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## $L^p$ IMPROVING PROPERTIES FOR MEASURES ON $\mathbb{R}^4$ SUPPORTED ON HOMOGENEOUS SURFACES IN SOME NON ELLIPTIC CASES

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Abstract

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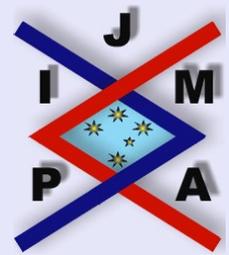


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## Abstract

In this paper we study convolution operators  $T_\mu$  with measures  $\mu$  in  $\mathbb{R}^4$  of the form  $\mu(E) = \int_B \chi_E(x, \varphi(x)) dx$ , where  $B$  is the unit ball of  $\mathbb{R}^2$ , and  $\varphi$  is a homogeneous polynomial function. If  $\inf_{h \in S^1} |\det(d_x^2 \varphi(h, \cdot))|$  vanishes only on a finite union of lines, we prove, under suitable hypothesis, that  $T_\mu$  is bounded from  $L^p$  into  $L^q$  if  $(\frac{1}{p}, \frac{1}{q})$  belongs to a certain explicitly described trapezoidal region.

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*Key words:* Singular measures,  $L^p$ -improving, convolution operators.

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# 1. Introduction

It is well known that a complex measure  $\mu$  on  $\mathbb{R}^n$  acts as a convolution operator on the Lebesgue spaces  $L^p(\mathbb{R}^n) : \mu * L^p \subset L^p$  for  $1 \leq p \leq \infty$ . If for some  $p$  there exists  $q > p$  such that  $\mu * L^p \subset L^q$ ,  $\mu$  is called  $L^p$ -improving. It is known that singular measures supported on smooth submanifolds of  $\mathbb{R}^n$  may be  $L^p$ -improving. See, for example, [2], [5], [8], [9], [7] and [4].

Let  $\varphi_1, \varphi_2$  be two homogeneous polynomial functions on  $\mathbb{R}^2$  of degree  $m \geq 2$  and let  $\varphi = (\varphi_1, \varphi_2)$ . Let  $\mu$  be the Borel measure on  $\mathbb{R}^4$  given by

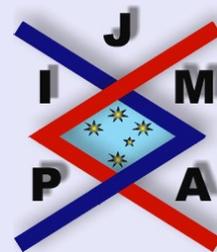
$$(1.1) \quad \mu(E) = \int_B \chi_E(x, \varphi(x)) dx,$$

where  $B$  denotes the closed unit ball around the origin in  $\mathbb{R}^2$  and  $dx$  is the Lebesgue measure on  $\mathbb{R}^2$ . Let  $T_\mu$  be the convolution operator given by  $T_\mu f = \mu * f$ ,  $f \in S(\mathbb{R}^4)$  and let  $E_\mu$  be the type set corresponding to the measure  $\mu$  defined by

$$E_\mu = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) : \|T_\mu\|_{p,q} < \infty, 1 \leq p, q \leq \infty \right\},$$

where  $\|T_\mu\|_{p,q}$  denotes the operator norm of  $T_\mu$  from  $L^p(\mathbb{R}^4)$  into  $L^q(\mathbb{R}^4)$  and where the  $L^p$  spaces are taken with respect to the Lebesgue measure on  $\mathbb{R}^4$ .

For  $x, h \in \mathbb{R}^2$ , let  $\varphi''(x)h$  be the  $2 \times 2$  matrix whose  $j$ -th column is  $\varphi''_j(x)h$ , where  $\varphi''_j(x)$  denotes the Hessian matrix of  $\varphi_j$  at  $x$ . Following [3, p. 152], we say that  $x \in \mathbb{R}^2$  is an elliptic point for  $\varphi$  if  $\det(\varphi''(x)h) \neq 0$  for all  $h \in \mathbb{R}^2 \setminus \{0\}$ . For  $A \subset \mathbb{R}^2$ , we will say that  $\varphi$  is strongly elliptic on  $A$  if  $\det(\varphi''_1(x)h, \varphi''_2(y)h) \neq 0$  for all  $x, y \in A$  and  $h \in \mathbb{R}^2 \setminus \{0\}$ .



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If every point  $x \in B \setminus \{0\}$  is elliptic for  $\varphi$ , it is proved in [4] that for  $m \geq 3$ ,  $E_\mu$  is the closed trapezoidal region  $\Sigma_m$  with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{m}{m+1}, \frac{m-1}{m+1})$  and  $(\frac{2}{m+1}, \frac{1}{m+1})$ .

Our aim in this paper is to study the case where the set of non elliptic points consists of a finite union of lines through the origin,  $L_1, \dots, L_k$ . We assume from now on, that for  $x \in R^2 - \{0\}$ ,  $\det(\varphi''(x)h)$  does not vanish identically, as a function of  $h$ . For each  $l = 1, 2, \dots, k$ , let  $\pi_{L_l}$  and  $\pi_{L_l^\perp}$  be the orthogonal projections from  $\mathbb{R}^2$  onto  $L_l$  and  $L_l^\perp$  respectively. For  $\delta > 0$ ,  $1 \leq l \leq k$ , let

$$V_\delta^l = \left\{ x \in B : 1/2 \leq |\pi_{L_l}(x)| \leq 1 \text{ and } \left| \pi_{L_l^\perp}(x) \right| \leq \delta |\pi_{L_l}(x)| \right\}.$$

It is easy to see (see Lemma 2.1 and Remark 3.2) that for  $\delta$  small enough, there exists  $\alpha_l \in \mathbb{N}$  and positive constants  $c$  and  $c'$  such that

$$c \left| \pi_{L_l^\perp}(x) \right|^{\alpha_l} \leq \inf_{h \in S^1} |\det(\varphi''(x)h)| \leq c' \left| \pi_{L_l}(x) \right|^{\alpha_l}$$

for all  $x \in V_\delta^l$ . Following the approach developed in [3], we prove, in Theorem 3.5, that if  $\alpha = \max_{1 \leq l \leq k} \alpha_l$  and if  $7\alpha \leq m + 1$ , then the interior of  $E_\mu$  agrees with the interior of  $\Sigma_m$ .

Moreover in Theorem 3.6 we obtain that  $\mathring{E}_\mu = \mathring{\Sigma}_m$  still holds in some cases where  $7\alpha > m + 1$ , if we require a suitable hypothesis on the behavior, near the lines  $L_1, \dots, L_k$ , of the map  $(x, y) \rightarrow \inf_{h \in S^1} |\det(\varphi''_1(x)h, \varphi''_2(y)h)|$ .

In any case, even though we can not give a complete description of the interior of  $E_\mu$ , we obtain a polygonal region contained in it.

Throughout the paper  $c$  will denote a positive constant not necessarily the same at each occurrence.



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## 2. Preliminaries

Let  $\varphi_1, \varphi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  be two homogeneous polynomial functions of degree  $m \geq 2$  and let  $\varphi = (\varphi_1, \varphi_2)$ . For  $\delta > 0$  let

$$(2.1) \quad V_\delta = \left\{ (x_1, x_2) \in B : \frac{1}{2} \leq |x_1| \leq 1 \text{ and } |x_2| \leq \delta |x_1| \right\}.$$

We assume in this section that, for some  $\delta_0 > 0$ , the set of the non elliptic points for  $\varphi$  in  $V_{\delta_0}$  is contained in the  $x_1$  axis.

For  $x \in \mathbb{R}^2$ , let  $P = P(x)$  be the symmetric matrix that realizes the quadratic form  $h \rightarrow \det(\varphi''(x)h)$ , so

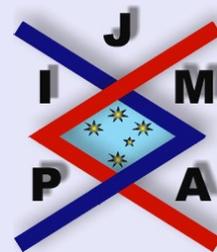
$$(2.2) \quad \det(\varphi''(x)h) = \langle P(x)h, h \rangle.$$

**Lemma 2.1.** *There exist  $\delta \in (0, \delta_0)$ ,  $\alpha \in \mathbb{N}$  and a real analytic function  $g = g(x_1, x_2)$  on  $V_\delta$  with  $g(x_1, 0) \neq 0$  for  $x_1 \neq 0$  such that*

$$(2.3) \quad \inf_{|h|=1} |\det(\varphi''(x)h)| = |x_2|^\alpha |g(x)|$$

for all  $x \in V_\delta$ .

*Proof.* Since  $P(x)$  is real analytic on  $V_\delta$  and  $P(x) \neq 0$  for  $x \neq 0$ , it follows that, for  $\delta$  small enough, there exists two real analytic functions  $\lambda_1(x)$  and  $\lambda_2(x)$  which are the eigenvalues of  $P(x)$ . Also,  $\inf_{|h|=1} |\det(\varphi''(x)h)| = \min\{|\lambda_1(x)|, |\lambda_2(x)|\}$  for  $x \in V_\delta$ . Since we have assumed that  $(1, 0)$  is not an elliptic point for  $\varphi$  and that  $P(x) \neq 0$  for  $x \neq 0$ , diminishing  $\delta$  if necessary, we can assume that  $\lambda_1(1, 0) = 0$  and that  $|\lambda_1(1, x_2)| \leq |\lambda_2(1, x_2)|$  for  $|x_2| \leq \delta$ .



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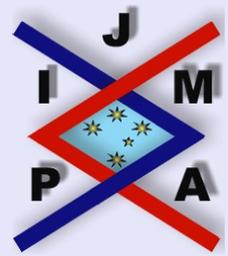


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Since  $P(x)$  is homogeneous in  $x$ , we have that  $\lambda_1(x)$  and  $\lambda_2(x)$  are homogeneous in  $x$  with the same homogeneity degree  $d$ . Thus  $|\lambda_1(x)| \leq |\lambda_2(x)|$  for all  $x \in V_\delta$ . Now,  $\lambda_1(1, x_2) = x_2^\alpha G(x_2)$  for some real analytical function  $G = G(x_2)$  with  $G(0) \neq 0$  and so  $\lambda_1(x_1, x_2) = x_1^d \lambda_1\left(1, \frac{x_2}{x_1}\right) = x_1^{d-\alpha} x_2^\alpha G\left(\frac{x_2}{x_1}\right)$ . Taking  $g(x_1, x_2) = x_1^{d-\alpha} G\left(\frac{x_2}{x_1}\right)$  the lemma follows.  $\square$

Following [3], for  $U \subset \mathbb{R}^2$  let  $J_U : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{\infty\}$  given by

$$J_U(h) = \inf_{x, x+h \in U} |\det(\varphi'(x+h) - \varphi'(x))|,$$

where the infimum of the empty set is understood to be  $\infty$ . We also set, as there, for  $0 < \alpha < 1$

$$R_\alpha^U(f)(x) = \int J_U(x-y)^{-1+\alpha} f(y) dy.$$

For  $r > 0$  and  $w \in \mathbb{R}^2$ , let  $B_r(w)$  denotes the open ball centered at  $w$  with radius  $r$ .

We have the following

**Lemma 2.2.** *Let  $w$  be an elliptic point for  $\varphi$ . Then there exist positive constants  $c$  and  $c'$  depending only on  $\|\varphi_1\|_{C^3(B)}$  and  $\|\varphi_2\|_{C^3(B)}$  such that if  $0 < r \leq c \inf_{|h|=1} |\det(\varphi''(w)h)|$  then*

$$(1) |\det(\varphi'(x+h) - \varphi'(x))| \geq \frac{1}{2} |\det(\varphi''(w)h)| \text{ if } x, x+h \in B_r(w).$$

$$(2) \left\| R_{\frac{1}{2}}^{B_r(w)}(f) \right\|_6 \leq c' r^{-\frac{1}{2}} \|f\|_{\frac{3}{2}}, f \in S(\mathbb{R}^4).$$

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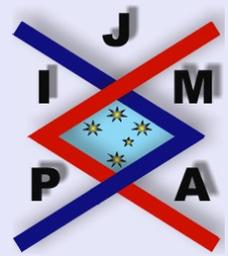


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*Proof.* Let  $F(h) = \det(\varphi'(x+h) - \varphi'(x))$  and let  $d_x^j F$  denotes the  $j$ -th differential of  $F$  at  $x$ . Applying the Taylor formula to  $F(h)$  around  $h = 0$  and taking into account that  $F(0) = 0$ ,  $d_0 F(h) = 0$  and that  $d_0^2 F(h, h) \equiv 2 \det(\varphi''(x)h)$  we obtain

$$\det(\varphi'(x+h) - \varphi'(x)) = \det(\varphi''(x)h) + \int_0^1 \frac{(1-t)^2}{2} d_{th}^3 F(h, h, h) dt.$$

Let  $H(x) = \det(\varphi''(x)h)$ . The above equation gives

$$\begin{aligned} \det(\varphi'(x+h) - \varphi'(x)) &= \det(\varphi''(w)h) + \int_0^1 d_{w+t(x-w)} H(h) dt \\ &+ \int_0^1 \frac{(1-t)^2}{2} d_{th}^3 F(h, h, h) dt. \end{aligned}$$

Then, for  $x, x+h \in B_r(w)$  we have

$$|\det(\varphi'(x+h) - \varphi'(x)) - \det(\varphi''(w)h)| \leq M|h|^3 \leq 2Mr|h|^2$$

with  $M$  depending only  $\|\varphi_1\|_{C^3(B)}$  and  $\|\varphi_2\|_{C^3(B)}$ . If we choose  $c \leq \frac{1}{4M}$ , we get, for  $0 < r < c \inf_{|h|=1} |\det(\varphi''(w)h)|$  that

$$|\det(\varphi'(x+h) - \varphi'(x))| \geq \frac{1}{2} |\det(\varphi''(w)h)|$$

and that

$$J_{B_r(w)}(h) \geq \frac{1}{2} |\det(\varphi''(w)h)| \geq \frac{1}{2c} r |h|^2$$

Thus  $\left\| R_{\frac{1}{2}}^{B_r(w)}(f) \right\|_6 \leq c'r^{-\frac{1}{2}} \|I_2(f)\|_6 \leq c'r^{-\frac{1}{2}} \|f\|_{\frac{3}{2}}$ , where  $I_\alpha$  denotes the Riesz potential on  $\mathbb{R}^4$ , defined as in [10, p. 117]. So the lemma follows from the Hardy–Littlewood–Sobolev theorem of fractional integration as stated e.g. in [10, p. 119].  $\square$

**Lemma 2.3.** *Let  $w$  be an elliptic point for  $\varphi$ . Then there exists a positive constant  $c$  depending only on  $\|\varphi_1\|_{C^3(B)}$  and  $\|\varphi_2\|_{C^3(B)}$  such that if  $0 < r \leq c \inf_{|h|=1} |\det(\varphi''(w)h)|$  then for all  $h \neq 0$  the map  $x \rightarrow \varphi(x+h) - \varphi(x)$  is injective on the domain  $\{x \in B : x, x+h \in B_r(w)\}$ .*

*Proof.* Suppose that  $x, y, x+h$  and  $y+h$  belong to  $B_r(w)$  and that

$$\varphi(x+h) - \varphi(x) = \varphi(y+h) - \varphi(y).$$

From this equation we get

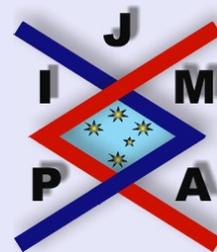
$$0 = \int_0^1 (\varphi'(x+th) - \varphi'(y+th)) h dt = \int_0^1 \int_0^1 d_{x+th+s(y-x)}^2 \varphi(y-x, h) ds dt.$$

Now, for  $z \in B_r(w)$ ,

$$\begin{aligned} |(d_z^2 \varphi - d_w^2 \varphi)(y-x, h)| &= \left| \int_0^1 d_{z+u(w-z)}^3 \varphi(w-z, y-x, h) du \right| \\ &\leq Mr |y-x| |h| \end{aligned}$$

then

$$\begin{aligned} 0 &= \int_0^1 \int_0^1 d_{x+th+s(y-x)}^2 \varphi(y-x, h) ds dt \\ &= d_w^2 \varphi(y-x, h) + \int_0^1 \int_0^1 [d_{x+th+s(y-x)}^2 \varphi - d_w^2 \varphi](y-x, h) ds dt. \end{aligned}$$



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So  $|d_w^2 \varphi(y-x, h)| \leq Mr|y-x||h|$  with  $M$  depending only on  $\|\varphi_1\|_{C^3(B)}$  and  $\|\varphi_2\|_{C^3(B)}$ .

On the other hand,  $w$  is an elliptic point for  $\varphi$  and so, for  $|u| = 1$ , the matrix  $A := \varphi''(w)u$  is invertible. Also  $A^{-1} = (\det A)^{-1} Ad(A)$ , then

$$|A^{-1}x| = |\det A|^{-1} |Ad(A)x| \leq \frac{\widetilde{M}}{|\det A|} |x|,$$

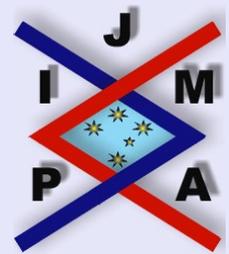
where  $\widetilde{M}$  depends only on  $\|\varphi_1\|_{C^2(B)}$  and  $\|\varphi_2\|_{C^2(B)}$ . Then, for  $|v| = 1$  and  $x = Av$ , we have  $|Av| \geq |\det A| / \widetilde{M}$ . Thus

$$\begin{aligned} |d_w^2 \varphi(y-x, h)| &\geq |y-x||h| \inf_{|u|=1, |v|=1} |d_w^2 \varphi(u, v)| \\ &= |y-x||h| \inf_{|u|=1, |v|=1} |\langle \varphi''(w)u, v \rangle| \\ &\geq \frac{1}{\widetilde{M}} |y-x||h| \inf_{|u|=1} |\det \varphi''(w)u|. \end{aligned}$$

If we choose  $r < \frac{1}{MM} \inf_{|u|=1} |\det \varphi''(w)u|$  the above inequality implies  $x = y$  and the lemma is proved.  $\square$

For any measurable set  $A \subset B$ , let  $\mu_A$  be the Borel measure defined by  $\mu_A(E) = \int_A \chi_E(x, \varphi(x)) dx$  and let  $T_{\mu_A}$  be the convolution operator given by  $T_{\mu_A} f = \mu_A * f$ .

**Proposition 2.4.** *Let  $w$  be an elliptic point for  $\varphi$ . Then there exist positive constants  $c$  and  $c'$  depending only on  $\|\varphi_1\|_{C^3(B)}$  and  $\|\varphi_2\|_{C^3(B)}$  such that if*



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$0 < r < c \inf_{|h|=1} |\det \varphi''(w) h|$  then

$$\left\| T_{\mu_{B_r(w)}} f \right\|_3 \leq c' r^{-\frac{1}{3}} \|f\|_{\frac{3}{2}}.$$

*Proof.* Taking account of Lemma 2.3, we can proceed as in Theorem 0 in [3] to obtain, as there, that

$$\left\| \mu_{B_r(w)} * f \right\|_3^3 \leq (A_1 A_2 A_3)^{\frac{1}{3}},$$

where

$$A_j = \int_{\mathbb{R}^2} F_j(x) \prod_{1 \leq m \leq 3, m \neq j} R_{\frac{1}{2}}^{B_r(w)} F_m(x) dx$$

and  $F_j(x) = \|f(x, \cdot)\|_{\frac{3}{2}}$

Then the proposition follows from Lemma 2.2 and an application of the triple Hölder inequality.  $\square$

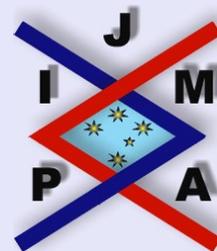
For  $0 < a < 1$  and  $j \in \mathbb{N}$  let

$$U_{a,j} = \{(x_1, x_2) \in B : |x_1| \geq a, 2^{-j} |x_1| \leq |x_2| \leq 2^{-j+1} |x_1|\}$$

and let  $U_{a,j,i}, i = 1, 2, 3, 4$  the connected components of  $U_{a,j}$ .

We have

**Lemma 2.5.** *Let  $0 < a < 1$ . Suppose that there exist  $\beta \in \mathbb{N}, j_0 \in \mathbb{N}$  and a positive constant  $c$  such that  $|\det(\varphi_1''(x) h, \varphi_2''(y) h)| \geq c 2^{-j\beta} |h|^2$  for all  $h \in \mathbb{R}^2, x, y \in U_{a,j,i}, j \geq j_0$  and  $i = 1, 2, 3, 4$ . Thus*



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(1) For all  $j \geq j_0, i = 1, 2, 3, 4$  if  $x$  and  $x + h$  belong to  $U_{a,j,i}$  then

$$|\det(\varphi'(x+h) - \varphi'(x))| \geq c2^{-j\beta} |h|^2.$$

(2) There exists a positive constant  $c'$  such that for all  $j \geq j_0, i = 1, 2, 3, 4$

$$\left\| R_{\frac{1}{2}}^{U_{a,j,i}}(f) \right\|_6 \leq c' 2^{\frac{j\beta}{2}} \|f\|_{\frac{3}{2}}.$$

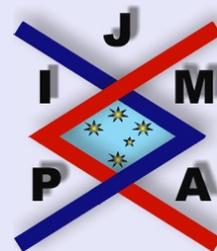
*Proof.* We fix  $i$  and  $j \geq j_0$ . For  $x \in U_{a,j,i}$  we have

$$\det(\varphi'(x+h) - \varphi'(x)) = \det\left(\int_0^1 \varphi''(x+sh) h ds\right).$$

For each  $h \in \mathbb{R}^2 \setminus \{0\}$  we have either  $\det(\varphi_1''(x)h, \varphi_2''(y)h) > c2^{-j\beta} |h|^2$  for all  $x, y \in U_{a,j,i}$  or  $\det(\varphi_1''(x)h, \varphi_2''(y)h) < -c2^{-j\beta} |h|^2$  for all  $x, y \in U_{a,j,i}$ .

We consider the first case. Let  $F(t) = \det\left(\int_0^t \varphi''(x+sh) h ds\right)$ . Then

$$\begin{aligned} F'(t) &= \det\left(\int_0^t \varphi_1''(x+sh) h ds, \varphi_2''(x+th)h\right) \\ &\quad + \det\left(\varphi_1''(x+th)h, \int_0^t \varphi_2''(x+sh) h ds\right) \\ &= \int_0^t \det(\varphi_1''(x+sh)h, \varphi_2''(x+th)h) ds \\ &\quad + \int_0^t \det(\varphi_1''(x+th)h, \varphi_2''(x+sh)h) ds \geq c2^{-j\beta} |h|^2 t. \end{aligned}$$



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Since  $F(0) = 0$  we get  $F(1) = \int_0^1 F'(t) dt \geq c2^{-j\beta} |h|^2$ . Thus

$$\det(\varphi'(x+h) - \varphi'(x)) = F(1) \geq c2^{-j\beta} |h|^2.$$

Then  $J_{U_{a,j,i}}(h) \geq c2^{-j\beta} |h|^2$ , and the lemma follows, as in Lemma 2.2, from the Hardy–Littlewood–Sobolev theorem of fractional integration. The other case is similar.  $\square$

For fixed  $x^{(1)}, x^{(2)} \in \mathbb{R}^2$ , let

$$B_{a,j,i}^{x^{(1)},x^{(2)}} = \{x \in \mathbb{R}^2 : x - x^{(1)} \in U_{a,j,i} \text{ and } x - x^{(2)} \in U_{a,j,i}\}, i = 1, 2, 3, 4.$$

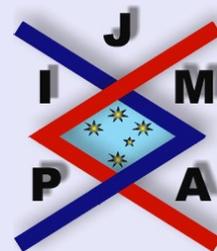
We have

**Lemma 2.6.** *Let  $0 < a < 1$  and let  $x^{(1)}, x^{(2)} \in \mathbb{R}^2$ . Suppose that there exist  $\beta \in \mathbb{N}$ ,  $j_0 \in \mathbb{N}$  and a positive constant  $c$  such that  $|\det(\varphi_1''(x)h, \varphi_2''(y)h)| \geq c2^{-j\beta} |h|^2$  for all  $h \in \mathbb{R}^2$ ,  $x, y \in U_{a,j,i}$ ,  $j \geq j_0$  and  $i = 1, 2, 3, 4$ . Then there exists  $j_1 \in \mathbb{N}$  independent of  $x^{(1)}, x^{(2)}$  such that for all  $j \geq j_1$ ,  $i = 1, 2, 3, 4$  and all nonnegative  $f \in S(\mathbb{R}^4)$  it holds that*

$$\int_{B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2} f(y - \varphi(x - x^{(1)}), y - \varphi(x - x^{(2)})) dx dy \leq \frac{m^2}{J_{U_{a,j,i}}(x^{(2)} - x^{(1)})} \int_{\mathbb{R}^4} f.$$

*Proof.* We assert that, if  $j \geq j_0$  then for each  $(z, w) \in \mathbb{R}^2 \times \mathbb{R}^2$  and  $i = 1, 2, 3, 4$ , the set

$$\left\{ (x, y) \in B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2 : z = y - \varphi(x - x^{(1)}) \text{ and } w = y - \varphi(x - x^{(2)}) \right\}$$



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is a finite set with at most  $m^2$  elements. Indeed, if  $z = y - \varphi(x - x^{(1)})$  and  $w = y - \varphi(x - x^{(2)})$  with  $x \in B_{a,j,i}^{x^{(1)},x^{(2)}}$ , Lemma 2.5 says that, for  $j$  large enough,

$$|\det(\varphi'(x - x^{(1)}) - \varphi'(x - x^{(2)}))| \geq c2^{-j\beta} |h|^2.$$

Thus the Bezout's Theorem (See e.g.[1, Lemma 11.5.1, p. 281]) implies that for each  $(z, w) \in \mathbb{R}^2 \times \mathbb{R}^2$  the set

$$\{x \in B_{a,j,i}^{x^{(1)},x^{(2)}} : \varphi(x - x^{(2)}) - \varphi(x - x^{(1)}) = z - w\}$$

is a finite set with at most  $m^2$  points. Since  $x$  determines  $y$ , the assertion follows.

For a fixed  $\eta > 0$  and for  $k = (k_1, \dots, k_4) \in Z^4$ , let

$$Q_k = \prod_{1 \leq n \leq 4} [k_n \eta, (1 + k_n) \eta].$$

Let  $\Phi_{k,j,i} : (B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2) \cap Q_k \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$  be the function defined by

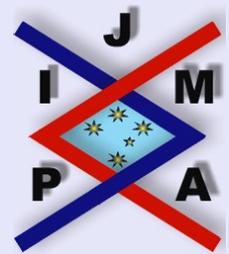
$$\Phi_{k,j,i}(x, y) = (y - \varphi(x - x^{(1)}), y - \varphi(x - x^{(2)}))$$

and let  $W_{k,j,i}$  its image. Also

$$\det(\Phi'_{k,j,i})(x, y) = \det(\varphi'(x - x^{(2)}) - \varphi'(x - x^{(1)})).$$

Thus

$$(2.4) \quad |\det(\Phi'_{k,j,i})(x, y)| \geq J_{U_{a,j,i}}(x^{(2)} - x^{(1)})$$



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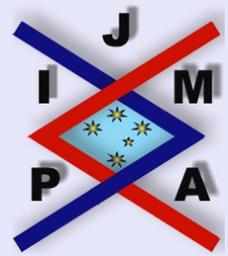


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for  $(x, y) \in \left( B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2 \right) \cap Q_k$ .

Since  $\Phi_{k,j,i}(x, y) = \Phi_{k,j,i}(\bar{x}, \bar{y})$  implies that  $\varphi(x - x^{(1)}) - \varphi(\bar{x} - x^{(1)}) = \varphi(x - x^{(2)}) - \varphi(\bar{x} - x^{(2)})$ , taking into account Lemma 2.1, from Lemma 2.3 it follows the existence of  $\tilde{j} \in N$  with  $\tilde{j}$  independent of  $x^{(1)}, x^{(2)}$  such that for  $j \geq \tilde{j}$  there exists  $\tilde{\eta} = \tilde{\eta}(j) > 0$  satisfying that for  $0 < \eta < \tilde{\eta}(j)$  the map  $\Phi_{k,j,i}$  is injective for all  $k \in Z^4$ . Let  $\Psi_{k,j,i} : W_{k,j,i} \rightarrow \left( B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2 \right) \cap Q_k$  its inverse.

Lemma 2.5 says that  $|\det(\Phi'_{k,j,i})| \geq c2^{-j\beta} |h|^2$  on  $\left( B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2 \right) \cap Q_k$ . We have

$$\begin{aligned} & \int_{B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2} f(y - \varphi(x - x^{(1)}), y - \varphi(x - x^{(2)})) dx dy \\ &= \sum_{k \in Z^4} \int_{\left( B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2 \right) \cap Q_k} f(y - \varphi(x - x^{(1)}), y - \varphi(x - x^{(2)})) dx dy \\ &= \sum_{k \in Z^4} \int_{W_{k,j,i}} f(z, w) \frac{1}{|\det(\Phi'_{k,j,i})(\Psi_{k,j,i}(z, w))|} dz dw \\ &\leq \frac{1}{J_{U_{a,j,i}}(x^{(2)} - x^{(1)})} \int_{\mathbb{R}^4} \sum_{k \in Z^4} \chi_{W_{k,j,i}}(v) f(v) dv \\ &\leq \frac{m^2}{J_{U_{a,j,i}}(x^{(2)} - x^{(1)})} \int_{\mathbb{R}^4} f \end{aligned}$$

where we have used (2.4). □

**Proposition 2.7.** Let  $0 < a < 1$ . Suppose that there exist  $\beta \in \mathbb{N}, j_0 \in \mathbb{N}$  and a positive constant  $c$  such that  $|\det(\varphi''_1(x)h, \varphi''_2(y)h)| \geq c2^{-j\beta} |h|^2$  for all

$h \in \mathbb{R}^2$ ,  $x, y \in U_{a,j,i}$ ,  $j \geq j_0$ ,  $i = 1, 2, 3, 4$ . Then, there exist  $j_1 \in \mathbb{N}$ ,  $c' > 0$  such that for all  $j \geq j_1$ ,  $f \in S(\mathbb{R}^4)$

$$\left\| T_{\mu_{U_{a,j}}} f \right\|_3 \leq c' 2^{\frac{j\beta}{3}} \|f\|_{\frac{3}{2}}.$$

*Proof.* For  $i = 1, 2, 3, 4$ , let

$$K_{a,j,i} = \left\{ (x, y, x^{(1)}, x^{(2)}, x^{(3)}) \right. \\ \left. \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 : x - x^{(s)} \in U_{a,j,i}, s = 1, 2, 3 \right\}.$$

We can proceed as in Theorem 0 in [3] to obtain, as there, that

$$\left\| \mu_{U_{a,j,i}} * f \right\|_3^3 = \int_{K_{a,j,i}} \prod_{1 \leq j \leq 3} f(x_j, y - \varphi(x - x_j)) dx dy dx^{(1)} dx^{(2)} dx^{(3)}$$

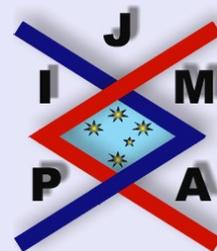
taking into account of Lemma 2.6 and reasoning, with the obvious changes, as in [3], Theorem 0, we obtain that

$$\left\| \mu_{U_{a,j,i}} * f \right\|_3^3 \leq m^2 (A_1 A_2 A_3)^{\frac{1}{3}}$$

with

$$A_j = \int_{\mathbb{R}^2} F_j(x) \prod_{1 \leq m \leq 3, m \neq j} R_{\frac{1}{2}}^{U_{a,j,i}} F_m(x) dx$$

and  $F_j(x) = \|f(x, \cdot)\|_{\frac{3}{2}}$ . Now the proof follows as in Proposition 2.4.  $\square$



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### 3. About the Type Set

**Proposition 3.1.** For  $\delta > 0$  let  $V_\delta$  be defined by (2.1). Suppose that the set of the non elliptic points for  $\varphi$  in  $V_\delta$  are those lying in the  $x_1$  axis and let  $\alpha$  be defined by (2.3). Then  $E_{\mu_{V_\delta}}$  contains the closed trapezoidal region with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha})$ ,  $(\frac{2}{7\alpha}, \frac{1}{7\alpha})$ , except perhaps the closed edge parallel to the principal diagonal.

*Proof.* We first show that  $(1 - \theta)(1, 1) + \theta(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha}) \in E_{\mu_{V_\delta}}$  if  $0 \leq \theta < 1$ .

If  $w = (w_1, w_2) \in U_{\frac{1}{2}, j}$  then  $2^{-j-1} \leq |w_2| \leq 2^{-j+1}$ . Thus, from Lemmas 2.2, 2.3 and Proposition 2.7, follows the existence of  $j_0 \in \mathbb{N}$  and of a positive constant  $c = c(\|\varphi_1\|_{C^3(B)}, \|\varphi_2\|_{C^3(B)})$  such that if  $r_j = c2^{-j\alpha}$ , then

$$\left\| T_{\mu_{B_{r_j}(w)}} f \right\|_3 \leq c' 2^{\frac{j\alpha}{3}} \|f\|_{\frac{3}{2}}$$

for some  $c' > 0$  and all  $j \geq j_0$ ,  $w \in U_{\frac{1}{2}, j}$ ,  $f \in S(\mathbb{R}^4)$ . For  $0 \leq t \leq 1$  let  $p_t, q_t$  be defined by  $(\frac{1}{p_t}, \frac{1}{q_t}) = t(\frac{2}{3}, \frac{1}{3}) + (1-t)(1, 1)$ . We have also  $\left\| T_{\mu_{B_{r_j}(w)}} f \right\|_1 \leq \pi c^2 2^{-2j\alpha} \|f\|_1$ , thus, the Riesz-Thorin theorem gives

$$\left\| T_{\mu_{B_r(w)}} f \right\|_{q_t} \leq c 2^{j(\frac{t\alpha}{3} - (1-t)2\alpha)} \|f\|_{p_t}.$$

Since  $U_{\frac{1}{2}, j}$  can be covered with  $N$  of such balls  $B_r(w)$  with  $N \simeq 2^{j(2\alpha-1)}$  we get that

$$\left\| T_{\mu_{U_{\frac{1}{2}, j}}} \right\|_{p_t, q_t} \leq c 2^{j(\frac{7}{3}\alpha t - 1)}.$$



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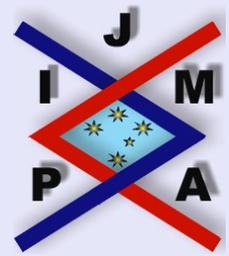


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Let  $U = \cup_{j \geq j_0} U_{\frac{1}{2}, j}$ . We have that  $\|T_{\mu U}\|_{p_t, q_t} \leq \sum_{j \geq j_0} \|T_{\mu U_{\frac{1}{2}, j}}\|_{p_t, q_t} < \infty$ , for  $t < \frac{3}{7\alpha}$ . Since for  $t = \frac{3}{7\alpha}$  we have  $\frac{1}{p_t} = 1 - \frac{1}{7\alpha}$  and  $\frac{1}{q_t} = 1 - \frac{2}{7\alpha}$  and since every point in  $V_\delta \setminus \overset{\circ}{U}$  is an elliptic point (and so, from Theorem 3 in [3],  $\|T_{\mu_{V_\delta \setminus \overset{\circ}{U}}}\|_{\frac{3}{2}, 3} < \infty$ ), we get that  $(1 - \theta)(1, 1) + \theta(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha}) \in E_{\mu_{V_\delta}}$  for  $0 \leq \theta < 1$ . On the other hand, a standard computation shows that the adjoint operator  $T_{\mu_{V_\delta}}^*$  is given by  $T_{\mu_{V_\delta}}^* f = (T_{\mu_{V_\delta}}(f^\vee))^\vee$ , where we write, for  $g : \mathbb{R}^4 \rightarrow C$ ,  $g^\vee(x) = g(-x)$ . Thus  $E_{\mu_{V_\delta}}$  is symmetric with respect to the nonprincipal diagonal. Finally, after an application of the Riesz-Thorin interpolation theorem, the proposition follows.  $\square$

For  $\delta > 0$ , let  $A_\delta = \{(x_1, x_2) \in B : |x_2| \leq \delta |x_1|\}$ .

*Remark 3.1.* For  $s > 0$ ,  $x = (x_1, \dots, x_4) \in \mathbb{R}^4$  we set  $s \bullet x = (sx_1, sx_2, s^m x_3, s^m x_4)$ . If  $E \subset \mathbb{R}^2$ ,  $F \subset \mathbb{R}^4$  we set  $sE = \{sx : x \in E\}$  and  $s \bullet F = \{s \bullet x : x \in F\}$ . For  $f : \mathbb{R}^4 \rightarrow C$ ,  $s > 0$ , let  $f_s$  denotes the function given by  $f_s(x) = f(s \bullet x)$ . A computation shows that

$$(3.1) \quad (T_{\mu_{2^{-j}V_\delta}} f) (2^{-j} \bullet x) = 2^{-2j} (T_{\mu_{V_\delta}} f_{2^{-j}}) (x)$$

for all  $f \in S(\mathbb{R}^4)$ ,  $x \in \mathbb{R}^4$ .

From this it follows easily that

$$\|T_{\mu_{2^{-j}V_\delta}}\|_{p, q} = 2^{-j(\frac{2(m+1)}{q} - \frac{2(m+1)}{p} + 2)} \|T_{\mu_{V_\delta}}\|_{p, q}.$$

This fact implies that

$$(3.2) \quad E_\mu \subset \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) : \frac{1}{q} \geq \frac{1}{p} - \frac{1}{m+1} \right\}$$

and that if  $\frac{1}{q} > \frac{1}{p} - \frac{1}{m+1}$  then  $\left( \frac{1}{p}, \frac{1}{q} \right) \in E_{\mu_{A_\delta}}$  if and only if  $\left( \frac{1}{p}, \frac{1}{q} \right) \in E_{\mu_{V_\delta}}$ .

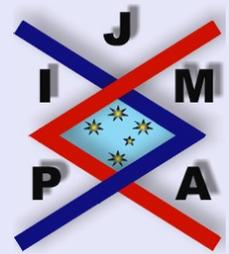
**Theorem 3.2.** *Suppose that for some  $\delta > 0$  the set of the non elliptic points for  $\varphi$  in  $A_\delta$  are those lying on the  $x_1$  axis and let  $\alpha$  be defined by (2.3). Then  $E_{\mu_{A_\delta}}$  contains the intersection of the two closed trapezoidal regions with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$ ,  $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$  and  $(0, 0)$ ,  $(1, 1)$ ,  $\left(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha}\right)$ ,  $\left(\frac{2}{7\alpha}, \frac{1}{7\alpha}\right)$  respectively, except perhaps the closed edge parallel to the diagonal.*

*Moreover, if  $7\alpha \leq m+1$  then the interior of  $E_{\mu_{A_\delta}}$  is the open trapezoidal region with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$  and  $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$ .*

*Proof.* Taking into account Proposition 3.1, the theorem follows from the facts of Remark 3.1.  $\square$

For  $0 < a < 1$  and  $\delta > 0$  we set  $V_{a,\delta} = \{(x_1, x_2) \in B : a \leq |x_1| \leq 1 \text{ and } |x_2| \leq \delta |x_1|\}$ . We have

**Proposition 3.3.** *Let  $0 < a < 1$ . Suppose that for some  $0 < a < 1$ ,  $j_0, \beta \in \mathbb{N}$  and some positive constant  $c$  we have  $|\det(\varphi_1''(x)h, \varphi_1''(y)h)| \geq c2^{-j\beta} |h|^2$  for all  $h \in \mathbb{R}^2$ ,  $x, y \in U_{a,j,i}$ ,  $j \geq j_0$  and  $i = 1, 2, 3, 4$ . Then, for  $\delta$  positive and small enough,  $E_{\mu_{V_{a,\delta}}}$  contains the closed trapezoidal region with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $\left(\frac{\beta+2}{\beta+3}, \frac{\beta+1}{\beta+3}\right)$ ,  $\left(\frac{2}{\beta+3}, \frac{1}{\beta+3}\right)$ , except perhaps the closed edge parallel to the principal diagonal.*




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*Proof.* Proposition 2.7 says that there exist  $j_1 \in N$  and a positive constant  $c$  such that for  $j \geq j_1$  and  $f \in S(\mathbb{R}^4)$

$$\left\| T_{\mu_{U_{a,j,i}}} f \right\|_3 \leq c 2^{\frac{j\beta}{3}} \|f\|_{\frac{3}{2}}.$$

Also, for some  $c > 0$  and all  $f \in S(\mathbb{R}^4)$  we have  $\left\| T_{\mu_{U_{a,j,i}}} f \right\|_1 \leq c 2^{-j} \|f\|_1$ .

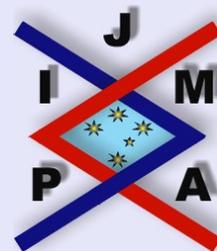
Then  $\left\| T_{\mu_{U_{a,j,i}}} f \right\|_{q_t} \leq c 2^{j(t\frac{\beta}{3} - (1-t))} \|f\|_{p_t}$  where  $p_t, q_t$  are defined as in the proof of Proposition 3.1. Let  $U = \cup_{j \geq j_1} U_{a,j}$ . Then  $\|T_{\mu_U} f\|_{p_t, q_t} < \infty$  if  $t < \frac{3}{\beta+3}$ . Now, the proof follows as in Proposition 3.1.  $\square$

**Theorem 3.4.** *Suppose that for some  $0 < a < 1$ ,  $j_0, \beta \in N$  and for some positive constant  $c$  we have  $|\det(\varphi_1''(x)h, \varphi_1''(y)h)| \geq c 2^{-j\beta} |h|^2$  for all  $x, y \in U_{a,j,i}$ ,  $j \geq j_0$  and  $i = 1, 2, 3, 4$ . Then for  $\delta$  positive and small enough,  $E_{\mu_{A_\delta}}$  contains the intersection of the two closed trapezoidal regions with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{m}{m+1}, \frac{m-1}{m+1})$ ,  $(\frac{2}{m+1}, \frac{1}{m+1})$  and  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{\beta+2}{\beta+3}, \frac{\beta+1}{\beta+3})$ ,  $(\frac{2}{\beta+3}, \frac{1}{\beta+3})$ , respectively, except perhaps the closed edge parallel to the diagonal.*

*Moreover, if  $\beta \leq m - 2$  then the interior of  $E_\mu$  is the open trapezoidal region with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{m}{m+1}, \frac{m-1}{m+1})$  and  $(\frac{2}{m+1}, \frac{1}{m+1})$ .*

*Proof.* Follows as in Theorem 3.2 using now Proposition 3.3 instead of Proposition 3.1.  $\square$

**Remark 3.2.** We now turn out to the case when  $\varphi$  is a homogeneous polynomial function whose set of non elliptic points is a finite union of lines through the origin,  $L_1, \dots, L_k$ .



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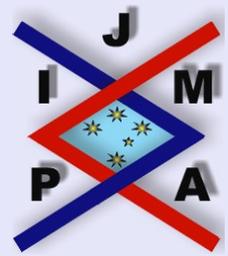


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For each  $l$ ,  $1 \leq l \leq k$ , let  $A_\delta^l = \{x \in \mathbb{R}^2 : |\pi_{L_l}^\perp x| \leq \delta |\pi_{L_l} x|\}$  where  $\pi_{L_l}$  and  $\pi_{L_l}^\perp$  denote the orthogonal projections from  $\mathbb{R}^2$  into  $L_l$  and  $L_l^\perp$  respectively. Thus each  $A_\delta^l$  is a closed conical sector around  $L_l$ . We choose  $\delta$  small enough such that  $A_\delta^l \cap A_\delta^i = \emptyset$  for  $l \neq i$ .

It is easy to see that there exists (a unique)  $\alpha_l \in N$  and positive constants  $c_l'$ ,  $c_l''$  such that

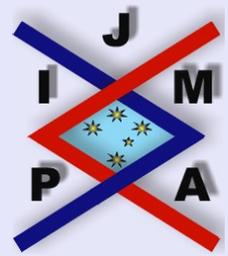
$$(3.3) \quad c_l' |\pi_{L_l}^\perp w|^{\alpha_l} \leq \inf_{|h|=1} |\det(\varphi''(w)h)| \leq c_l'' |\pi_{L_l}^\perp x|^{\alpha_l}$$

for all  $w \in A_\delta^l$ . Indeed, after a rotation the situation reduces to that considered in Remark 3.1.

**Theorem 3.5.** *Suppose that the set of non elliptic points is a finite union of lines through the origin,  $L_1, \dots, L_k$ . For  $l = 1, 2, \dots, k$ , let  $\alpha_l$  be defined by (3.3), and let  $\alpha = \max_{1 \leq l \leq k} \alpha_l$ . Then  $E_\mu$  contains the intersection of the two closed trapezoidal regions with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{m}{m+1}, \frac{m-1}{m+1})$ ,  $(\frac{2}{m+1}, \frac{1}{m+1})$  and  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha})$ ,  $(\frac{2}{7\alpha}, \frac{1}{7\alpha})$ , respectively, except perhaps the closed edge parallel to the diagonal.*

Moreover, if  $7\alpha \leq m+1$  then the interior of  $E_\mu$  is the interior of the trapezoidal regions with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{m}{m+1}, \frac{m-1}{m+1})$ ,  $(\frac{2}{m+1}, \frac{1}{m+1})$ .

*Proof.* For  $l = 1, 2, \dots, k$ , let  $A_\delta^l$  be as above. From Theorem 3.2, we obtain that  $E_{\mu, A_\delta^l}$  contains the intersection of the two closed trapezoidal regions with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{m}{m+1}, \frac{m-1}{m+1})$ ,  $(\frac{2}{m+1}, \frac{1}{m+1})$  and  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{7\alpha_l-1}{7\alpha_l}, \frac{7\alpha_l-2}{7\alpha_l})$ ,  $(\frac{2}{7\alpha_l}, \frac{1}{7\alpha_l})$  respectively, except perhaps the closed edge parallel to the diagonal.



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Since every  $x \in B \setminus \cup_l A_\delta^l$  is an elliptic point for  $\varphi$ , Theorem 0 in [3] and a compactness argument give that  $\|T_{\mu_D}\|_{\frac{3}{2},3} < \infty$  where  $D = \{x \in B \setminus \cup_l A_\delta^l : \frac{1}{2} \leq |x|\}$ . Then (using the symmetry of  $E_{\mu_D}$ , the fact of that  $\mu_D$  is a finite measure and the Riesz-Thorin theorem)  $E_{\mu_D}$  is the closed triangle with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{2}{3}, \frac{1}{3})$ . Now, proceeding as in the proof of Theorem 3.2 we get that  $\|T_{\mu_{B \setminus \cup_l A_\delta^l}}\|_{p,q} < \infty$  if  $\frac{1}{q} > \frac{1}{p} - \frac{1}{m+1}$ . Then the first assertion of the theorem is true. The second one follows also using the facts of Remark 3.1.  $\square$

For  $0 < a < 1$ , we set

$$U_{a,j}^l = \{x \in \mathbb{R}^2 : a \leq |\pi_{L^l}(x)| \leq 1$$

$$\text{and } 2^{-j} |\pi_{L^l}(x)| \leq |\pi_{L^l}^{\perp}(x)| \leq 2^{-j+1} |\pi_{L^l}(x)|\}$$

let  $U_{a,j,i}^l$ ,  $i = 1, 2, 3, 4$  be the connected components of  $U_{a,j}^l$ .

**Theorem 3.6.** *Suppose that the set of non elliptic points for  $\varphi$  is a finite union of lines through the origin,  $L_1, \dots, L_k$ . Let  $0 < a < 1$  and let  $j_0 \in \mathbb{N}$  such that*

*For  $l = 1, 2, \dots, k$ , there exists  $\beta_l \in \mathbb{N}$  satisfying  $|\det(\varphi_1''(x)h, \varphi_1''(y)h)| \geq c2^{-j\beta_j} |h|^2$  for all  $x, y \in U_{a,j,i}^l$ ,  $j \geq j_0$  and  $i = 1, 2, 3, 4$ . Let  $\beta = \max_{1 \leq j \leq k} \beta_j$ . Then  $E_\mu$  contains the intersection of the two closed trapezoidal regions with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{m}{m+1}, \frac{m-1}{m+1})$ ,  $(\frac{2}{m+1}, \frac{1}{m+1})$  and  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{\beta+2}{\beta+3}, \frac{\beta+1}{\beta+3})$ ,  $(\frac{2}{\beta+3}, \frac{1}{\beta+3})$ , respectively, except perhaps the closed edge parallel to the diagonal.*

*Moreover, if  $\beta \leq m - 2$  then the interior of  $E_\mu$  is the interior of the trapezoidal region with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{m}{m+1}, \frac{m-1}{m+1})$ ,  $(\frac{2}{m+1}, \frac{1}{m+1})$ .*

*Proof.* Follows as in Theorem 3.5, using now Theorem 3.4 instead of Theorem 3.2.  $\square$

*Example 3.1.*  $\varphi(x_1, x_2) = (x_1^2x_2 - x_1x_2^2, x_1^2x_2 + x_1x_2^2)$

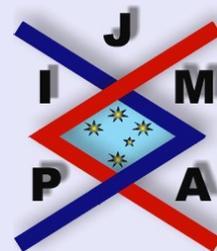
It is easy to check that the set of non elliptic points is the union of the coordinate axes. Indeed, for  $h = (h_1, h_2)$  we have  $\det \varphi''(x_1, x_2) h = 8x_2^2h_1^2 + 8x_1x_2h_1h_2 + 8x_1^2h_2^2$  and this quadratic form in  $(h_1, h_2)$  has non trivial zeros only if  $x_1 = 0$  or  $x_2 = 0$ . The associated symmetric matrix to the quadratic form is

$$\begin{bmatrix} 8x_2^2 & 4x_1x_2 \\ 4x_1x_2 & 8x_1^2 \end{bmatrix}$$

and for  $x_1 \neq 0$  and  $|x_2| \leq \delta|x_1|$  with  $\delta$  small enough, its eigenvalue of lower absolute value is  $\lambda_1(x_1, x_2) = 4x_1^2 + 4x_2^2 - 4\sqrt{(x_2^4 - x_1^2x_2^2 + x_1^4)}$ . Thus  $\lambda_1(x_1, x_2) \simeq 6x_2^2$  for such  $(x_1, x_2)$ . Similarly, for  $x_2 \neq 0$  and  $|x_1| \leq \delta|x_2|$  with  $\delta$  small enough, the eigenvalue of lower absolute value is comparable with  $6x_1^2$ . Then, in the notation of Theorem 3.5, we obtain  $\alpha = 2$  and so  $E_\mu$  contains the closed trapezoidal region with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{13}{14}, \frac{6}{7})$  and  $(\frac{1}{7}, \frac{1}{14})$  except perhaps the closed edge parallel to the principal diagonal. Observe that, in this case, Theorem 3.6 does not apply. In fact, for  $x = (x_1, x_2)$ ,  $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$  and  $h = (h_1, h_2)$  we have

$$\begin{aligned} \det(\varphi_1''(x)h, \varphi_2''(\tilde{x})h) &= 4h_1^2(x_2\tilde{x}_1 - \tilde{x}_2x_1 + 2x_2\tilde{x}_2) \\ &\quad + 4h_1h_2(x_1\tilde{x}_2 + \tilde{x}_1x_2) + 4h_2^2(x_1\tilde{x}_2 - x_2\tilde{x}_1 + 2x_2\tilde{x}_1). \end{aligned}$$

Take  $x_1 = \tilde{x}_1 = 1$  and let  $A = A(x_2, \tilde{x}_2)$  the matrix of the above quadratic form in  $(h_1, h_2)$ . For  $x_2 = 2^{-j}$ ,  $\tilde{x}_2 = 2^{-j+1}$  we have  $\det A < 0$  for  $j$  large




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enough but if we take  $x_2 = 2^{-j+1}$  and  $\tilde{x}_2 = 2^{-j}$ , we get  $\det A > 0$  for  $j$  large enough, so, for all  $j$  large enough,  $\det A = 0$  for some  $2^{-j} \leq x_2, \tilde{x}_2 \leq 2^{-j+1}$ . Thus, for such  $x_2, \tilde{x}_2$ ,

$$\inf_{|(h_1, h_2)|=1} \det (\varphi_1''(1, x_2)(h_1, h_2), \varphi_2''(1, \tilde{x}_2)(h_1, h_2)) = 0.$$

*Example 3.2.* Let us show an example where Theorem 3.6 characterizes  $\overset{\circ}{E}_\mu$ . Let

$$\varphi(x_1, x_2) = (x_1^3 x_2 - 3x_1 x_2^3, 3x_1^2 x_2^2 - x_2^4).$$

In this case the set of non elliptic points for  $\varphi$  is the  $x_1$  axis. Indeed,

$$\det(\varphi''(x_1, x_2)(h_1, h_2)) = 18(x_1^2 + x_2^2)((h_2 x_1 + x_2 h_1)^2 + 2x_2^2 h_1^2 + 6h_2^2 x_2^2).$$

In order to apply Theorem 3.6, we consider the quadratic form in  $h = (h_1, h_2)$

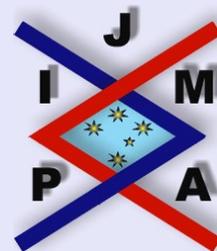
$$\det(\varphi_1''(x_1, x_2)h, \varphi_2''(\tilde{x}_1, \tilde{x}_2)h).$$

If  $x = (x_1, x_2)$  and  $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ , let  $A = A(x, \tilde{x})$  its associated symmetric matrix. An explicit computation of  $A$  shows that, for a given  $0 < a < 1$  and for all  $j$  large enough and  $i = 1, 2, 3, 4$ , if  $x$  and  $\tilde{x}$  belong to  $U_{a,j,i}$ , then

$$a^2 \leq \text{tr}(A) \leq 20$$

thus, if  $\lambda_1(x, \tilde{x})$  denotes the eigenvalue of lower absolute value of  $A(x, \tilde{x})$ , we have, for  $x, \tilde{x} \in W_a$  that

$$c_1 |\det A| \leq |\lambda_1(x, \tilde{x})| \leq c_2 |\det A|$$




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where  $c_1, c_2$  are positive constants independent of  $j$ . Now, a computation gives

$$\det A = 324 \left( -x_1^2 \tilde{x}_1^2 - 9x_2^2 \tilde{x}_2^2 - 12x_1 x_2 \tilde{x}_1 \tilde{x}_2 + 2x_1^2 \tilde{x}_2^2 \right) \\ \times \left( x_2^2 \tilde{x}_1^2 - 2x_2^2 \tilde{x}_2^2 - 4x_1 x_2 \tilde{x}_1 \tilde{x}_2 + x_1^2 \tilde{x}_2^2 \right).$$

Now we write  $\tilde{x}_2 = tx_2$ , with  $\frac{1}{2} \leq t \leq 2$ . Then

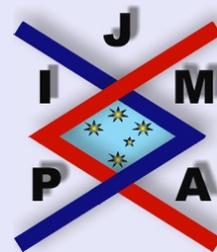
$$\det A = 324x_2^2 \left[ -x_1^2 \tilde{x}_1^2 - 9t^2 x_2^4 - 12tx_1 x_2^2 \tilde{x}_1 + 2t^2 x_2^2 x_1^2 \right] \\ \times \left[ \tilde{x}_1^2 - 2t^2 x_2^2 - 4tx_1 \tilde{x}_1 + t^2 x_1^2 \right].$$

Note that the the first bracket is negative for  $x, \tilde{x} \in W_a$  if  $j$  is large enough. To study the sign of the second one, we consider the function  $F(t, x_1, \tilde{x}_1) = \tilde{x}_1^2 - 4tx_1 \tilde{x}_1 + t^2 x_1^2$ . Since  $F$  has a negative maximum on  $\{1\} \times \{1\} \times [\frac{1}{2}, 2]$ , it follows easily that we can choose  $a$  such that for  $x, \tilde{x} \in W_a$  and  $j$  large enough, the same assertion holds for the second bracket. So  $\det A$  is comparable with  $2^{-2j}$ , thus the hypothesis of the Theorem 3.6 are satisfied with  $\beta = 2$  and such  $a$ . Moreover, we have  $\beta = m - 2$ , then we conclude that the interior of  $E_\mu$  is the open trapezoidal region with vertices  $(0, 0), (1, 1), (\frac{3}{5}, \frac{4}{5}), (\frac{2}{5}, \frac{1}{5})$ .

On the other hand, in a similar way than in Example 3.1 we can see that  $\alpha = 2$  (in fact  $\det A(x, x) = 648(x_1^2 + 9x_2^2)(x_1^2 + x_2^2)^2 x_2^2$ ), so in this case Theorem 3.6 gives a better result (a precise description of  $\overset{\circ}{E}_\mu$ ) than that given by Theorem 3.5, that asserts only that  $\overset{\circ}{E}_\mu$  contains the trapezoidal region with vertices  $(0, 0), (1, 1), (\frac{13}{14}, \frac{6}{7})$  and  $(\frac{1}{7}, \frac{1}{14})$ .

*Example 3.3.* The following is an example where Theorem 3.5 characterizes  $\overset{\circ}{E}_\mu$ . Let

$$\varphi(x_1, x_2) = (x_2 \operatorname{Re}(x_1 + ix_2))^{12}, x_2 \operatorname{Im}(x_1 + ix_2)^{12}.$$




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A computation gives that for  $x = (x_1, x_2)$  and  $h = (h_1, h_2)$

$$\det(\varphi''(x)h) = 288(x_1^2 + x_2^2)^{10} (66x_2^2h_1^2 + 11x_1x_2h_1h_2 + (x_1^2 + 78x_2^2)h_2^2)$$

and this quadratic form in  $(h_1, h_2)$  does not vanish for  $h \neq 0$  unless  $x_2 = 0$ . So the set of non elliptic points for  $\varphi$  is the  $x_1$  axis. Moreover, its associate symmetric matrix

$$A = A(x) = 288(x_1^2 + x_2^2)^{10} \begin{bmatrix} 66x_2^2 & \frac{11}{2}x_1x_2 \\ \frac{11}{2}x_1x_2 & x_1^2 + 78x_2^2 \end{bmatrix}$$

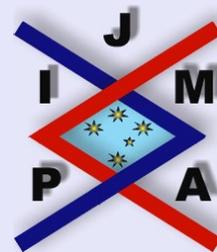
satisfies  $c_1 \leq \text{tr}A(x) \leq c_2$  for  $x \in B$ ,  $\frac{1}{2} \leq |x_1|$ , and  $|x_2| \leq \delta|x_1|$ ,  $\delta > 0$  small enough.

Thus if  $\lambda_1 = \lambda_1(x)$  denotes the eigenvalue of lower absolute value of  $A(x)$ , we have, for  $x$  in this region, that

$$k_1 |\det A| \leq |\lambda_1| \leq k_2 |\det A|,$$

where  $k_1$  and  $k_2$  are positive constants.

Since  $\det A(1, x_2) = (288)^2(1 + x_2^2)^{20} \left(\frac{143}{4}x_2^2 + 5148x_2^4\right)$ , we have that  $\alpha = 2$ . So  $7\alpha = m + 1$  and, from Theorem 3.5, we conclude that the interior of  $E_\mu$  is the open trapezoidal region with vertices  $(0, 0)$ ,  $(1, 1)$ ,  $(\frac{13}{14}, \frac{6}{7})$ ,  $(\frac{1}{7}, \frac{1}{14})$ .



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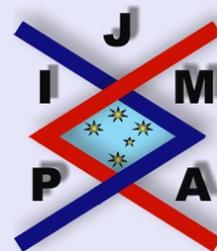
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## References

- [1] J. BOCHNAK, M. COSTE AND M. F. ROY, *Real Algebraic Geometry*, Springer, 1998.
- [2] M. CHRIST. Endpoint bounds for singular fractional integral operators, *UCLA Preprint*, (1988).
- [3] S. W. DRURY AND K. GUO, Convolution estimates related to surfaces of half the ambient dimension. *Math. Proc. Camb. Phil. Soc.*, **110** (1991), 151–159.
- [4] E. FERREYRA, T. GODOY AND M. URICUOLO. The type set for some measures on  $\mathbb{R}^{2n}$  with  $n$  dimensional support, *Czech. Math. J.*, (to appear).
- [5] A. IOSEVICH AND E. SAWYER, Sharp  $L^p - L^q$  estimates for a class of averaging operators, *Ann Inst. Fourier*, **46**(5) (1996), 359–1384.
- [6] T. KATO, *Perturbation Theory for Linear Operators*, Second edition. Springer Verlag, Berlin Heidelberg- New York, 1976.
- [7] D. OBERLIN, Convolution estimates for some measures on curves, *Proc. Amer. Math. Soc.*, **99**(1) (1987), 56–60.
- [8] F. RICCI, Limitatezza  $L^p - L^q$  per operatori di convoluzione definiti da misure singolari in  $\mathbb{R}^n$ , *Bollettino U.M.I.*, (7) 11-A (1997), 237–252.
- [9] F. RICCI AND E. M. STEIN, Harmonic analysis on nilpotent groups and singular integrals III, fractional integration along manifolds, *J. Funct. Anal.*, **86** (1989), 360–389.



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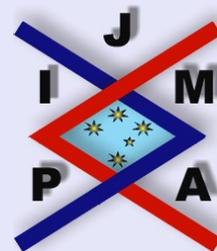
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[10] E. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.



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