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A GENERALIZATION OF THE PRE-GRÜSS INEQUALITY AND APPLICATIONS TO SOME QUADRATURE FORMULAE

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ABSTRACT. A generalization of the pre-Grüss inequality is presented. It is applied to estimations of remainders of some quadrature formulas.

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1. INTRODUCTION

In recent years a number of authors have written about generalizations of Ostrowski's inequality. For example, this topic is considered in [1], [2], [5], [7], [9] and [12]. In this way some new types of inequalities are formed, such as inequalities of Ostrowski-Grüss type, inequalities of Ostrowski-Chebyshev type, etc. An important role in forming these inequalities is played by the pre-Grüss inequality. This paper develops a new approach to the topic obtaining better results than the approach using the pre-Grüss inequality. It presents new, improved versions of the mid-point and trapezoidal inequality. The mid-point inequality is considered in [1], [2], [3], [7] and [9], while the trapezoidal inequality is considered in [4], [5], [7] and [9].

In [11] we can find the pre-Grüss inequality:

$$T(f,g)^2 \le T(f,f)T(g,g), \tag{1.1}$$

where $f, g \in L_2(a,b)$ and $T(f,g)$ is the Chebyshev functional:

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1.2}$$

If there exist constants $\gamma, \delta, \Gamma, \Delta \in \mathbb{R}$ such that

$$\delta \le f(t) \le \Delta \text{ and } \gamma \le g(t) \le \Gamma,\ t \in [a,b]$$

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then, using (1.1), we get the Grüss inequality:

$$|T(f,g)| \le \frac{(\Delta-\delta)(\Gamma-\gamma)}{4}. \tag{1.3}$$

Specially, we have

$$T(f,f) \le \frac{(\Delta-\delta)^2}{4}. \tag{1.4}$$

Using the above inequalities we get the following inequalities:

$$\begin{aligned}\left| f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)dt \right| &\le \frac{(b-a)^2}{2\sqrt{3}} \left[\frac{1}{b-a} \|f'\|_2^2 - \left(\frac{f(b)-f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}} \\ &\le \frac{(b-a)^2}{4\sqrt{3}} (\Gamma-\gamma)\end{aligned} \tag{1.5}$$

where $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function whose derivative $f' \in L_2(a,b)$ and $\gamma \le f'(t) \le \Gamma$, $t \in [a,b]$. As usual, $\|\cdot\|_2$ is the norm in $L_2(a,b)$. Further,

$$\begin{aligned}\left| \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)dt \right| &\le \frac{(b-a)^2}{2\sqrt{3}} \left[\frac{1}{b-a} \|f'\|_2^2 - \left(\frac{f(b)-f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}} \\ &\le \frac{(b-a)^2}{4\sqrt{3}} (\Gamma-\gamma)\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}&\left| \frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}(b-a) - \int_a^b f(t)dt \right| \\ &\qquad \le \frac{(b-a)^2}{4\sqrt{3}} \left[\frac{1}{b-a} \|f'\|_2^2 - \left(\frac{f(b)-f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}} \\ &\qquad \le \frac{(b-a)^2}{8\sqrt{3}} (\Gamma-\gamma)\end{aligned} \tag{1.7}$$

where the function f satisfies the above conditions. The inequalities (1.5)-(1.7) are considered (and proved) in [2], [9] and [12].

In this paper we generalize (1.1). We use the generalization to improve the above inequalities.

2. Main Results

Lemma 2.1. *Let $f, g, \Psi_i \in L_2(a,b)$, $i = 0,1,2,...,n$, where $\Psi_i^0 = \Psi_i(t)/\|\Psi_i\|_2$ are orthonormal functions. If $S_n(f,g)$ is defined by*

$$S_n(f,g) = \int_a^b f(t)g(t)dt - \sum_{i=0}^{n} \int_a^b f(s)\Psi_i^0(s)ds \int_a^b g(s)\Psi_i^0(s)ds$$

then we have

$$|S_n(f,g)| \le S_n(f,f)^{\frac{1}{2}} S_n(g,g)^{\frac{1}{2}}.$$

The proof follows by the known inequality holding in inner product spaces $(H, \langle\cdot,\cdot\rangle)$

$$\left| \langle x,y\rangle - \sum_{i=0}^{n} \langle x,l_i\rangle \langle l_i,y\rangle \right|^2 \le \left(\|x\|^2 - \sum_{i=0}^{n} |\langle x,l_i\rangle|^2 \right) \left(\|y\|^2 - \sum_{i=0}^{n} |\langle l_i,y\rangle|^2 \right),$$

where $x, y \in H$ and $\{l_i\}_{i=0,n}$ is an orthonormal family in H, i.e., $(l_i, l_j) = \delta_{ij}$ for $i, j \in \{0, \ldots, n\}$.

We here use only the case $n = 1$. We choose $\Psi_0^0(t) = 1/\sqrt{b-a}$, $\Psi_1(t) = \Psi(t)$ and denote $S_1(g,h) = S_\Psi(g,h)$ such that

$$S_\Psi(g,h) = \int_a^b g(t)h(t)dt - \frac{1}{b-a}\int_a^b g(t)dt \int_a^b h(t)dt - \int_a^b g(t)\Psi_0(t)dt \int_a^b h(t)\Psi_0(t)dt \tag{2.1}$$

where $g, h, \Psi \in L_2(a,b)$, $\Psi_0(t) = \Psi(t)/\left\|\Psi\right\|_2$ and

$$\int_a^b \Psi(t)dt = 0. \tag{2.2}$$

Lemma 2.2. *With the above notations we have*

$$|S_\Psi(g,h)| \leq S_\Psi(g,g)^{\frac{1}{2}} S_\Psi(h,h)^{\frac{1}{2}}. \tag{2.3}$$

It is obvious that

$$S_\Psi(g,h) = (b-a)T(g,h) - \int_a^b g(t)\Psi_0(t)dt \int_a^b h(t)\Psi_0(t)dt \tag{2.4}$$

so that $S_\Psi(g,h)$ is a generalization of the Chebyshev functional.

We also define the functions:

$$\Phi(t) = \begin{cases} t - \frac{2a+b}{3}, & t \in \left[a, \frac{a+b}{2}\right] \\ t - \frac{a+2b}{3}, & t \in \left(\frac{a+b}{2}, b\right] \end{cases} \tag{2.5}$$

and

$$\chi(t) = \begin{cases} t - \frac{5a+b}{6}, & t \in \left[a, \frac{a+b}{2}\right] \\ t - \frac{a+5b}{6}, & t \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{2.6}$$

It is not difficult to verify that

$$\int_a^b \Phi(t)dt = \int_a^b \chi(t)dt = 0 \tag{2.7}$$

and

$$\left\|\Phi\right\|_2^2 = \left\|\chi\right\|_2^2 = \frac{(b-a)^3}{36}. \tag{2.8}$$

We define

$$\Phi_0(t) = \frac{\Phi(t)}{\left\|\Phi\right\|_2}, \ \chi_0(t) = \frac{\chi(t)}{\left\|\chi\right\|_2}. \tag{2.9}$$

Integrating by parts, we have

$$\begin{aligned} Q(f;a,b) &= \int_a^b \Phi_0(t)f'(t)dt \\ &= \frac{2}{\sqrt{b-a}}\left[f(a) + f\left(\frac{a+b}{2}\right) + f(b) - \frac{3}{b-a}\int_a^b f(t)dt\right] \end{aligned} \tag{2.10}$$

and

$$(2.11)\qquad P(f;a,b) = \int_a^b \chi_0(t) f'(t)dt = \frac{1}{\sqrt{b-a}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)-\frac{6}{b-a}\int_a^b f(t)dt\right].$$

Remark 2.3. It is obvious that

$$(2.12)\qquad S_\Psi(g,g)=(b-a)T(g,g)-\left(\int_a^b g(t)\Psi_0(t)dt\right)^2 \le (b-a)T(g,g).$$

Theorem 2.4. *(Mid-point inequality) Let $I\subset\mathbb{R}$ be a closed interval and $a,b\in Int\, I$, $a<b$. If $f: I\to\mathbb{R}$ is an absolutely continuous function whose derivative $f'\in L_2(a,b)$ then we have*

$$(2.13)\qquad \left|f\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)dt\right| \le \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}}C_1,$$

where

$$(2.14)\qquad C_1=\left\{\|f'\|_2^2-\frac{[f(b)-f(a)]^2}{b-a}-[Q(f;a,b)]^2\right\}^{\frac{1}{2}}$$

and $Q(f;a,b)$ is defined by (2.10).

Proof. We define

$$(2.15)\qquad p(t)=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right] \\ t-b, & t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Then we have

$$(2.16)\qquad \int_a^b p(t)dt=0$$

and

$$(2.17)\qquad \|p\|_2^2=\int_a^b p(t)^2dt=\frac{(b-a)^3}{12}.$$

We now calculate

$$(2.18)\qquad \int_a^b p(t)\Phi(t)dt=\int_a^{\frac{a+b}{2}}(t-a)\left(t-\frac{2a+b}{3}\right)dt+\int_{\frac{a+b}{2}}^b (t-b)\left(t-\frac{a+2b}{3}\right)dt=0.$$

Integrating by parts, we have

$$(2.19)\qquad \int_a^b p(t)f'(t)dt = \int_a^{\frac{a+b}{2}}(t-a)f'(t)dt+\int_{\frac{a+b}{2}}^b (t-b)f'(t)dt = f\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)dt.$$

Using (2.16), (2.18) and (2.19) we get

$$S_\Phi(p,f') = \int_a^b p(t)f'(t)dt - \frac{1}{b-a}\int_a^b p(t)dt\int_a^b f'(t)dt - \int_a^b f'(t)\Phi_0(t)dt\int_a^b p(t)\Phi_0(t)dt = f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)dt. \tag{2.20}$$

From (2.20) and (2.3) it follows that

$$\left|f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)dt\right| \le S_\Phi(f',f')^{\frac{1}{2}} S_\Phi(p,p)^{\frac{1}{2}}. \tag{2.21}$$

From (2.16)-(2.18) we get

$$S_\Phi(p,p) = \|p\|_2^2 - \frac{1}{b-a}\left(\int_a^b p(t)dt\right)^2 - \left(\int_a^b p(t)\Phi_0(t)dt\right)^2 = \frac{(b-a)^3}{12}. \tag{2.22}$$

We also have

$$C_1^2 = S_\Phi(f',f'). \tag{2.23}$$

From (2.21)-(2.23) we easily find that (2.13) holds. □

Remark 2.5. It is not difficult to see that (2.13) is better than the first estimation in (1.5).

Theorem 2.6. *(Trapezoidal inequality) Under the assumptions of Theorem 2.4 we have*

$$\left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)dt\right| \le \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}} C_2, \tag{2.24}$$

where

$$C_2 = \left\{\|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - [P(f;a,b)]^2\right\}^{\frac{1}{2}} \tag{2.25}$$

and $P(f;a,b)$ is defined by (2.11).

Proof. Let $p(t)$ be defined by (2.15). We calculate

$$\int_a^b p(t)\chi(t)dt = \int_a^{\frac{a+b}{2}} (t-a)\left(t - \frac{5a+b}{6}\right)dt + \int_{\frac{a+b}{2}}^b (t-b)\left(t-\frac{a+5b}{6}\right)dt = \frac{(b-a)^3}{24}. \tag{2.26}$$

Integrating by parts, we have

$$\int_a^b f'(t)\chi(t)dt = \int_a^{\frac{a+b}{2}} \left(t-\frac{5a+b}{6}\right)f'(t)dt + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+5b}{6}\right)f'(t)dt = \frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}(b-a) - \int_a^b f(t)dt. \tag{2.27}$$

Using (2.16), (2.19), (2.26), (2.27) and (2.8) we get

$$
\begin{aligned}
(2.28)\qquad S_\chi(f',p) &= \int_a^b p(t)f'(t)dt - \frac{1}{b-a}\int_a^b f'(t)dt\int_a^b p(t)dt \\
&\quad - \int_a^b p(t)\chi_0(t)dt\int_a^b f'(t)\chi_0(t)dt \\
&= f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)dt \\
&\quad - \frac{3}{2}\left[\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}(b-a) - \int_a^b f(t)dt\right] \\
&= -\frac{1}{2}(b-a)\frac{f(a)+f(b)}{2} + \frac{1}{2}\int_a^b f(t)dt.
\end{aligned}
$$

From (2.3) and (2.28) it follows that

$$
(2.29)\qquad \left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)dt\right| \le 2S_\chi(f',f')^{\frac{1}{2}}S_\chi(p,p)^{\frac{1}{2}}.
$$

We have

$$
\begin{aligned}
(2.30)\qquad S_\chi(p,p) &= \|p\|_2^2 - \frac{1}{b-a}\left(\int_a^b p(t)dt\right)^2 - \left(\int_a^b p(t)\chi_0(t)dt\right)^2 \\
&= \frac{(b-a)^3}{48}
\end{aligned}
$$

and

$$
(2.31)\qquad C_2^2 = S_\chi(f',f').
$$

From (2.29)-(2.31) we easily get (2.24). □

Remark 2.7. We see that (2.24) is better than the first estimation in (1.6).

We now consider a simple quadrature rule of the form

$$
\begin{aligned}
(2.32)\qquad &\frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}(b-a) - \int_a^b f(t)dt \\
&= \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right](b-a) - \int_a^b f(t)dt = R(f).
\end{aligned}
$$

It is not difficult to see that (2.32) is a convex combination of the mid-point quadrature rule and the trapezoidal quadrature rule. In [5] it is shown that (2.32) has a better estimation of error than the well-known Simpson quadrature rule (when we estimate the error in terms of the first derivative f' of integrand f). We here have a similar case.

Theorem 2.8. *Under the assumptions of Theorem 2.4 we have*

$$
(2.33)\qquad \left|\frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}(b-a) - \int_a^b f(t)dt\right| \le \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}}C_3,
$$

where

$$
(2.34)\qquad C_3 = \left[\|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - \frac{1}{b-a}\left(f(a)-2f\left(\frac{a+b}{2}\right)+f(b)\right)^2\right]^{\frac{1}{2}}.
$$

Proof. We define

$$\eta(t)=\begin{cases}1, & t\in\left[a,\frac{a+b}{2}\right]\\ -1, & t\in\left(\frac{a+b}{2},b\right]\end{cases}, \tag{2.35}$$

$$\eta_0(t)=\frac{\eta(t)}{\|\eta\|_2}. \tag{2.36}$$

We easily find that

$$\int_a^b \eta(t)dt=0,\quad \|\eta\|_2^2=b-a. \tag{2.37}$$

Let $p(t)$ be defined by (2.15). Then we have

$$\int_a^b p(t)\eta(t)dt=\int_a^{\frac{a+b}{2}}(t-a)dt-\int_{\frac{a+b}{2}}^b(t-b)dt=\frac{(b-a)^2}{4}. \tag{2.38}$$

We also have

$$\int_a^b f'(t)\eta(t)dt=-f(a)+2f\left(\frac{a+b}{2}\right)-f(b). \tag{2.39}$$

From (2.37)-(2.39) we get

$$\begin{aligned} S_\eta(f',p) &= \int_a^b p(t)f'(t)dt-\frac{1}{b-a}\int_a^b f'(t)dt\int_a^b p(t)dt \\ &\quad -\int_a^b p(t)\eta_0(t)dt\int_a^b f'(t)\eta_0(t)dt \\ &= f\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)dt \\ &\quad -\frac{b-a}{4}\left[-f(a)+2f\left(\frac{a+b}{2}\right)-f(b)\right] \\ &= \frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}(b-a)-\int_a^b f(t)dt. \end{aligned} \tag{2.40}$$

From (2.3) and (2.40) it follows that

$$\left|\frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}(b-a)-\int_a^b f(t)dt\right|\le S_\eta(f',f')^{\frac12}S_\eta(p,p)^{\frac12}. \tag{2.41}$$

We now calculate

$$\begin{aligned} S_\eta(p,p) &= \|p\|_2^2-\frac{1}{b-a}\left(\int_a^b p(t)dt\right)^2-\left(\int_a^b p(t)\eta_0(t)dt\right)^2 \\ &= \frac{(b-a)^3}{48}. \end{aligned} \tag{2.42}$$

We also have

$$C_3^2=S_\eta(f',f'). \tag{2.43}$$

From (2.41)-(2.43) we easily get (2.33). □

Remark 2.9. It is not difficult to see that (2.33) is better than the first estimation in (1.7).

Finally, in [12] we can find the next inequality

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}(b-a)-\int_a^b f(t)dt\right| \le \frac{(b-a)^2}{12}(\Gamma-\gamma), \tag{2.44}$$

where $f : I \to \mathbb{R}$, ($I \subset \mathbb{R}$ is an open interval, $a < b$, $a, b \in I$) is a differentiable function, f' is integrable and there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \le f'(t) \le \Gamma$, $t \in [a, b]$.

Inequality (2.44) is a variant of the Simpson's inequality. On the other hand, we have

$$\left|\frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}(b-a)-\int_a^b f(t)dt\right| \le \frac{(b-a)^2}{8\sqrt{3}}(\Gamma-\gamma). \tag{2.45}$$

Inequality (2.45) follows from (2.33), since

$$S_\eta(f', f') \le (b-a)\left(\frac{\Gamma-\gamma}{2}\right)^2 \tag{2.46}$$

and (2.46) follows from (2.4) and (1.4).

Form (2.44) and (2.45) we see that the simple 3-point quadrature rule (2.32) has a better estimation of error than the well-known 3-point Simpson quadrature rule. Note that the estimations are expressed in terms of the first derivative f' of integrand.

Finally, the following remark is valid.

Remark 2.10. The considered case $n = 1$ illustrates how to apply Lemma 2.1 to quadrature formulas. It is also shown that the derived results are better than some recently obtained results. We can use Lemma 2.1 to derive further improvements of the obtained results. However, in such a case we must require

$$\int_a^b g(t)\Psi_i^0(t)dt = 0,\ i = 0, 1, 2, ..., n.$$

Thus, the construction of such a finite sequence $\{\Psi_i^0\}_0^n$ can be complicated. However, if we really need better error bounds, without taking into account possible complications, then we can apply the procedure described in this section.

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