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NORM INEQUALITIES IN STAR ALGEBRAS

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ABSTRACT. A norm inequality is proved for elements of a star algebra so that the algebra is noncommutative. In particular, a relation between maximal and minimal extensions of regular norm on a C^*-algebra is established.

Key words and phrases: W^*-algebras, C^*-algebras, Self-adjoint elements (operators), Maximal and minimal extensions of a regular norm, Furuta type inequalities.

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Let H be a Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. A subset of $B(H)$ is a W^*-algebra on H if X is a C^*-algebra which is closed in the weak operator topology, see [1]. Also, a W^*-algebra is a C^*-subalgebra of $B(H)$ which is weakly closed. In particular, a W^*-algebra is an algebra of operators. We note that a C^*-algebra acting on H is commutative if and only if zero is the only nilpotent element of the algebra.

Let X be a W^*-algebra and X_{SA} be the set of self-adjoint elements of X, that is if $T \in S(X) \Longrightarrow T = T^*$, where T^* is the adjoint of T. Here we prove the following theorem.

Theorem 1. *A unital W^*-algebra X of operators is noncommutative if $\forall A, B \in X_{SA}$,*

$$\|A\| = 1 = \|B\| \Longrightarrow \|A + B\| > 1 + \|AB\| .$$

Proof. Since X is noncommutative, there exists an operator T in X such that $T^2 = 0$. Suppose X_1 is the range of T and X_2 is the orthogonal complement of X_1. Then $H = X_1 \oplus X_2$. Let S be an operator with $\|S\| = 1$. Then

$$T = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T^* = \begin{pmatrix} 0 & 0 \\ S^* & 0 \end{pmatrix}.$$

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We use these representations for T and T^* to define the operators A and B as follows: For $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_1 + \sigma_2 = 1$, we have

$$\begin{aligned} A &= \begin{pmatrix} SS^* & 0 \\ 0 & 0 \end{pmatrix} = TT^*, \\ B &= \begin{pmatrix} \sigma_1 SS^* & \sigma_2 S \\ \sigma_2 S^* & \sigma_1 S^* S \end{pmatrix} = \sigma_1 (TT^* + T^*T) + \sigma_2 (T + T^*). \end{aligned}$$

Clearly, $A = A^*$, $B = B^*$ and $A, B \in X$. Next, we consider the following two cases.

Case 1. Let $\sigma_1 = \sigma_2 = \frac{1}{2}$ and

$$B = \frac{1}{2} \begin{pmatrix} SS^* & S \\ S^* & S^*S \end{pmatrix} = \frac{1}{2} (TT^* + T^*T + T + T^*).$$

It is not difficult to see that $\|A\| = 1$ and $\|B\| \leq 1$. To obtain $\|B\| \geq 1$, let $\|S\| = 1$ then $\exists a_n \in X \ni$: $\|a_n\| = 1$. Also,

$$\begin{aligned} \|SS^* a_n - a_n\|^2 &= \|SS^* a_n\|^2 - 2\|S^* a_n\|^2 + \|a_n\|^2 \\ &\leq 2\left(\|a_n\|^2 - \|S^* a_n\|^2\right) \end{aligned}$$

and if $\|S^* a_n\| \to 1$ then $SS^* a_n - a_n \to 0$. Further, $(Bb - b) \to 0$, where $b = (a_n + S^* a_n)$ and hence, $\|B\| \geq 1$, which concludes that $\|B\| = 1$.

Let

$$\begin{aligned} AB &= \frac{1}{2} \begin{pmatrix} SS^* & S \\ S^* & S^*S \end{pmatrix} \quad \text{and} \\ A + B &= \begin{pmatrix} SS^* + \frac{SS^*}{2} & \frac{S}{2} \\ \frac{S^*}{2} & \frac{S^*S}{2} \end{pmatrix} = \frac{1}{2} (TT^* + T^*T + T + T^*). \end{aligned}$$

Choose a_n as above and $b_n = \frac{S^* a_n}{(2\mu - 1)}$, where $\mu > 1$. Let

$$\mu_1 = \sigma_1 + \frac{1}{2} + \left(\sigma_2 + \frac{1}{4}\right)^{\frac{1}{2}}$$

so that it satisfies the equation

$$(\mu_1 - \sigma_1 - 1)(\mu_1 - \sigma_1) = \sigma_2^2.$$

Then

$$[(A + B)(a_n + b_n) - \mu (a_n + b_n)] \to 0$$

and $\|A + B\| \geq \mu > 1$. If we choose σ_1 and σ_2 so that

$$\sigma_1 + \frac{1}{2} + \left(\sigma_2 + \frac{1}{4}\right)^{\frac{1}{2}} > 1 + \left(\sigma_1^2 + \sigma_2^2\right)^{\frac{1}{2}},$$

then we have

$$\|A + B\| > 1 + \|AB\|.$$

For example, it is sufficient to take $\sigma_1 = \frac{2}{3}$ and $\sigma_2 = \frac{1}{3}$. We note that $\sigma_1 > \sigma_2$. If $\sigma_1 < \sigma_2$ then the above inequality fails. Since $\mu > 1 + \sqrt{2}\sigma_1$ the proof in this case is complete.

Case 2. Let $\sigma_1 \neq \sigma_2$. Then $\|AB\| \leq a_0$, (by mimicking the proof of Case 1), where

$$a_0 = \sup\left\{\sigma_1 \|a\| + \sigma_2 \|b\| : \|a\|^2 + \|b\|^2 = 1\right\} = \sqrt{\sigma_1^2 + \sigma_2^2}.$$

Let $b_n(\mu_1 - \sigma_1) = \sigma_2 S^* a_n$, where μ_1 depends on σ_1 and σ_2. Then $\|A+B\| \geq \mu_1$ and one can have the following form of μ_1, that is, $2\mu_1 = (2\sigma_1 + 1) + \sqrt{1+\sigma_1^2}$. Hence, $\mu_1 > 1 + a_0$ and this concludes the proof of the theorem.

□

Remark 2. If X is commutative then for $A, B \in X_{SA}$ with $\|A\| = 1 = \|B\|$, we have $0 \leq I - B - A + AB$, where I is the identity operator. Thus $\|A+B\| \leq 1 + \|AB\|$.

Let $0 < p, q, r$ be real numbers such that $q(2r+1) \geq (2r+p)$ and $q \geq 1$. If two bounded linear operators $A, B \in B(H)$ on a Hilbert space H satisfy $0 \leq B \leq A$ then $(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{p}{q}} B^{\frac{2r}{q}}$. This inequality is called the Furuta inequality and can be found in [3]. Recently, Kotaro and others in [6] have extended this inequality in a unital hermitian Banach $*$-algebras with continuous involution. We give a slightly different version of these inequalities in the following corollary.

Corollary 3. *Suppose that X_{C^*} is a C^*-algebra acting on H. Let λ, μ and σ be three real numbers with $\sigma > 0$, $\lambda > 0$. Then there exists operators T_1, T_2 and T_3 in $X \ni$: $\lambda T_1 + \mu T_2 + \sigma T_3 \geq 0 \Longleftrightarrow \lambda\sigma \geq \mu^2$.*

Proof. We recall that an operator $O \in B(H)$ is positive if $\langle Oh, h\rangle \geq 0$ for every vector h. Using the techniques of Theorem 1, the following operators belong to X_{C^*}. That is,

$$T_1 = \begin{pmatrix} SS^* & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \sqrt{SS^*}S \\ S^*\sqrt{SS^*} & 0 \end{pmatrix}, \text{ and}$$
$$T_3 = \begin{pmatrix} 0 & 0 \\ 0 & S^*S \end{pmatrix}$$

are in X_{C^*}. In this case we have

$$\lambda T_1 + \mu T_2 + \sigma T_3 = \begin{pmatrix} \lambda SS^* & \mu\sqrt{SS^*}S \\ \mu S^*\sqrt{SS^*} & \sigma S^*S \end{pmatrix}.$$

Let $\lambda T_1 + \mu T_2 + \sigma T_3 = \Lambda$. Then we observe that the determinant of Λ is zero if $\lambda\sigma = \mu^2$. If $\varepsilon < 0$ and $h \in H$, then we have

$$\|(\Lambda - \varepsilon)h\|^2 = \|\Lambda h\|^2 - 2\varepsilon\langle \Lambda h, h\rangle + \varepsilon^2\|h\|^2 \geq -2\varepsilon\langle\Lambda h, h\rangle + \varepsilon^2\|h\|^2 \geq +\varepsilon^2\|h\|^2.$$

Thus $\varepsilon \notin SP_{ap}(\Lambda)$, the approximate point spectrum of Λ. This means that $(\Lambda - \varepsilon)$ is left invertible. Since $(\Lambda - \varepsilon)$ is hermitian, it must also be right invertible. That is, $\varepsilon \notin SP(\Lambda)$ and so $\Lambda \geq 0 \Longleftrightarrow \lambda\sigma \geq \mu^2$.

Alternatively, for $a \in X_1$ and $b \in X_2$, we have

$$\langle \Lambda(a+b), (a+b)\rangle = \left\|\sqrt{\sigma}Sb + \mu\sqrt{\frac{SS^*}{\sigma}}a\right\|^2 + \left(\lambda - \frac{\mu^2}{\sigma}\right)\|S^*a\|^2.$$

Since $a \in X_1$, therefore

$$\exists b_n \in X_2 \ni: \sqrt{\sigma}(Sb_n) + \mu\sqrt{\frac{SS^*}{\sigma}}a \to 0 \Longrightarrow \Lambda \geq 0 \Longleftrightarrow \lambda - \frac{\mu^2}{\sigma} \geq 0.$$

Hence the proof of the corollary is complete. □

Remark 4. By reducing the matrix Λ into a product of three matrices the above corollary can also be proved. That is, $\Lambda = L^*DL$, where

$$L = \begin{pmatrix} I & W \\ 0 & I \end{pmatrix} \quad \text{and} \quad W = \frac{\mu}{\sigma}\sqrt{SS^*}S\sqrt{S^*S}.$$

By the partial commutation relation, we have

$$\sqrt{SS^*}S = S\sqrt{S^*S}$$

and hence

$$D = \begin{pmatrix} \left(\lambda - \frac{\mu^2}{\sigma}\right) SS^* & 0 \\ 0 & \sigma S^* S \end{pmatrix}.$$

Under the above assumption about σ and S^*S, the Sylvester type test applies. That is, Λ is positive (semi definite) if and only if $\sigma > 0$ and $\lambda - \frac{\mu^2}{\sigma} \geq 0$.

Let $A, B > 0$ be invertible operators on H. In this case a Furuta type inequality is obtainable by replacing 1 with 0 in the original Furuta inequality in Remark 2. In fact, we have

$$A^{2r} \geq \left(A^r B^p A^r\right)^{\frac{2r}{(2r+p)}}.$$

Also, if $A \geq B \geq 0$, $\ni$: $A > 0$, then for each $\alpha \in [0,1]$ and $p \geq 1$ we have

$$\left\{A^{\frac{r}{2}} \left(A^{\frac{-\alpha}{2}} B^{\alpha} A^{\frac{-\alpha}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{(1+r-\alpha)}{[(p-\alpha)s+r]}} \leq A^{(1+r-\alpha)} \text{ for } s \geq 1 \text{ and } r \geq \alpha.$$

For more details, see [3]. The following examples give an application of these inequalities in case of C^*-algebras.

Example 0.1. Let X be a commutative C^*-algebra acting on H. If we take

$$\begin{aligned} A &= 6T_1 + 0T_2 + 3T_3 \text{ and} \\ B &= 3T_1 + 2T_2 + T_3 \end{aligned}$$

then

$$A - B = 3T_1 - 2T_2 + T_3$$

and by the Corollary 3 we have $A - B \geq 0$. We further note that A^2 is not greater than or equal to B^2, since for $b \in X_2$,

$$\left\langle \left(A^2 - B^2\right) b, b\right\rangle + \left\langle (S^*S)^2 b, b\right\rangle = 0.$$

Example 0.2. Let

$$\begin{aligned} A &= 2T_1, \quad B = T_1 + T_2 + T_3 \text{ and} \\ C &= 4T_1 + T_2 + T_3. \end{aligned}$$

Then $A \geq 0$ and $B + C \geq 0$. Further, by the Corollary 3, we have $B + C - A \geq 0$. Let $\Psi \leq B$ and $\Phi \leq C$, where $A = \Psi + \Phi$. Then $\Psi \leq A$. Hence, from Corollary 3, for $a \in X_1$ and $b \in X_2$, we have

$$\langle (T_1 + T_2 + T_3)(a + b), (a + b)\rangle = \left\|\sqrt{SS^*}a + Sb\right\|^2.$$

Also, $b_n \in X_2 \Longrightarrow \Psi = 0$, because $\left(Sb_n + \sqrt{SS^*}a\right) \to 0$. Thus

$$A = \Phi = 2T_1 \leq 4T_1 + T_2 + T_3.$$

Example 0.3. Let

$$\begin{aligned} A &= \frac{1}{3}T_1, \quad B = T_1 + T_2 + T_3 \text{ and} \\ C &= 4T_1 + 2T_2 + T_3. \end{aligned}$$

Then $A \geq 0$ and $B + C \geq 0$. Next, by Corollary 3, we get $B + C - A \geq 0$. Now by Example 0.2 it follows that $A = \Phi = \frac{1}{3}T_1 \leq C$. This contradicts Corollary 3, since for $(C - A)$, $\lambda\sigma < \mu^2$.

Remark 5. The algebra norm $\|\cdot\|$ on a non-unital Banach algebra $\mathfrak{J}$ can be extended to an algebra norm on the unitization $\mathfrak{J}^+ = Ce + \mathfrak{J}$, (where e is the unit in the algebra) in many ways. In particular, the following two norms,

$$l_1 - \text{norm} = \|(\lambda)\, e + a\|_1 = |\lambda| + \|a\|$$

and the operator norm,

$$\|(\lambda)\, e + a\|_{OP} = \sup\left\{\|(\lambda)\, b + ab\|\,, \|(\lambda)\, b + ba\|\,;\; b \in \mathfrak{J},\; \|b\| \le 1\right\},\; \lambda \in C,\; a \in \mathfrak{J}$$

are the maximal and the minimal extensions of the original norm respectively, if it is a regular norm, that is,

$$\|a\| = \sup\left\{\|ab\|\,, \|ba\|\,;\; b \in \mathfrak{J},\; \|b\| \le 1\right\},$$

see [4]. The unitization $\mathfrak{J}^+$ is complete under both $\|\cdot\|_1$ and $\|\cdot\|_{OP}$, so by the two norm lemma, [2, II, 2.5] these two norms are equivalent. If $\mathfrak{J}$ is a C^*-algebra, $a \in \mathfrak{J}$ is self-adjoint, and λ is complex, then $\|(\lambda)\, e + a\|_1 \le 3\,\|(\lambda)\, e + a\|_{OP}$. So far the constant 3 is the best possible in this case.

Recently, in [5], we extended the above result to locally m-convex algebras. Now we prove the following corollary.

Corollary 6. *If is a commutative non-unital C^*-algebra $A, B \in Y_{SA}$ with $\|A\| = 1 = \|B\|$, and λ is complex, then*

$$\|(\lambda I + \Psi)\|_1 \le \lambda_0 + 3\,\|\Phi\|_{OP}\,,$$

where $\lambda_0 > 0,\ \Psi = A + B$, and $\Phi = AB$.

Proof. Since

$$\|(\lambda I + A + B)\|_1 \le 3\,\|(\lambda I + A + B)\|_{OP}\,,$$

we have

$$\frac{1}{3}\,\|(\lambda I + A + B)\|_1 \le |\lambda| + \|(A + B)\|_{OP} \Longrightarrow \|(\lambda I + \Psi)\|_1 \le \lambda_0 + 3\,\|\Phi\|_{OP}\,,$$

where$\lambda_0 = 3\,(|\lambda| + 1) > 0$. This concludes the proof of the corollary. □

Remark 7. $\|(\lambda i + \Psi)\|_{OP} \le \frac{\lambda_0}{3} + \|\Phi\|_{OP}\,.$

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