \{\zeta_{,\alpha}z_{,\beta}\},\quad P(\zeta,\zeta) = P(\zeta), \tag{2.11}$$

where $m = \{m_{\alpha\beta}\}$ and $k = \{k_{\alpha\beta}\}$, $\alpha,\beta = 1,2$ are 2×2 tensors.

Let us also introduce the functional framework. We assume that $u.v \in [H^1(\Omega)]^2$ and that $\zeta, z \in Z$, where

$$Z = \{z | z \in H^2(\Omega), z = 0 \text{ on } \Gamma\}.$$

Taking into account expression (E), (2.9), the boundary conditions and the inequalities defining the multivalued operator ∂ we obtain the following problem: find $u \in [H^1(\Omega)]^2$ and $\zeta \in Z$ such as to satisfy the hemivariational inequality (HI):

$$\alpha(\zeta, z-\zeta) + hR(\varepsilon(u) + \frac{1}{2}P(\zeta), P(\zeta, z-\zeta)) + \int_{\Omega'} J^0(\zeta, z-\zeta)d\Omega \geq \int_\Omega \overline{\overline{f}}(z-\zeta)d\Omega.\ \forall z \in Z$$

and the variational equality (VE) :

$$R(\varepsilon(u) + \frac{1}{2}P(\zeta), \varepsilon(v-u)) = 0,\ \forall v \in [H^1(\Omega)]^2.$$

Further we shall eliminate the in-plane displacement of the plate. To this end we note first that $R(.,.)$ as defined in (2.10) is a continuous symmetric, coercive bilinear form on $[L^2(\Omega)]^4$ and that $P : [H^2]^2 \to [L^2(\Omega)]^4$ of (2.11) is a completely continuous operator (see [20] p. 219). Thus the equality (VE) and the Lax-Milgram theorem imply that to every deflection $\zeta \in Z$, there corresponds a plane displacement $u(\zeta) \in [H^1(\Omega)]^2$. Indeed, due to Korn's inequality $R(\varepsilon(u), \varepsilon(v))$ is a bilinear coercive form on the quotient space $[H^1(\Omega)]^2/\bar{R}$, where $\bar{R}$ is the space of in-plane rigid displacements defined by

$$\bar{R} = \{\bar{r}/\bar{r} \in [H^1(\Omega)]^2, \bar{r}_1 = \alpha_1 + bx_2, \bar{r}_2 = \alpha_2 - bx_2, \alpha_1, \alpha_2, b \in \mathbb{R}\}. \tag{2.12}$$

From (VE) it results that

$$\varepsilon(u(\zeta)) : Z \to [L^2(\Omega)]^4 \tag{2.13}$$

is uniquely determined and is completely quadratic function of ζ, since $\varepsilon(u(\zeta))$ is a linear continuous function of $P(\zeta)$. We also introduce the completely continuous quadratic function $G : Z \to [L^2(\Omega)]^4$ which is defined by

$$\zeta \to G(\zeta) = \varepsilon(u(\zeta)) + \frac{1}{2}P(\zeta) \tag{2.14}$$

and satisfies the equation

$$R(G(\zeta), \varepsilon(u(\zeta))) = 0. \tag{2.15}$$

We now define the operator: $A : Z \to Z$ and $C : Z \to Z$ such that

$$\alpha(\zeta, z) = (A\zeta, z) \tag{2.16}$$

and

$$hR(G(\zeta), P(\zeta.z)) = (C(\zeta), z). \tag{2.17}$$

A is a continuous linear operator, C a completely continuous operator and $(\cdot, \cdot)$ denotes the scalar product in Z. Thus the following problem results:

find $\zeta \in Z$, so as to satisfy the hemivariational inequality

$$a(\zeta, z - \zeta) + (C(\zeta), z - \zeta) + \int_{\Omega'} j^0(\zeta, z - \zeta) d\Omega \geq \int_{\Omega} \overline{\overline{f}}(z - \zeta) d\Omega \ \forall z \in Z. \tag{2.18}$$

The last hemivariational inequality characterizes the position of equilibrium of the studied problem. Note that the second member of (2.18) can be expressed by means of a linear, self-adjoint and compact operator B. For the explicit form of B, we refer to [20] (equation 7.2.13).

Therefore, this problem can be viewed as, and actually is, a particular case of (GHI_0).

Remark 2.1. Notice that if we take $J = 0$, (GHI_0) covers the Generalized variational and quasi variational inequalities. Some other mathematical problems contained in (GHI_0) can be found in [4].

3. MAIN CONVERGENCE RESULTS

In this section, we present our stability results. By means of a general version of the celebrated Minty's Lemma, we proceed first by the epi-convergence method. In the sequel, unless another framework is specified, the space X is a Banach space with dual X^* equipped with the weak* topology denoted by w^*. The symbols $\to$ will stand for the strong convergence both in X and X^*. We first recall the following definitions:

Definition 3.1. A sequence $f_n : X \to (-\infty, +\infty)$ is said to be equi-lower semidifferentiable iff for every $x \in X$ there exists a ball B around x such that for every $\varepsilon > 0$ we can find $\delta > 0$ so as

$$f_n(z) \geq f_n(y) + \langle u, z - y \rangle - \varepsilon \|z - y\| \tag{3.1}$$

for every $y \in B$, every n, every $u \in \partial^- f_n(y)$ and every z such that $\|z - y\| \leq \delta$. Where ∂^- denotes the lower semigradient given for some function g and $x \in X$ by: $u \in \partial^- g(x)$ iff $u \in X^*$ and

$$\liminf_{y \to x} (g(y) - g(x) - \langle u, y - x \rangle)/\|y - x\| \geq 0.$$

Definition 3.2. A sequence $f_n : X \to (-\infty, +\infty)$ is called strongly epi-convergent to $f : X \to (-\infty, +\infty)$ iff $v_n \to v$ implies $f(v) \leq \liminf_n f_n(v_n)$, and for every $v \in X$ there exists a sequence $v_n \to v$ such that: $\limsup_n f_n(v_n) \leq f(v)$.

3.1. Epi-convergence approach.

Having our applications in mind, we make the following assumptions:

- (H_0) X is separable and has a equivalent norm that is Fréchet differentiable off 0;
- (H_1)
  - i) Φ_0 is monotone, that is for each $u, v \in K$, $\Phi_0(u, v) + \Phi_0(v, u) \leq 0$;
  - ii) Φ_0 is upper hemicontinuous i.e., for all $u, v, w \in X$, the map $t \in [0, 1] \mapsto \Phi_0(tu + (1 - t)v, w)$ is upper semicontinuous;
  - iii) Φ_0 is convex on the second argument and $\Phi_0(u, u) = 0$ for all $u \in X$;
- (H_2) Ψ_0 is convex on the second argument and $\Psi_0(u, u) = 0$ for all $u \in X$;
- (H_3) φ_0 is proper and convex;

(H_4) Φ_n is monotone for each n and (Φ_n) lower-converges to $\Phi_0 : \forall\, u \in X, v \in X$, $\forall u_n \to u$ and $\forall (v_n)_n \to v$ it results $\Phi_0(u,v) \le \liminf_n \Phi_n(u_n, v_n)$;

(H_5) (Ψ_n) upper-converges to $\Psi_0 : \forall\, u \in X, v \in X$, $\forall u_n \to u$ and $\forall v_n \to v$ for a subsequence n_k one has $\limsup_k \Psi_{n_k}(u_{n_k}, v_{n_k}) \le \Psi_0(u,v)$;

(H_6) the sequence $(\varphi_n)_n$ is strongly epi-convergent to φ_0;

(H_7) The sequence J_n is locally equi-Lipschitz, that is for every ball B in X there exists $M > 0$ such that

$$|J_n(u) - J_n(v)| \le M\|u - v\|$$

for all $u, v \in B$ and all n;

(H_8) $(J_n)_n$ is equi-lower semidifferentiable and strongly epi-convergent to J_0;

Remark 3.1. We should notice that we do not need to make appeal to the assumption (H_1) $i)$ since it is included in (H_4). Indeed, for any $u, v \in X$ and for some $u_n \to u$ and $v_n \to v$, let us remark that

$$\begin{aligned}\Phi_0(u,v) + \Phi_0(v,u) &\le \liminf_n \Phi_n(u_n, v_n) + \liminf_n \Phi_n(v_n, n_n)\\ &\le \liminf_n [\Phi_n(u_n, v_n) + \Phi(v_n, u_n)]\\ &\le 0\ .\end{aligned}$$

In the following theorem we denote by S_n the set of solutions to (GHI_n).

Theorem 3.2. *Suppose that assumptions $(H_0) - (H_8)$ are verified. Then, we have*

$$s - \liminf_n S_n \subset S_0.$$

Remark 3.3. The result of Theorem 3.2 means that whenever a sequence u_n of solutions to (GHI_n) is strongly converging to u, u is a solution to (GHI_0).

To prove this theorem, we first collect some lemmas.

Lemma 3.4. [8] *Let g be a real Lipschitz function of rank k near x. Then, the function $v \to g^0(x; v)$ is positively homogeneous and subadditive (thus convex), continuous and Lipschitz of rank k on X.*

Lemma 3.5 (Minty's)**.** *Let f be an extended real-valued bifunction such that f is convex in the second argument and $f(v,v) = 0$ for each $v \in X$. Assume moreover that (H_1) hold, then the following statements are equivalent.*

$a)$ *There exists $u \in X$ such that for every $v \in X$,*

$$\Phi_0(u,v) + f(u,v) \ge 0\ .$$

$b)$ *There exists $u \in X$ such that for every $v \in X$,*

$$\Phi_0(v,u) \le f(u,v).$$

Proof. $a) \Rightarrow b)$ Let $u \in X$ such that $a)$ is satisfied. Thus we have

$$-\Phi_0(u,v) \le f(u,v).$$

since Φ_0 is monotone, it follows that $\Phi_0(v,u) \le -\Phi_0(u,v)$, $\forall v \in X$. Therefore, for every $v \in X$ we have

$$\Phi_0(v,u) \le f(u,v),$$

which means that $b)$ is verified.

$b) \Rightarrow a)$ Let u be a solution in $b)$ and fix $v \in X$ and $t \in]0,1[$. Then, using $(H_1)\, iii)$ and the convexity of $f(u,.)$, for $w_t = tu + (1-t)v$, we have

$$\begin{aligned} 0 = \Phi_0(w_t, w_t) &\leq (1-t)\Phi_0(w_t, v) + t\Phi_0(w_t, u) \\ &\leq (1-t)\Phi_0(w_t, v) + tf(u, w_t) \\ &\leq (1-t)\Phi_0(w_t, v) + t(1-t)[f(u,v)]. \end{aligned}$$

Because $f(u,u) = 0$. Therefore,

$$-t[f(u,v)] \leq \Phi_0(w_t, v).$$

Hence, by upper hemicontinuity of Φ_0, we end at

$$-[f(u,v)] \leq \limsup_{t\to 1} \Phi(w_t, v) \leq \Phi_0(u,v)$$

which leads to

$$0 \leq \Phi_0(u,v) + f(u,v).$$

Finally, v being arbitrary chosen in X, the last inequality means that $a)$ is satisfied □

Remark 3.6. Notice that a particular case of Lemma 3.5 is the variational Minty's Lemma given in [13, p. 249] as follows:

$$\text{find } u \in X \text{ such that} : \ \langle l, v-u\rangle \leq \langle A(u), v-u\rangle \ \text{ for all } v \in X$$

is equivalent to

$$\text{find } u \in X \text{ such that} : \ \langle l, v-u\rangle \leq \langle A(v), v-u\rangle \ \text{ for all } v \in X$$

where A is an hemicontinuous and monotone operator from a Banach space X into its topological dual X^*, and $l \in X^*$.

Lemma 3.7. [8] *Let g be as stated in Lemma 3.4. Then, $\partial g(x)$ is a nonempty, convex, week*−compact subset of X^* and $\|\zeta\| \leq k$ for each $\zeta \in \partial g(x)$.*

Lemma 3.8. *Assumption (H_7) holds. Then, the sequence of set-valued map $(\partial J_n)_n$ is uniformly bounded.*

Proof. Let u_n be a bounded sequence. $(u_n)_n$ is then contained in a ball $B = B(0,r)$ where $r > 0$. Let also M be a positive constant such that

$$|J_n(u) - J_n(v)| \leq M\|u - v\| \ \text{ for all } \ u, v \in B \ \text{ and all } \ n.$$

Therefore by Lemma 3.7 we deduce that: $\|\partial J_n(u_n)\| \leq M$, that is whenever $\xi \in \partial J_n(u_n)$ we have $\|\xi\| \leq M$. This means that ∂J_n is uniformly bounded. □

Lemma 3.9. [8] *Consider g as stated in Lemma 3.4. Then, For every v in X, one has*

$$g^0(x;v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial g(x)\}.$$

Lemma 3.10. *Under (H_0), (H_7) and (H_8) , for any $u, v \in X$ and any $u_n \to u, v_n \to v$ there exists a subsequence $(n_k)_k$ such that*

$$\limsup_k J^0_{n_k}(u_{n_k}; v_{n_k}) \leq J^0_0(u; v).$$

Remark 3.11. To simplify the notation we consider, without loss of generality, the inequality of Lemma 3.10 as:

$$\limsup_n J^0_n(u_n; v_n) \leq J^0_0(u; v).$$

Proof. Let $u_n \to u$, $\xi_n \in \partial J_n(u_n)$ and let $v_n \to v$. As, for each n, $\partial J_n(u_n)$ is weakly compact, there exists a map $\xi_n : X \to X^*$ defined, for each $w \in X$, as follows: $\xi_n(w) \in \partial J_n(u_n)$ such that

$$(\xi_n(w), w) = \max_{\xi \in \partial J_n(u_n)} (\xi, w) = J_n^0(u_n; w).$$

Since u_n is bounded, by Lemma 3.8 it results that $(\xi_n(v_n))_n$ is bounded. Therefore, $(\xi_n(v_n))_n$ has a weakly converging subsequence also denoted by $(\xi_n(v_n))_n$. Let $\xi(v) \in X^*$ be the weak* limit of $\xi_n(v_n)$. On the other hand, (H_7) implies that $(J_n)_n$ is locally equi-bounded. Then, by (H_0) and (H_8), we apply [23, Theorem 1] and obtain

$$\limsup_n gph\ \partial J_n \subset gph\ \partial J_0 \text{ in } (X, s) \times (X^*, w^*),$$

which implies that $\xi(v) \in \partial J_0(u)$. Hence, taking Lemma 3.9 into account, we end at

$$\begin{aligned}\limsup_n J_n^0(u_n; v_n) &= \limsup_n \max_{\xi \in \partial J_n(u_n)} \langle \xi, v_n \rangle \\ &= \limsup_n \langle \xi_n(v_n), v_n \rangle \\ &= \langle \xi(v), v \rangle \\ &\leq \max_{\xi \in \partial J_0(u)} \langle \xi, v \rangle = J_0^0(u; v).\end{aligned}$$

□

Proof of Theorem 3.2. Let $u_n \in s - \liminf S_n$ and u be the strong limit of u_n. We wish to prove that $u \in S_0$. To this end, fix $v \in X$. By (H_6) there exists a sequence $(v_n)_n$ such that $v_n \to v$ and $\limsup_n \varphi_n(v_n) \leq \varphi_0(v)$. As u_n is a solution to (GHI_n), by monotonicity of Φ_n it follows:

$$\Phi_n(v_n, u_n) \leq \Psi_n(v_n, v_n) + J_n^0(u_n; v_n - u_n) + \varphi_n(v_n) - \varphi_n(u_n).$$

hence, taking into account $(H_4) - (H_6)$ and Lemma 3.10, there exists $(n_k)_k$ such that

$$\begin{aligned}\Phi_0(v, u) &\leq \liminf_k \Phi_{n_k}(v_{n_k}, v_{n_k}) \\ &\leq \limsup_k J_{n_k}^0(u_{n_k}; v_{n_k} - u_{n_k}) - \liminf_k \varphi_{n_k}(u_{n_k}) \\ &\qquad + \limsup_k \varphi_{n_k}(v_{n_k}) + \limsup_k \Psi_{n_k}(v_{n_k}, v_{n_k}) \\ &\leq \Psi_0(u, v) + J_0^0(u; v - u) + \varphi_0(v) - \varphi_0(u).\end{aligned}$$

therefore,

$$\Phi_0(v, u) \leq \Psi(u, v) + J_0^0(u; v - u) + \varphi_0(v) - \varphi_0(u).$$

Since φ_0 is proper, it follows that $u \in dom(\varphi)$. Hence, as $J_0^0(u; . - u)$ is convex (Lemma 3.4), we can take $f = \Psi_0(u, v) + J_0^0(u; v - u) + \varphi_0(v) - \varphi_0(u)$ in Lemma 3.5 and obtain

$$0 \leq \Phi_0(u, v) + \Psi_0(u, v) + J_0^0(u; v - u) + \varphi_0(v) - \varphi_0(u).$$

Now, v being arbitrary chosen, we conclude that u is a solution to (GHI_0). The proof is therefore complete. □

Remark 3.12. Let us mention that, if we take $J_0 = 0$ in (GHI_0), this result is not affected if the sequence of solutions is weakly converging. In this case we shall obtain:

$$w - \liminf S_n \subset S_0.$$

In fact, we have made recourse to strong convergence in (H_8) because of the presence of Clarke's derivative in (GHI_0).

From Theorem 3.2 we deduce the following variant of the stability results in [12, 14].

Corollary 3.13. *Let T and T_n, for each $n \geq 1$, be operators from X to X^*. Suppose that:*

a) *T is hemicontinuous on X;*
b) *T_n is monotone;*
c) *$(T_n)_n$ converges to T_n in the sense that: for any $u \in X$, any sequence u_n strongly converging to u we have $T_n u_n \rightharpoonup Tu$;*
d) *(H_6) is satisfied.*

Then, if a sequence $(\overline{u}_n)$ of solutions to the variational inequality:

$$(VI_n) \quad \textit{find } u \in X \ \textit{ such that } (T_n u, v-u) + \varphi_n(v) - \varphi_n(u) \geq 0 \ \ \forall v \in X$$

converging to a point $\overline{u}$, $\overline{u}$ is a solution to the variational inequality:

$$(VI) \quad \textit{find } u \in X \ \textit{ such that } (Tu, v-u) + \varphi(v) - \varphi(u) \geq 0 \ \ \forall v \in X.$$

Proof. It suffices to take, for each $u, v \in X$, $\Phi(u,v) = (Tu, v-u)$ and $\Phi_n(u,v) = (T_n u, v-u)$. The result is hence an easy consequence of Theorem 3.2. □

The paragraph below presents a stability result without recourse to (H_0) and (H_8).

3.2. **Distances approach.** In this paragraph, we first present the stability result for the equilibrium problem. Further, we derive the result for (GHI_0). In this respect, we suppose that X is a normed vector space with norm $\|\cdot\|$ and assume that $\varphi_0 = 0$. We shall also consider a sequence of bifunctions $F_n : X \times X \to \mathbb{R}$ and the following equilibrium problems: for any $n \geq 0$ find $\overline{u}_n \in X$ such that:

$$(EP_n) \ \ F_n(\overline{u}_n, v) \geq 0 \text{ for all } \ v \in X$$

To carry out our stability analysis, we need the following monotonicity assumption:

(A_1) $F_n(u,v) + F_n(v,u) \leq -M\|u-v\|^2$ for all $u, v \in X$, $n \geq 1$, where $M > 0$.

F_n will be said $-M$-monotone.

Remark 3.14. Let $C : X \to X^*$ be a r-Lipschitz operator, (where $r > 0$) and $B : X \to X^*$ be a linear bounded operator. Let us define the bifunctions h and h_1 given by

$$h(u,v) = \langle Cu, v-u \rangle \text{ and } h_1(u,v) = \langle Bu; v-u \rangle.$$

It is easily shown that h is r-monotone and h_1 is $\|B\|$-monotone.

Let us give now an-essential-example of bifunction satisfying a relaxed monotonicity assumption of (A_1). Let $X = H$ be a Hilbert space, I the identity mapping on H and J_0 a real locally Lipschitz function on X.

Lemma 3.15. *Suppose that, for some $\alpha \in \mathbb{R}$, $\partial J_0 + \alpha I$ is monotone. Then, the bifunction g defined, for all $u, v \in H$, by $g(u,v) = J_0^0(u; v-u)$ is α-monotone.*

Proof. Straightforward. □

Remark 3.16. Let us remark that in lemma 3.15, we can easily check that ∂J_0 is strongly monotone if $\alpha < 0$, monotone if $\alpha = 0$ and weakly nonmonotone if $\alpha > 0$. It is known from convex analysis that the monotonicity of ∂J_0 leads to convexity of J_0. Then, whenever $\alpha \leq 0$ the problem (GHI_0) comes back to the generalized variational inequality, since in this case J_0 is a convex , whereas if $\alpha > 0$ the function J_0 is not necessarily convex.

Remark 3.17. A special case of J_0 is when it is defined as follows:

$$J_0(u) = \int_\Omega j(u(x))dx$$

Here, the space X is supposed imbedded in $L^p(\Omega)$ with Ω an open bound subset of $\mathbb{R}^n$, and

$$j(t) = \int_0^t \beta(s)ds; \quad \beta \in L^\infty_{loc}(\mathbb{R}).$$

Notice that in [1], the authors provided some condition on β so as to satisfy the monotonicity condition of Lemma 3.15. Precisely, they considered the following property:

$$t_1 \le t_2 \Rightarrow \beta^+(t_1) < \beta_-(t_2) + \gamma(t_2 - t_1)^r, \tag{3.2}$$

where $\gamma, r > 0$ and β^+ and β_- are given by

$$\beta^+(t) = \lim_{\delta\to 0} \underset{|s-t|\le\delta}{ess\ sup}\ \beta(s),\ \ \beta_-(t) = \lim_{\delta\to 0} \underset{|s-t|\le\delta}{ess\ inf}\ \beta(s) \text{ for some } t \in \mathbb{R}.$$

Using this assumption, it is argued in [1] that J_0 is K-monotone for some constant $K > 0$.

Remark 3.18. The condition of Lemma 3.15 is nothing else than a relaxed form of monotonicity for ∂J_0 but it keeps the nonconvex framework for the energy function J_0. After we have finished this work we have realized that a such condition was used by Naniewicz and Panagiotopoulos in [16] (Chapter 7) for existence results.

Before stating the main result of this paragraph, we introduce the following "distance" between two bifunctions f and g as follows:

$$\rho_\tau(f,g) := \sup_{u\ne v, \|u\|\le\tau} \frac{|(f-g)(u,v)|}{\|u-v\|}$$

where $\tau > 0$.

Remark 3.19. Let A and B be two operators from X toX^*. We associate to A and B two bifunctions as follows:

$$f_A(u,v) = (Au, v-u); \quad f_B(u,v) = (Bu, v-u).$$

It is readily shown that

$$\rho_\tau(A,B) := \rho_\tau(f_A, f_B) \le d_\tau(A,B)$$

where d_τ is the classical "distance" defined by

$$d_\tau(A,B) := \max_{\|u\|\le\tau} \|A(u) - B(u)\|.$$

Assume that the set of solutions to (EP_0), also denoted by S_0, is nonempty and bounded and let $\tau > 0$ such that $S_0 \subset B(0,\tau)$. We claim the following:

Theorem 3.20. *Assume that assumption (A_1) holds and the sequence F_n converge, following ρ_τ, to F. Then, whenever the solution $\overline{u}_n$ to (EP_n) exists it must be unique and strongly convergent to the unique solution $\overline{u}_0$ to (EP_0) and we have*

$$\|\overline{u}_n - \overline{u}_0\| \le \frac{1}{M}\rho_\tau(F_n, F_0).$$

Proof. Let us first establish, for each $i \ge 1$, the following estimation:

$$\|\overline{u}_i - \overline{u}_0\| \le \frac{1}{M}\rho_\tau(F_i, F_0).$$

Let $i \ge 1$. Since, for $j = 0, i$, $\overline{u}_j$ is a solutions to (EP_j), we have

$$F_j(\overline{u}_j, v) \ge 0 \ \text{ for all } \ v \in X$$

thus, we make in (EP_j), $v = \overline{u}_m$ for $m = 0, i$ and $m \ne j$, and adding the two relations we obtain:

$$F_i(\overline{u}_i, \overline{u}_0) + F_0(\overline{u}_0, \overline{u}_i) \ge 0.$$

Therefore

$$F_i(\overline{u}_i,\overline{u}_0) - F_0(\overline{u}_i,\overline{u}_0) + F_0(\overline{u}_i,\overline{u}_0) + F_0(\overline{u}_0,\overline{u}_i) \geq 0.$$

Taking into account (A_1), we deduce that

$$\begin{aligned} M\|\overline{u}_i - \overline{u}_0\|^2 &\leq F_0(\overline{u}_0,\overline{u}_i) - F_i(\overline{u}_0,\overline{u}_i) \\ &\leq \rho_\tau(F_i,F_0)\|\overline{u}_i - \overline{u}_0\| \end{aligned}$$

which leads to

(3.3) $$M\|\overline{u}_i - \overline{u}_0\| \leq \rho_\tau(F_i,F_0).$$

Now for $n \geq 1$, the uniqueness of solution $\overline{u}_n$ to (EP_n) comes (3.3). Furthermore, we have

$$\|\overline{u}_n - \overline{u}_0\| \leq \frac{1}{M}\rho_\tau(F_n,F)$$

therefore, we conclude that

$$e(\overline{u}_n,S) := \sup_{w\in S}\|\overline{u}_n - w\| \to 0 \text{ as } n \text{ goes to } +\infty.$$

Hence, $\overline{u}_n$ strongly converges to some u which must be the unique solution to (EP_0). The proof is then finished. □

We are now in a position to derive a result with estimation of solutions to (GHI_0). We hence claim the following:

Corollary 3.21. *Assume that X is Hilbert space, for each $n \geq 1$ Φ_n is $-\gamma$-monotone for some $\gamma > 0$, $\partial J_n + \alpha I$ is monotone for some $\alpha > 0$, $(\Psi_n)_n$ is c-monotone and $\gamma > \alpha + c$. Then, if $\tau > 0$ is such that $S_0 \subset B(0,\tau)$, whenever the solution $\overline{u}_n$ to $(GHI)_n$ exists is unique and the following estimation holds:*

$$\|\overline{u}_n - \overline{u}_0\| \leq \frac{1}{(\gamma-\alpha-c)}[\rho_\tau(\Phi_n,\Phi_0) + \rho_\tau(\Psi_n,\Psi_0) + \rho_\tau(g_n,g_0)].$$

If moreover, the sequences (Φ_n), (Ψ_n) and (g_n) converge with respect to ρ_τ, then $\overline{u}_n$ strongly converges to $\overline{u}_0$.

Here we have adopted the notation:

$$g_n(u,v) := J_n^0(u; v-u).$$

Proof. Let us take:

$$F_n(u,v) = \Phi_n(u,v) + J_n^0(u,v-u) + \Psi_n(u,v).$$

Using Lemma 3.15, we see that F_n is $(\gamma-\alpha-c)$-monotone. The result is hence direct from Theorem 3.20. □

4. APPLICATION

4.1. **Equilibrium of the von kármán plates.** We treat here a mathematical problem which simply models the equilibrium problem of the von kármán plates presented In Section 2. In this way, let V be a Hilbert space with scalar product $(\cdot,\cdot)$ and the associated norm $\|.\|$. Space V is supposed densely and compactly imbedded into $L^p(\Omega,\mathbb{R})$ for some $p \geq 2$. Here Ω is a bounded domain in $\mathbb{R}^N$. We shall consider a bilinear form $a : V \times V \to \mathbb{R}$, a nonlinear operator $C : V \to V$, a function $\beta \in L^\infty_{loc}(\mathbb{R})$ and the locally Lipschitz function j defined by: $j(t) = \int_0^t \beta(s)ds,\ t \in \mathbb{R}$. The problem is formulated as a hemivariational inequality: find $\overline{u} \in V$ so as to satisfy:

(EVKP) $$a(u,v) + (Cu,v) + \int_\Omega j^0(u(x);v(x))dx \geq 0\ \forall v \in V$$

which is equivalently expressed as

$$((EVKP)_{equi}) \qquad a(u,v-u)+(Cu,v-u)+\int_\Omega j^0(u(x);v(x)-u(x))dx \geq 0\ \forall v\in V.$$

Indeed, suppose that u is a solution to ($EVKP$) and let $v \in V$. By making $v' = v - u$ in ($EVKP$) we see that u solves $(EVKP)_{equi}$. If u solves $(EVKP)_{equi}$, for any $v \in V$ we take $v' = v + u$ to see that u is a solution to ($EVKP$).

Remark 4.1. The solutions to this problem have been provided in [15] by use of critical point theory and other results are also established for a similar form of ($EVKP$) by means of Ky Fan's minimax inequality in [1].

Let us remark that if we set $J : L^p(\Omega) \to \mathbb{R}$ defined by $J(u) = \int_\Omega j(u(x))dx\ u \in V$, the problem ($EVKP$) can be regarded in the form of (GHI). Moreover, it is possible to prove that these two problems are equivalent under suitable assumptions. The following lemma argue the passage from (GHI) to ($EVKP$).

Lemma 4.2. *Assume that for some $\alpha_1 \in \mathbb{R}$ and $\alpha_2 > 0$, we have*

$$(H) \qquad |\beta(s)| \leq \alpha_1 + \alpha_2|s|^{p-1}, \forall s \in \mathbb{R}.$$

Then every solution to

$$(GHI) \qquad a(u,v)+(Cu,v)+J^0(u;v) \geq 0\ \forall v \in V$$

is also a solution to ($EVKP$).

Proof. We should first mention that, in view of assumption (H), J is well defined and locally Lipschitz on $L^p(\Omega)$ (see [7]). Now let u be a solution to (GHI). Let us remark that, following Example 1 in [7], the assumption (H) ensures that

$$\forall s \in \mathbb{R},\ \ \forall \xi \in \partial j(t), |\xi| \leq \alpha_1 + 2^{p-1}\alpha_2|s|^{p-1}.$$

Hence, since V is dense in $L^p(\Omega)$ we can apply Theorem 2.7.5 of [8] and Theorem 2.2 of [7] to conclude that:

$$\partial J_{/V}(u) \subset \int_\Omega \partial j(u(x))dx.$$

On the other hand, since u is a solution to (GHI), it follows that

$$-\alpha(u,v)-(Cu,v) \leq J^0(u;v)\ \forall v \in V.$$

Therefore, by definition Clarke's gradient, it results that:

$$-\alpha(u,.)-(Cu,.) \in \partial J_{/V}(u) \subset \int_\Omega \partial j(u(x))dx.$$

Which is interpreted as:

$$\begin{aligned} -\alpha(u,v)-(Cu,v) &\leq \int_\Omega \max_{z\in\partial j(u_(x))} z(v(x))dx \\ &\leq \int_\Omega j^0(u(x),v(x))dx\ \forall v \in V. \end{aligned}$$

u is henceforth a solution to ($EVKP$). □

Now, by varying a, C and J we consider the perturbed problem: find $\overline{u}_n \in V$ so as to satisfy:

$$(EVKP)_n \ \ a_n(\overline{u}_n,v)+(C_n\overline{u}_n,v)+J_n^0(\overline{u}_n;v)dx \geq 0\ \forall v \in V.$$

Consequently from Theorem 3.2 we have the following stability result for ($EVKP$).

Corollary 4.3. *Assume that:*

i) *a is positive, that is $a(u,u) \geq 0\ \forall u \in V$ and continuous;*

ii) *a_n is positive for each n and for all $u, v \in V$, all $u_n \to u$ and all $v_n \to v$ it results*

$$a(u,v) \leq \liminf_n a_n(u_n, v_n);$$

iii) *$(C_n)_n$ converges to C, that is for all $u, v \in V$, all $u_n \to u$ and all $v_n \to v$ it results*

$$\limsup_n C_n(u_n, v_n) \leq (Cu, v);$$

vi) *Assume that (H_0) holds and $(J_n)_n$ satisfies assumptions (H_7) and (H_8) of Theorem 3.2.*

Then whenever the sequence $(\overline{u}_n)_n$ of solutions to $(EVKP_n)_{equi}$ strongly converges to $\overline{u}$, $\overline{u}$ is a solution to $(EVKP)_{equi}$.

We now apply the result of the second approach.

Corollary 4.4. *Assume that*

$h_1)$ *for each n, a_n is γ-coercive, that is $a_n(u,u) \geq \gamma\|u\|^2\ \forall u \in V$;*

$h_2)$ *for each n, C_n is Lipschitz of rank $c > 0$;*

$h_3)$ *$\partial J_n + \alpha I$ is monotone, for each n, for some $\alpha > 0$;*

$h_4)$ *the sequences $(a_n)_n, (C_n)_n$ and $(g_n)_n$ ρ_τ-converges to a, C and g respectively where τ is such that the solutions to $(EVKP)_{equi}$ are contained in $B(0,\tau)$.*

Then, if $\gamma > \alpha + c$, the solution u_n to $(EVKP_n)_{equi}$ is unique and strongly converging to the unique solution u to $(EVKP)_{equi}$ and the following estimation holds:

$$\|u_n - u\| \leq \frac{1}{(\gamma - \alpha - c)}[\rho_\tau(a_n, a) + \rho_\tau(C_n, C) + \rho_\tau(g_n, g)].$$

Here $\rho_\tau(a_n, a) := \rho_\tau(f_{a_n}, f_a)$ with $f_{a_n}(u,v) = a_n(u, v-u)$ and $f_a(u,v) = a(u, v-u)$.

Remark 4.5. Assume moreover, for each n, that a_n is continuous. Then, thanks to remark 3.19 the estimation of the last corollary leads to

$$\|u_n - u\| \leq \frac{1}{(\gamma - \alpha - c)}[\|a_n - a\| + d_\tau(C_n, C) + \rho_\tau(g_n, g)].$$

5. COMMENTS

1. The Theorem of Zolezzi [23] has been a crucial argument in our Theorem 3.2. To make our result more powerful, one should improve the result of [23] in two directions. First to extend it to the case where the space is equipped with the weak topology instead of the strong one. Further, to look whether it is possible to delete the assumption of semi-differentiability on J_n since the hemivariational inequalities do not involve (in their general formulation) any type of differentiability energy functions J_n.
2. The distance approach was presented for variational inequalities in the paper by Doktor and Kucera [9], wherein the authors have dealt with the following two monotone variational inequalities in a Hilbert space H: given two operators $A_1, A_2 : H \to H$, two closed convex subsets $\mathcal{K}_1, \mathcal{K}_2$ and $f_1, f_2 \in H$, one seek $u \in \mathcal{K}_n$ so as to satisfy

$$(VI_n) \qquad \langle A_n u, v - u\rangle \geq \langle f_n, v - u\rangle \text{ for all } v \in \mathcal{K}_n.$$

They have obtained the following estimate between the solutions u_1 and u_2:

$$\|u_1 - u_2\| \leq c[\varrho(\mathcal{K}_1, \mathcal{K}_2) + \|f_1 - f_2\| + a(A_1, A_2)], \tag{5.1}$$

for appropriate distances ϱ, a and a positive constant c. These connection between solutions have been based on the fact that solutions to (VI_n) are characterized by means of the projection $P_{\mathcal{K}_n}$ into $\mathcal{K}_n$ as follows: u solves (VI_n) if and only if

$$u = P_{\mathcal{K}_n}(u - \gamma(A_n u - f_n)),$$

γ being an arbitrary positive number. Thus, estimate (5.1) has been concluded thanks to the Lipschitz property of A_n and nonexpansivity of the projection mapping P. This technique is not valid for hemivariational inequalities because we cannot find a Lipschitz operator $B : V \to V^*$, which satisfies

$$\langle B(u), v\rangle := \int_\Omega j^0(u(x); v(x))dx \quad \forall u, v \in \mathcal{K}.$$

It is the case only when the Clarke's derivative coincides with the Gâteaux derivative which corresponds to the smooth energy functional J as it has been shown in [2].

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