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## L'HOSPITAL TYPE RULES FOR MONOTONICITY: APPLICATIONS TO PROBABILITY INEQUALITIES FOR SUMS OF BOUNDED RANDOM VARIABLES

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## Abstract

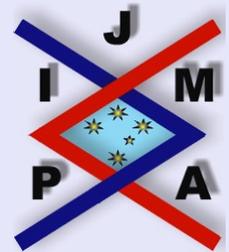
This paper continues a series of results begun by a l'Hospital type rule for monotonicity, which is used here to obtain refinements of the Eaton-Pinelis inequalities for sums of bounded independent random variables.

*2000 Mathematics Subject Classification:* Primary: 26A48, 26D10, 60E15; Secondary: 26D07, 62H15, 62F04, 62F35, 62G10, 62G15

*Key words:* L'Hospital's Rule, Monotonicity, Probability inequalities, Sums of independent random variables, Student's statistic

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# 1. Introduction

In [8], the following criterion for monotonicity was given, which reminds one of the l'Hospital rule for computing limits.

**Proposition 1.1.** *Let  $-\infty \leq a < b \leq \infty$ . Let  $f$  and  $g$  be differentiable functions on an interval  $(a, b)$ . Assume that either  $g' > 0$  everywhere on  $(a, b)$  or  $g' < 0$  on  $(a, b)$ . Suppose that  $f(a+) = g(a+) = 0$  or  $f(b-) = g(b-) = 0$  and  $\frac{f'}{g'}$  is increasing (decreasing) on  $(a, b)$ . Then  $\frac{f}{g}$  is increasing (respectively, decreasing) on  $(a, b)$ . (Note that the conditions here imply that  $g$  is nonzero and does not change sign on  $(a, b)$ .)*

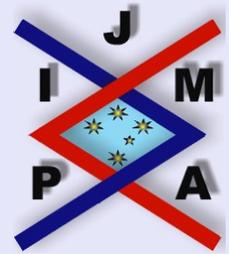
Developments of this result and applications were given: in [8], applications to certain information inequalities; in [10], extensions to non-monotonic ratios of functions, with applications to certain probability inequalities arising in bioequivalence studies and to convexity problems; in [9], applications to monotonicity of the relative error of a Padé approximation for the complementary error function.

Here we shall consider further applications, to probability inequalities, concerning the Student  $t$  statistic.

Let  $\eta_1, \dots, \eta_n$  be independent zero-mean random variables such that  $\mathbb{P}(|\eta_i| \leq 1) = 1$  for all  $i$ , and let  $a_1, \dots, a_n$  be any real numbers such that  $a_1^2 + \dots + a_n^2 = 1$ . Let  $\nu$  stand for a standard normal random variable.

In [3] and [4], a multivariate version of the following inequality was given:

$$(1.1) \quad \mathbb{P}(|a_1\eta_1 + \dots + a_n\eta_n| \geq u) < c \cdot \mathbb{P}(|\nu| \geq u) \quad \forall u \geq 0,$$



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where

$$c := \frac{2e^3}{9} = 4.463 \dots;$$

cf. Corollary 2.6 in [4] and the comment in the middle of page 359 therein concerning the Hunt inequality. For subsequent developments, see [5], [6], and [7].

Inequality (1.1) implies a conjecture made by Eaton [2]. In turn, (1.1) was obtained in [4] based on the inequality

$$(1.2) \quad \mathbb{P}(|a_1\eta_1 + \dots + a_n\eta_n| \geq u) \leq Q(u) \quad \forall u \geq 0,$$

where

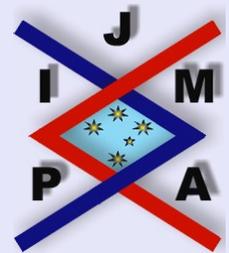
$$(1.3) \quad Q(u) := \min \left[ 1, \frac{1}{u^2}, W(u) \right]$$

$$(1.4) \quad = \begin{cases} 1 & \text{if } 0 \leq u \leq 1, \\ \frac{1}{u^2} & \text{if } 1 \leq u \leq \mu_1, \\ W(u) & \text{if } u \geq \mu_1, \end{cases}$$

$$\mu_1 := \frac{\mathbb{E}|\nu|^3}{\mathbb{E}|\nu|^2} = 2\sqrt{\frac{2}{\pi}} = 1.595 \dots;$$

$$W(u) := \inf \left\{ \frac{\mathbb{E}(|\nu| - t)_+^3}{(u - t)^3} : t \in (0, u) \right\};$$

cf. Lemma 3.5 in [4]. The bound  $Q(u)$  possesses a certain optimality property; cf. (3.7) in [4] and the definition of  $Q_r(u)$  therein. In [1],  $Q(u)$  is denoted by



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$B_{EP}(u)$ , called the Eaton-Pinelis bound, and tabulated, along with other related bounds; various statistical applications are given therein.

Let

$$\varphi(u) := \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \quad \Phi(u) := \int_{-\infty}^u \varphi(s) ds, \quad \text{and} \quad \bar{\Phi}(u) := 1 - \Phi(u)$$

denote, as usual, the density, distribution function, and tail function of the standard normal law.

It follows from [4] (cf. Lemma 3.6 therein) that the ratio

$$(1.5) \quad r(u) := \frac{Q(u)}{c \cdot \mathbb{P}(|\nu| \geq u)} = \frac{Q(u)}{c \cdot 2\bar{\Phi}(u)}, \quad u \geq 0,$$

of the upper bounds in (1.2) and (1.1) is less than 1 for all  $u \geq 0$ , so that (1.2) indeed implies (1.1). Moreover, it was shown in [4] that  $r(u) \rightarrow 1$  as  $u \rightarrow \infty$ ; cf. Proposition A.2 therein. Other methods of obtaining (1.1) are given in [5] and [6].

In Section 2 of this paper, we shall present monotonicity properties of the ratio  $r$ , from which it follows, once again, that

$$(1.6) \quad r < 1 \quad \text{on} \quad (0, \infty).$$

Combining the bounds (1.1) and (1.2) and taking (1.3) into account, one has the following improvement of the upper bound provided by (1.1):

$$(1.7) \quad \mathbb{P}(|a_1\eta_1 + \cdots + a_n\eta_n| \geq u) \leq V(u) := \min \left[ 1, \frac{1}{u^2}, c \cdot \mathbb{P}(|\nu| \geq u) \right] \quad \forall u \geq 0.$$



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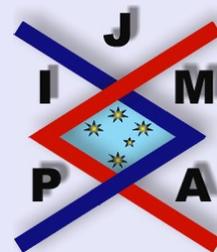
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Monotonicity properties of the ratio

$$(1.8) \quad R := \frac{Q}{V}$$

of the upper bounds in (1.2) and (1.7) will be studied in Section 3.

Our approach is based on Proposition 1.1. Mainly, we follow here lines of [3].



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## 2. Monotonocity Properties of the Ratio $r$ given by (1.5)

### Theorem 2.1.

1. There is a unique solution to the equation  $2\bar{\Phi}(d) = d \cdot \varphi(d)$  for  $d \in (1, \mu_1)$ ; in fact,  $d = 1.190\dots$

2. The ratio  $r$  is

(a) increasing on  $[0, 1]$  from  $r(0) = \frac{1}{c} = 0.224\dots$  to  $r(1) = \frac{1}{c \cdot 2\bar{\Phi}(1)} = 0.706\dots$ ;

(b) decreasing on  $[1, d]$  from  $r(1) = 0.706\dots$  to  $r(d) = \frac{1}{c \cdot 2\bar{\Phi}(d)} = 0.675\dots$ ;

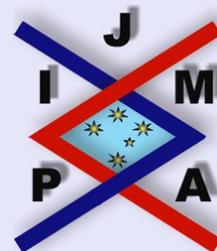
(c) increasing on  $[d, \infty)$  from  $r(d) = 0.675\dots$  to  $r(\infty) = 1$ .

*Proof.*

1. Consider the function

$$h(u) := 2\bar{\Phi}(u) - u\varphi(u).$$

One has  $h(1) = 0.07\dots > 0$ ,  $h(\mu_1) = -0.06\dots < 0$ , and  $h'(u) = (u^2 - 3)\varphi(u)$ . Hence,  $h'(u) < 0$  for  $u \in [1, \mu_1]$ , since  $\mu_1 < \sqrt{3}$ . This implies part 1 of the theorem.



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2.

(a) Part 2(a) of the theorem is immediate from (1.5) and (1.4).

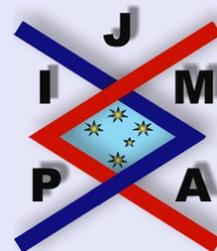
(b) For  $u > 0$ , one has

$$\frac{d}{du} (u^2 \bar{\Phi}(u)) = uh(u),$$

where  $h$  is the function considered in the proof of part 1 of the theorem. Since  $h > 0$  on  $[1, d)$  and  $r(u) = \frac{1}{2cu^2 \bar{\Phi}(u)}$  for  $u \in [1, \mu_1]$ , part 2(b) now follows.

(c) Since  $h < 0$  on  $(d, \mu_1]$ , it also follows from above that  $r$  is increasing on  $[d, \mu_1]$ . It remains to show that  $r$  is increasing on  $[\mu_1, \infty)$ . This is the main part of the proof, and it requires some notation and facts from [4]. Let

$$(2.1) \quad \begin{aligned} C &:= \frac{1}{\int_0^\infty e^{-s^2/2} ds}, \\ \gamma(u) &:= \int_u^\infty (s-u)^3 e^{-s^2/2} ds, \\ \gamma^{(j)}(u) &:= \frac{d^j \gamma(u)}{du^j} \quad (\gamma^{(0)} := \gamma), \\ \mu(t) &:= t - \frac{3\gamma(t)}{\gamma'(t)}, \\ F(t, u) &:= C \frac{\gamma(t)}{(u-t)^3}, \quad t < u; \end{aligned}$$



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cf. notation on pages 361–363 in [4], in which we presently take  $r = 1$ .

Then  $\forall j \in \{0, 1, 2, 3, 4, 5\}$

$$(2.2) \quad (-1)^j \gamma^{(j)} > 0 \quad \text{on} \quad (0, \infty),$$

$$(2.3) \quad (-1)^j \gamma^{(j)}(u) = 6u^{j-4} e^{-u^2/2} (1 + o(1)) \quad \text{as} \quad u \rightarrow \infty,$$

$$(2.4) \quad \gamma^{(4)}(u) = 6e^{-u^2/2} \quad \text{and} \quad \gamma^{(5)}(u) = -6ue^{-u^2/2};$$

cf. Lemma 3.3 in [4]. Moreover, it was shown in [4] (see page 363 therein) that on  $[0, \infty)$

$$(2.5) \quad \mu' > 0,$$

so that the formula

$$t \leftrightarrow u = \mu(t)$$

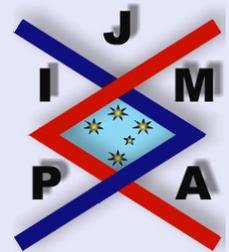
defines an increasing correspondence between  $t \geq 0$  and  $u \geq \mu(0) = \mu_1$ , so that the inverse map

$$\mu^{-1} : [\mu_1, \infty) \rightarrow [0, \infty)$$

is correctly defined and is a bijection. Finally, one has (cf. (3.11) in [4] and (1.4) and (2.1) above)

$$(2.6) \quad \forall u \geq \mu_1 \quad Q(u) = W(u) = F(t, u) = -\frac{C}{27} \frac{\gamma'(t)^3}{\gamma(t)^2};$$

here and in the rest of this proof,  $t$  stands for  $\mu^{-1}(u)$  and, equivalently,  $u$  for  $\mu(t)$ .



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Now equation (2.6) implies

$$(2.7) \quad Q'(u) = \frac{\frac{dQ(\mu(t))}{dt}}{\frac{d\mu(t)}{dt}} = -\frac{C}{27} \frac{\gamma'(t)^4}{\gamma(t)^3}.$$

for  $u \geq \mu_1$ ; here we used the formula

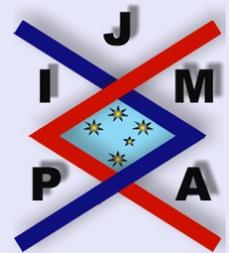
$$(2.8) \quad \mu'(t) = \frac{3\gamma(t)\gamma''(t) - 2\gamma'(t)^2}{\gamma'(t)^2}.$$

Next,

$$\begin{aligned} \gamma'(t)\mu(t) &= t\gamma'(t) - 3\gamma(t) \\ &= -3 \int_t^\infty [t(s-t)^2 + (s-t)^3] e^{-s^2/2} ds \\ &= -3 \int_t^\infty (s-t)^2 s e^{-s^2/2} ds \\ &= -6 \int_t^\infty (s-t) e^{-s^2/2} ds \\ &= -\gamma''(t); \end{aligned}$$

for the fourth of the five equalities here, integration by parts was used. Hence, on  $[0, \infty)$ ,

$$(2.9) \quad \mu = -\frac{\gamma''}{\gamma'},$$



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whence

$$\mu' = \frac{\gamma''^2 - \gamma'\gamma'''}{\gamma'^2};$$

this and (2.5) yield

$$(2.10) \quad \gamma''^2 - \gamma'\gamma''' > 0.$$

Let (cf. (1.5) and use (2.7))

$$(2.11) \quad \rho(u) := \frac{Q'(u)}{c \cdot 2\bar{\Phi}'(u)} = \frac{C}{54c} \frac{\gamma'(t)^4}{\gamma(t)^3 \varphi(\mu(t))}.$$

Using (2.11) and then (2.9) and (2.8), one has

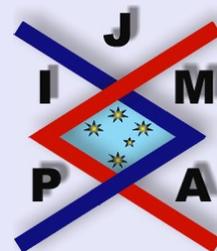
$$(2.12) \quad \begin{aligned} \frac{d \ln \rho(u)}{dt} &= \frac{d}{dt} \left( 4 \ln |\gamma'(t)| - 3 \ln \gamma(t) + \frac{\mu(t)^2}{2} \right) \\ &= -\frac{3D(t)^2 \gamma''(t)^2}{\gamma(t) \gamma'(t)^3} \end{aligned}$$

for all  $t > 0$ , where

$$D := \frac{\gamma'^2}{\gamma''} - \gamma.$$

Further, on  $(0, \infty)$ ,

$$(2.13) \quad D' = \frac{\gamma'}{\gamma''^2} (\gamma''^2 - \gamma'\gamma''') < 0,$$



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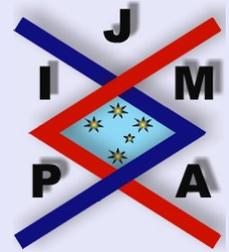
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in view of (2.2) and (2.10). On the other hand, it follows from (2.3) that  $D(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Hence, (2.13) implies that on  $(0, \infty)$

$$(2.14) \quad D > 0.$$

Now (2.12), (2.14), and (2.2) imply that  $\rho$  is increasing on  $(\mu_1, \infty)$ . Also, it follows from (2.6) and (2.3) that  $Q(u) \rightarrow 0$  as  $u \rightarrow \infty$ ; it is obvious that  $c \cdot 2\bar{\Phi}(u) \rightarrow 0$  as  $u \rightarrow \infty$ . It remains to refer to (1.5), (2.11), Proposition 1.1, and also (for  $r(\infty) = 1$ ) to Proposition A.2 [4].

□



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### 3. Monotonicity Properties of the Ratio $R$ given by (1.8)

#### Theorem 3.1.

1. There is a unique solution to the equation

$$(3.1) \quad \frac{1}{z^2} = c \cdot \mathbb{P}(|\nu| \geq z)$$

for  $z > \mu_1$ ; in fact,  $z = 1.834\dots$

2.

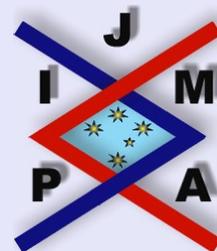
$$(3.2) \quad V(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq 1, \\ \frac{1}{u^2} & \text{if } 1 \leq u \leq z, \\ c \cdot \mathbb{P}(|\nu| \geq u) & \text{if } u \geq z. \end{cases}$$

3. (a)  $R = 1$  on  $[0, \mu_1]$ ;

(b)  $R$  is decreasing on  $[\mu_1, z]$  from  $R(\mu_1) = 1$  to  $R(z) = 0.820\dots$ ;

(c)  $R$  is increasing on  $[z, \infty)$  from  $R(z) = 0.820\dots$  to  $R(\infty) = 1 [= r(\infty)]$ .

Thus, the upper bound  $V$  is quite close to the optimal Eaton-Pinelis bound  $Q = B_{EP}$  given by (1.3), exceeding it by a factor of at most  $\frac{1}{R(z)} = 1.218\dots$ . In addition,  $V$  is asymptotic (at  $\infty$ ) to and as universal as  $Q$ . On the other hand,  $V$  is much more transparent and tractable than  $Q$ .



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*Proof of Theorem 3.1.*

1. Consider the function

$$(3.3) \quad \lambda(u) := \frac{c\mathbb{P}(|\nu| \geq u)}{\frac{1}{u^2}} = 2cu^2\bar{\Phi}(u).$$

Then

$$\lambda'(u) = 2cuh(u),$$

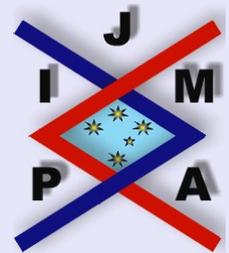
where  $h$  is the same as in the beginning of the proof of Theorem 2.1 on page 7, with  $h'(u) = (u^2 - 3)\varphi(u)$ , so that  $\sqrt{3}$  is the only root of the equation  $h'(u) = 0$ . Since  $h(\mu_1) = -0.06\dots < 0$ ,  $h(\sqrt{3}) = -0.07\dots < 0$ , and  $h(\infty) = 0$ , it follows that  $h < 0$  on  $[\mu_1, \infty)$ , and then so is  $\lambda'$ . Hence,  $\lambda$  is decreasing on  $[\mu_1, \infty)$  from  $\lambda(\mu_1) = 1.2\dots$  to  $\lambda(\infty) = 0$ . Now part 1 of the theorem follows.

2. It also follows from the above that  $\lambda \geq 1$  on  $[\mu_1, z]$  and  $\lambda \leq 1$  on  $[z, \infty)$ .

In addition, by (3.3), (1.5), and (1.4), one has  $\lambda = \frac{1}{r}$  on  $[1, \mu_1]$ , whence  $\lambda > 1$  on  $[1, \mu_1]$  by (1.6). Thus,  $\lambda \geq 1$  on  $[1, z]$  and  $\lambda \leq 1$  on  $[z, \infty)$ ; in particular,  $c\mathbb{P}(|\nu| \geq 1) = \lambda(1) \geq 1$ . Now part 2 of the theorem follows.

3. (a) Part 3(a) of the theorem is immediate from (1.4), (3.2), and the inequality  $z > \mu_1$ .

(b) Of all the parts of the theorem, part 3(b) is the most difficult to prove. In view of (3.2), the inequalities  $z > \mu_1 > 1$ , (2.6), and (2.9), one has



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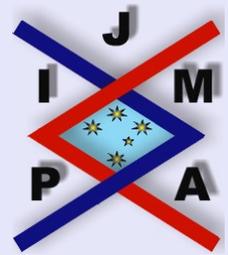


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$$(3.4) \quad R(u) = u^2 Q(u) = -\frac{C}{27} \frac{\gamma'(t)\gamma''(t)^2}{\gamma(t)^2} \quad \forall u \in [\mu_1, z];$$

here and to the rest of this proof,  $t$  again stands for  $\mu^{-1}(u)$  and, equivalently,  $u$  for  $\mu(t)$ . It follows that for all  $u \in [\mu_1, z]$  or, equivalently, for all  $t \in [0, \mu^{-1}(z)]$ ,

$$(3.5) \quad \frac{d}{dt} \ln R(u) = L(t) := \frac{\gamma''(t)}{\gamma'(t)} + 2 \frac{\gamma'''(t)}{\gamma''(t)} - 2 \frac{\gamma'(t)}{\gamma(t)}.$$

Comparing (2.1) and (2.9), one has for all  $t > 0$

$$(3.6) \quad \frac{\gamma''(t)}{\gamma'(t)} = 3 \frac{\gamma(t)}{\gamma'(t)} - t = -\left(t + \frac{3}{\kappa(t)}\right),$$

where

$$(3.7) \quad \kappa(t) := -\frac{\gamma'(t)}{\gamma(t)};$$

similarly,

$$(3.8) \quad \frac{\gamma'''(t)}{\gamma''(t)} = 2 \frac{\gamma'(t)}{\gamma''(t)} - t = \frac{2}{\frac{\gamma''(t)}{\gamma'(t)}} - t;$$

this and (3.6) yield

$$(3.9) \quad \frac{\gamma'''(t)}{\gamma''(t)} = -\frac{(t^2 + 2)\kappa(t) + 3t}{t\kappa(t) + 3}.$$

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Now (3.5), (3.6), and (3.9) lead to

$$(3.10) \quad L(t) = -\frac{N(t, \kappa(t))}{\kappa(t) (t\kappa(t) + 3)},$$

where

$$N(t, k) := -2t k^3 + (3t^2 - 2) k^2 + 12t k + 9.$$

Next, for  $t > 0$ ,

$$-\frac{1}{6t} \frac{\partial N}{\partial k} = k^2 - \left(t - \frac{2}{3t}\right) k - 2,$$

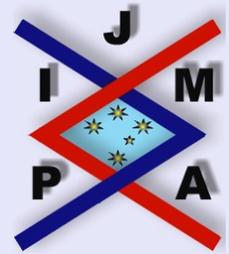
which is a monic quadratic polynomial in  $k$ , the product of whose roots is  $-2$ , negative, so that one has  $k_1(t) < 0 < k_2(t)$ , where  $k_1(t)$  and  $k_2(t)$  are the two roots. It follows that  $\frac{\partial N}{\partial k} > 0$  on  $(0, k_2(t))$  and  $\frac{\partial N}{\partial k} < 0$  on  $(k_2(t), \infty)$ .

Hence,  $N(t, k)$  is increasing in  $k \in (0, k_2(t))$  and decreasing in  $k \in (k_2(t), \infty)$ . On the other hand, it follows from (3.7) and (2.2) that

$$(3.11) \quad \kappa(t) > 0 \quad \forall t > 0.$$

Therefore,

$$(3.12) \quad (\kappa(t) < \kappa^*(t) \quad \forall t > 0) \\ \implies (N(t, \kappa(t)) > \min(N(t, 0), N(t, \kappa^*(t))) \quad \forall t > 0);$$



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at this point,  $\kappa^*$  may be any function which majorizes  $\kappa$  on  $(0, \infty)$ .

Let us now show the function  $\kappa^*(t) := t + 2$  is such a majorant of  $\kappa(t)$ . Toward this end, introduce

$$\gamma^{(-1)}(t) := -\frac{1}{4} \int_t^\infty (s-t)^4 e^{-s^2/2} ds,$$

so that

$$(\gamma^{(-1)})' = \gamma.$$

Similarly to (3.6) and (3.8),

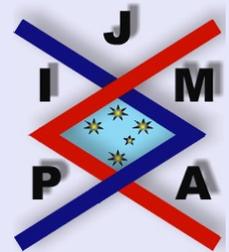
$$(3.13) \quad \kappa(t) = -\frac{\gamma'(t)}{\gamma(t)} = -4\frac{\gamma^{(-1)}(t)}{\gamma(t)} + t.$$

Again with  $\gamma^{(0)} := \gamma$ , one has for  $t > 0$

$$\frac{(-\gamma^{(j-1)})'}{(\gamma^{(j)})'} = \frac{-\gamma^{(j)}}{\gamma^{(j+1)}} \quad \forall j \in \{0, 1, \dots\},$$

and, in view of (2.4),  $\frac{-\gamma^{(4)}(t)}{\gamma^{(5)}(t)} = \frac{1}{t}$  is decreasing in  $t > 0$ . In addition, (2.3) implies that  $\gamma^{(j)}(t) \rightarrow 0$  as  $t \rightarrow \infty$ , for every  $j \in \{-1, 0, 1, \dots\}$ . Using now Proposition 1.1 repeatedly, 5 times, one sees that  $\frac{-\gamma^{(-1)}}{\gamma}$  is decreasing on  $(0, \infty)$ , whence  $\forall t > 0$

$$\frac{-\gamma^{(-1)}(t)}{\gamma(t)} < \frac{-\gamma^{(-1)}(0)}{\gamma(0)} = \frac{3\sqrt{2\pi}}{16} < \frac{1}{2}.$$



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This and (3.13) imply that

$$\kappa(t) < t + 2 \quad \forall t > 0.$$

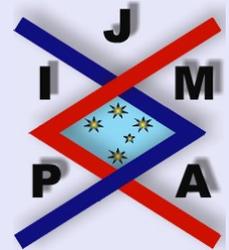
Hence, in view of (3.12),

$$N(t, \kappa(t)) > \min(N(t, 0), N(t, t + 2)) \quad \forall t > 0.$$

But  $N(t, 0) = 9 > 0$  and  $N(t, t + 2) = (t^2 - 1)^2 \geq 0$  for all  $t$ . Therefore,  $N(t, \kappa(t)) > 0 \quad \forall t > 0$ . Recalling now (3.5), (3.10) and (3.11), one concludes that  $R$  is decreasing on  $[\mu_1, z]$ . To compute  $R(z)$ , use (3.4). Now part 3(b) of the theorem is proved.

- (c) In view of (1.5) and (3.2), one has  $R = r$  on  $[z, \infty)$ . Part 3(c) of the theorem now follows from part 2(c) of Theorem 2.1 and inequalities  $d < \mu_1 < z$ .

□



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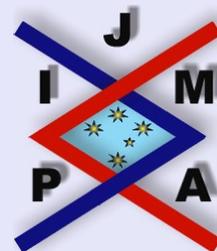
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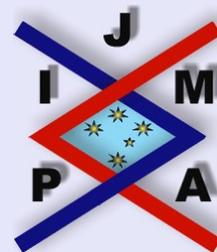
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