Journal of Inequalities in Pure and Applied Mathematics

L^p IMPROVING PROPERTIES FOR MEASURES ON $\mathbb{R}^4$ SUPPORTED ON HOMOGENEOUS SURFACES IN SOME NON ELLIPTIC CASES

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ISSN (electronic): 1443-5756
052-00

volume 2, issue 3, article 37,
2001.

*Received 08 January, 2001;
accepted 05 June, 2001.*

Communicated by: L. Pick

Abstract

In this paper we study convolution operators T_μ with measures μ in $\mathbb{R}^4$ of the form $\mu(E) = \int_B \chi_E(x, \varphi(x))\, dx$, where B is the unit ball of $\mathbb{R}^2$, and φ is a homogeneous polynomial function. If $\inf_{h \in S^1} \left|\det\left(d_x^2 \varphi(h, .)\right)\right|$ vanishes only on a finite union of lines, we prove, under suitable hypothesis, that T_μ is bounded from L^p into L^q if $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to a certain explicitly described trapezoidal region.

2000 Mathematics Subject Classification: 42B20, 42B10.
Key words: Singular measures, L^p-improving, convolution operators.

Partially supported by Agencia Cordoba Ciencia, Secyt-UNC and Conicet

Contents

1. Introduction

It is well known that a complex measure μ on $\mathbb{R}^n$ acts as a convolution operator on the Lebesgue spaces $L^p(\mathbb{R}^n) : \mu * L^p \subset L^p$ for $1 \leq p \leq \infty$. If for some p there exists $q > p$ such that $\mu * L^p \subset L^q$, μ is called L^p- improving. It is known that singular measures supported on smooth submanifolds of $\mathbb{R}^n$ may be L^p- improving. See, for example, [2], [5], [8], [9], [7] and [4].

Let φ_1, φ_2 be two homogeneous polynomial functions on $\mathbb{R}^2$ of degree $m \geq 2$ and let $\varphi = (\varphi_1, \varphi_2)$. Let μ be the Borel measure on $\mathbb{R}^4$ given by

$$\mu(E) = \int_B \chi_E(x, \varphi(x))\, dx, \tag{1.1}$$

where B denotes the closed unit ball around the origin in $\mathbb{R}^2$ and dx is the Lebesgue measure on $\mathbb{R}^2$. Let T_μ be the convolution operator given by $T_\mu f = \mu * f$, $f \in S(\mathbb{R}^4)$ and let E_μ be the type set corresponding to the measure μ defined by

$$E_\mu = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) : \|T_\mu\|_{p,q} < \infty, 1 \leq p, q \leq \infty \right\},$$

where $\|T_\mu\|_{p,q}$ denotes the operator norm of T_μ from $L^p(\mathbb{R}^4)$ into $L^q(\mathbb{R}^4)$ and where the L^p spaces are taken with respect to the Lebesgue measure on $\mathbb{R}^4$.

For $x, h \in \mathbb{R}^2$, let $\varphi''(x)h$ be the 2×2 matrix whose $j - th$ column is $\varphi_j''(x)h$, where $\varphi_j''(x)$ denotes the Hessian matrix of φ_j at x. Following [3, p. 152], we say that $x \in \mathbb{R}^2$ is an elliptic point for φ if $\det(\varphi''(x)h) \neq 0$ for all $h \in \mathbb{R}^2 \setminus \{0\}$. For $A \subset \mathbb{R}^2$, we will say that φ is strongly elliptic on A if $\det(\varphi_1''(x)h, \varphi_2''(y)h) \neq 0$ for all $x, y \in A$ and $h \in \mathbb{R}^2 \setminus \{0\}$.

If every point $x \in B \backslash \{0\}$ is elliptic for φ, it is proved in [4] that for $m \geq 3$, E_μ is the closed trapezoidal region Σ_m with vertices $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$ and $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$.

Our aim in this paper is to study the case where the set of non elliptic points consists of a finite union of lines through the origin, $L_1, \ldots, L_k$. We assume from now on, that for $x \in R^2 - \{0\}$, $\det(\varphi''(x)h)$ does not vanish identically, as a function of h. For each $l = 1, 2, \ldots, k$, let π_{L_l} and $\pi_{L_l^\perp}$ be the orthogonal projections from $\mathbb{R}^2$ onto L_l and $L_l^\perp$ respectively. For $\delta > 0$, $1 \leq l \leq k$, let

$$V_\delta^l = \left\{ x \in B : 1/2 \leq |\pi_{L_l}(x)| \leq 1 \text{ and } \left|\pi_{L_l^\perp}(x)\right| \leq \delta |\pi_{L_l}(x)| \right\}.$$

It is easy to see (see Lemma 2.1 and Remark 3.2) that for δ small enough, there exists $\alpha_l \in \mathbb{N}$ and positive constants c and c' such that

$$c \left|\pi_{L_l^\perp}(x)\right|^{\alpha_l} \leq \inf_{h \in S^1} |\det(\varphi''(x)h)| \leq c' \left|\pi_{L_l^\perp}(x)\right|^{\alpha_l}$$

for all $x \in V_\delta^l$. Following the approach developed in [3], we prove, in Theorem 3.5, that if $\alpha = \max_{1 \leq l \leq k} \alpha_l$ and if $7\alpha \leq m+1$, then the interior of E_μ agrees with the interior of Σ_m.

Moreover in Theorem 3.6 we obtain that $\overset{\circ}{E}_\mu = \overset{\circ}{\Sigma}_m$ still holds in some cases where $7\alpha > m+1$, if we require a suitable hypothesis on the behavior, near the lines $L_1, \ldots, L_k$, of the map $(x,y) \to \inf_{h \in S^1} |\det(\varphi_1''(x)h, \varphi_2''(y)h)|$.

In any case, even though we can not give a complete description of the interior of E_μ, we obtain a polygonal region contained in it.

Throughout the paper c will denote a positive constant not necessarily the same at each occurrence.

2. Preliminaries

Let $\varphi_1, \varphi_2 : \mathbb{R}^2 \to \mathbb{R}$ be two homogeneous polynomials functions of degree $m \geq 2$ and let $\varphi = (\varphi_1, \varphi_2)$. For $\delta > 0$ let

$$V_\delta = \left\{ (x_1, x_2) \in B : \frac{1}{2} \leq |x_1| \leq 1 \text{ and } |x_2| \leq \delta |x_1| \right\}. \tag{2.1}$$

We assume in this section that, for some $\delta_0 > 0$, the set of the non elliptic points for φ in V_{δ_0} is contained in the x_1 axis.

For $x \in \mathbb{R}^2$, let $P = P(x)$ be the symmetric matrix that realizes the quadratic form $h \to \det(\varphi''(x)\, h)$, so

$$\det(\varphi''(x)\, h) = \langle P(x)\, h, h \rangle . \tag{2.2}$$

Lemma 2.1. *There exist $\delta \in (0, \delta_0)$, $\alpha \in \mathbb{N}$ and a real analytic function $g = g(x_1, x_2)$ on V_δ with $g(x_1, 0) \neq 0$ for $x_1 \neq 0$ such that*

$$\inf_{|h|=1} |\det(\varphi''(x)\, h)| = |x_2|^\alpha |g(x)| \tag{2.3}$$

for all $x \in V_\delta$.

Proof. Since $P(x)$ is real analytic on V_δ and $P(x) \neq 0$ for $x \neq 0$, it follows that, for δ small enough, there exists two real analytic functions $\lambda_1(x)$ and $\lambda_2(x)$ wich are the eigenvalues of $P(x)$. Also, $\inf_{|h|=1} |\det(\varphi''(x)\, h)| = \min\{|\lambda_1(x)|, |\lambda_2(x)|\}$ for $x \in V_\delta$. Since we have assumed that $(1, 0)$ is not an elliptic point for φ and that $P(x) \neq 0$ for $x \neq 0$, diminishing δ if necessary, we can assume that $\lambda_1(1, 0) = 0$ and that $|\lambda_1(1, x_2)| \leq |\lambda_2(1, x_2)|$ for $|x_2| \leq \delta$.

Since $P(x)$ is homogeneous in x, we have that $\lambda_1(x)$ and $\lambda_2(x)$ are homogeneous in x with the same homogeneity degree d. Thus $|\lambda_1(x)| \leq |\lambda_2(x)|$ for all $x \in V_\delta$. Now, $\lambda_1(1, x_2) = x_2^\alpha G(x_2)$ for some real analytical function $G = G(x_2)$ with $G(0) \neq 0$ and so $\lambda_1(x_1, x_2) = x_1^d \lambda_1\left(1, \frac{x_2}{x_1}\right) = x_1^{d-\alpha} x_2^\alpha G\left(\frac{x_2}{x_1}\right)$. Taking $g(x_1, x_2) = x_1^{d-\alpha} G\left(\frac{x_2}{x_1}\right)$ the lemma follows. □

Following [3], for $U \subset \mathbb{R}^2$ let $J_U : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ given by

$$J_U(h) = \inf_{x,\, x+h \in U} |\det(\varphi'(x+h) - \varphi'(x))|,$$

where the infimum of the empty set is understood to be ∞. We also set, as there, for $0 < \alpha < 1$

$$R_\alpha^U(f)(x) = \int J_U(x-y)^{-1+\alpha} f(y)\, dy.$$

For $r > 0$ and $w \in \mathbb{R}^2$, let $B_r(w)$ denotes the open ball centered at w with radius r.

We have the following

Lemma 2.2. *Let w be an elliptic point for φ. Then there exist positive constants c and c' depending only on $\|\varphi_1\|_{C^3(B)}$ and $\|\varphi_2\|_{C^3(B)}$ such that if $0 < r \leq c \inf_{|h|=1} |\det(\varphi''(w)h)|$ then*

(1) $|\det(\varphi'(x+h) - \varphi'(x))| \geq \frac{1}{2} |\det(\varphi''(w)h)|$ *if* $x, x+h \in B_r(w)$.

(2) $\left\| R_{\frac{1}{2}}^{B_r(w)}(f) \right\|_6 \leq c' r^{-\frac{1}{2}} \|f\|_{\frac{3}{2}},\ f \in S(\mathbb{R}^4)$.

Proof. Let $F(h) = \det(\varphi'(x+h) - \varphi'(x))$ and let $d_x^j F$ denotes the $j-$th differential of F at x. Applying the Taylor formula to $F(h)$ around $h = 0$ and taking into account that $F(0) = 0$, $d_0 F(h) = 0$ and that $d_0^2 F(h,h) \equiv 2\det(\varphi''(x)h)$ we obtain

$$\det(\varphi'(x+h) - \varphi'(x)) = \det(\varphi''(x)h) + \int_0^1 \frac{(1-t)^2}{2} d_{th}^3 F(h,h,h)\,dt.$$

Let $H(x) = \det(\varphi''(x)h)$. The above equation gives

$$\det(\varphi'(x+h) - \varphi'(x)) = \det(\varphi''(w)h) + \int_0^1 d_{w+t(x-w)} H(h)\,dt + \int_0^1 \frac{(1-t)^2}{2} d_{th}^3 F(h,h,h)\,dt.$$

Then, for $x, x+h \in B_r(w)$ we have

$$|\det(\varphi'(x+h) - \varphi'(x)) - \det(\varphi''(w)h)| \leq M|h|^3 \leq 2Mr|h|^2$$

with M depending only $\|\varphi_1\|_{C^3(B)}$ and $\|\varphi_2\|_{C^3(B)}$. If we choose $c \leq \frac{1}{4M}$, we get, for $0 < r < c \inf_{|h|=1} |\det(\varphi''(w)h)|$ that

$$|\det(\varphi'(x+h) - \varphi'(x))| \geq \frac{1}{2}|\det(\varphi''(w)h)|$$

and that

$$J_{B_r(w)}(h) \geq \frac{1}{2}|\det(\varphi''(w)h)| \geq \frac{1}{2c} r|h|^2$$

Thus $\left\| R_{\frac{1}{2}}^{B_r(w)}(f) \right\|_6 \leq c' r^{-\frac{1}{2}} \|I_2(f)\|_6 \leq c' r^{-\frac{1}{2}} \|f\|_{\frac{3}{2}}$, where I_α denotes the Riesz potential on $\mathbb{R}^4$, defined as in [10, p. 117]. So the lemma follows from the Hardy–Littlewood–Sobolev theorem of fractional integration as stated e.g. in [10, p. 119]. □

Lemma 2.3. *Let w be an elliptic point for φ. Then there exists a positive constant c depending only on $\|\varphi_1\|_{C^3(B)}$ and $\|\varphi_2\|_{C^3(B)}$ such that if $0 < r \leq c \inf_{|h|=1} |\det(\varphi''(w)h)|$ then for all $h \neq 0$ the map $x \to \varphi(x+h) - \varphi(x)$ is injective on the domain $\{x \in B : x, x+h \in B_r(w)\}$.*

Proof. Suppose that $x, y, x+h$ and $y+h$ belong to $B_r(w)$ and that

$$\varphi(x+h) - \varphi(x) = \varphi(y+h) - \varphi(y).$$

From this equation we get

$$0 = \int_0^1 (\varphi'(x+th) - \varphi'(y+th))\, h dt = \int_0^1 \int_0^1 d^2_{x+th+s(y-x)} \varphi(y-x, h)\, ds dt.$$

Now, for $z \in B_r(w)$,

$$\begin{aligned} \left| \left(d_z^2 \varphi - d_w^2 \varphi \right)(y-x, h) \right| &= \left| \int_0^1 d^3_{z+u(w-z)} \varphi(w-z, y-x, h)\, du \right| \\ &\leq Mr\, |y-x|\, |h| \end{aligned}$$

then

$$\begin{aligned} 0 &= \int_0^1 \int_0^1 d^2_{x+th+s(y-x)} \varphi(y-x, h)\, ds dt \\ &= d_w^2 \varphi(y-x, h) + \int_0^1 \int_0^1 \left[d^2_{x+th+s(y-x)} \varphi - d_w^2 \varphi \right](y-x, h)\, ds dt. \end{aligned}$$

So $|d_w^2\varphi(y-x,h)| \leq Mr|y-x||h|$ with M depending only on $\|\varphi_1\|_{C^3(B)}$ and $\|\varphi_2\|_{C^3(B)}$.

On the other hand, w is an elliptic point for φ and so, for $|u|=1$, the matrix $A := \varphi''(w)u$ is invertible. Also $A^{-1} = (\det A)^{-1} Ad(A)$, then

$$\left|A^{-1}x\right| = |\det A|^{-1}|Ad(A)x| \leq \frac{\widetilde{M}}{|\det A|}|x|,$$

where $\widetilde{M}$ depends only on $\|\varphi_1\|_{C^2(B)}$ and $\|\varphi_2\|_{C^2(B)}$. Then, for $|v|=1$ and $x = Av$, we have $|Av| \geq |\det A| / \widetilde{M}$. Thus

$$\begin{aligned} \left|d_w^2\varphi(y-x,h)\right| &\geq |y-x||h| \inf_{|u|=1,|v|=1}\left|d_w^2\varphi(u,v)\right| \\ &= |y-x||h| \inf_{|u|=1,|v|=1}|\langle\varphi''(w)u,v\rangle| \\ &\geq \frac{1}{\widetilde{M}}|y-x||h| \inf_{|u|=1}|\det\varphi''(w)u|. \end{aligned}$$

If we choose $r < \frac{1}{M\widetilde{M}} \inf_{|u|=1}|\det\varphi''(w)u|$ the above inequality implies $x = y$ and the lemma is proved. □

For any measurable set $A \subset B$, let μ_A be the Borel measure defined by $\mu_A(E) = \int_A \chi_E(x,\varphi(x))\,dx$ and let T_{μ_A} be the convolution operator given by $T_{\mu_A}f = \mu_A * f$.

Proposition 2.4. *Let w be an elliptic point for φ. Then there exist positive constants c and c' depending only on $\|\varphi_1\|_{C^3(B)}$ and $\|\varphi_2\|_{C^3(B)}$ such that if*

$0 < r < c \inf_{|h|=1} |\det \varphi''(w) h|$ *then*

$$\left\| T_{\mu_{B_r(w)}} f \right\|_3 \le c' r^{-\frac{1}{3}} \|f\|_{\frac{3}{2}} .$$

Proof. Taking account of Lemma 2.3, we can proceed as in Theorem 0 in [3] to obtain, as there, that

$$\left\| \mu_{B_r(w)} * f \right\|_3^3 \le (A_1 A_2 A_3)^{\frac{1}{3}} ,$$

where

$$A_j = \int_{\mathbb{R}^2} F_j(x) \prod_{1 \le m \le 3, m \ne j} R_{\frac{1}{2}}^{B_r(w)} F_m(x)\, dx$$

and $F_j(x) = \|f(x,.)\|_{\frac{3}{2}}$

Then the proposition follows from Lemma 2.2 and an application of the triple Hölder inequality. □

For $0 < a < 1$ and $j \in N$ let

$$U_{a,j} = \left\{ (x_1, x_2) \in B : |x_1| \ge a,\ 2^{-j} |x_1| \le |x_2| \le 2^{-j+1} |x_1| \right\}$$

and let $U_{a,j,i}$, $i = 1,2,3,4$ the connected components of $U_{a,j}$.

We have

Lemma 2.5. *Let* $0 < a < 1$. *Suppose that there exist* $\beta \in \mathbb{N}$, $j_0 \in \mathbb{N}$ *and a positive constant* c *such that* $|\det(\varphi_1''(x) h, \varphi_2''(y) h)| \ge c 2^{-j\beta} |h|^2$ *for all* $h \in \mathbb{R}^2$, $x, y \in U_{a,j,i}$, $j \ge j_0$ *and* $i = 1,2,3,4$. *Thus*

(1) *For all* $j \geq j_0, i = 1,2,3,4$ *if* x *and* $x+h$ *belong to* $U_{a,j,i}$ *then*

$$\left|\det\left(\varphi'(x+h) - \varphi'(x)\right)\right| \geq c2^{-j\beta}\left|h\right|^2.$$

(2) *There exists a positive constant* c' *such that for all* $j \geq j_0$, $i = 1,2,3,4$

$$\left\|R_{\frac{1}{2}}^{U_{a,j,i}}(f)\right\|_6 \leq c' 2^{\frac{j\beta}{2}} \left\|f\right\|_{\frac{3}{2}}.$$

Proof. We fix i and $j \geq j_0$. For $x \in U_{a,j,i}$ we have

$$\det\left(\varphi'(x+h) - \varphi'(x)\right) = \det\left(\int_0^1 \varphi''(x+sh)\, h ds\right).$$

For each $h \in \mathbb{R}^2 \setminus \{0\}$ we have either $\det\left(\varphi_1''(x)\, h, \varphi_2''(y)\, h\right) > c2^{-j\beta}\left|h\right|^2$ for all $x, y \in U_{a,j,i}$ or $\det\left(\varphi_1''(x)\, h, \varphi_2''(y)\, h\right) < -c2^{-j\beta}\left|h\right|^2$ for all $x, y \in U_{a,j,i}$. We consider the first case. Let $F(t) = \det\left(\int_0^t \varphi''(x+sh)\, h ds\right)$. Then

$$\begin{aligned} F'(t) &= \det\left(\int_0^t \varphi_1''(x+sh)\, h ds, \varphi_2''(x+th)\, h\right) \\ &\quad + \det\left(\varphi_1''(x+th)\, h, \int_0^t \varphi_2''(x+sh)\, h ds\right) \\ &= \int_0^t \det\left(\varphi_1''(x+sh)\, h, \varphi_2''(x+th)\, h\right) ds \\ &\quad + \int_0^t \det\left(\varphi_1''(x+th)\, h, \varphi_2''(x+sh)\, h\right) ds \geq c2^{-j\beta}\left|h\right|^2 t. \end{aligned}$$

Since $F(0)=0$ we get $F(1)=\int_0^1 F'(t)\,dt \geq c2^{-j\beta}|h|^2$. Thus

$$\det\left(\varphi'(x+h)-\varphi'(x)\right)=F(1)\geq c2^{-j\beta}|h|^2.$$

Then $J_{U_{a,j,i,}}(h)\geq c2^{-j\beta}|h|^2$, and the lemma follows, as in Lemma 2.2, from the Hardy–Littlewood–Sobolev theorem of fractional integration. The other case is similar. □

For fixed $x^{(1)}, x^{(2)}\in\mathbb{R}^2$, let

$$B_{a,j,i}^{x^{(1)},x^{(2)}}=\left\{x\in\mathbb{R}^2 : x-x^{(1)}\in U_{a,j,i} \text{ and } x-x^{(2)}\in U_{a,j,i}\right\}, i=1,2,3,4.$$

We have

Lemma 2.6. *Let $0<a<1$ and let $x^{(1)}, x^{(2)}\in\mathbb{R}^2$. Suppose that there exist $\beta\in\mathbb{N}$, $j_0\in\mathbb{N}$ and a positive constant c such that $\left|\det\left(\varphi_1''(x)\,h,\varphi_2''(y)\,h\right)\right|\geq c2^{-j\beta}|h|^2$ for all $h\in\mathbb{R}^2$, $x,y\in U_{a,j,i}$, $j\geq j_0$ and $i=1,2,3,4$. Then there exists $j_1\in\mathbb{N}$ independent of $x^{(1)}, x^{(2)}$ such that for all $j\geq j_1$, $i=1,2,3,4$ and all nonnegative $f\in S(\mathbb{R}^4)$ it holds that*

$$\int_{B_{a,j,i}^{x^{(1)},x^{(2)}}\times\mathbb{R}^2} f\left(y-\varphi\left(x-x^{(1)}\right),y-\varphi\left(x-x^{(2)}\right)\right)dx\,dy$$

$$\leq\frac{m^2}{J_{U_{a,j,i}}\left(x^{(2)}-x^{(1)}\right)}\int_{\mathbb{R}^4} f.$$

Proof. We assert that, if $j\geq j_0$ then for each $(z,w)\in\mathbb{R}^2\times\mathbb{R}^2$ and $i=1,2,3,4$, the set

$$\left\{(x,y)\in B_{a,j,i}^{x^{(1)},x^{(2)}}\times\mathbb{R}^2 : z=y-\varphi\left(x-x^{(1)}\right) \text{ and } w=y-\varphi\left(x-x^{(2)}\right)\right\}$$

is a finite set with at most m^2 elements. Indeed, if $z = y - \varphi\left(x - x^{(1)}\right)$ and $w = y - \varphi\left(x - x^{(2)}\right)$ with $x \in B_{a,j,i}^{x^{(1)},x^{(2)}}$, Lemma 2.5 says that, for j large enough,

$$\left|\det\left(\varphi'\left(x - x^{(1)}\right) - \varphi'\left(x - x^{(2)}\right)\right)\right| \geq c2^{-j\beta}\left|h\right|^2.$$

Thus the Bezout's Theorem (See e.g.[1, Lemma 11.5.1, p. 281]) implies that for each $(z, w) \in \mathbb{R}^2 \times \mathbb{R}^2$ the set

$$\left\{x \in B_{a,j,i}^{x^{(1)},x^{(2)}} : \varphi\left(x - x^{(2)}\right) - \varphi\left(x - x^{(1)}\right) = z - w\right\}$$

is a finite set with at most m^2 points. Since x determines y, the assertion follows.

For a fixed $\eta > 0$ and for $k = (k_1, ..., k_4) \in Z^4$, let

$$Q_k = \prod_{1 \leq n \leq 4} \left[k_n \eta, (1 + k_n)\eta\right].$$

Let $\Phi_{k,j,i} : \left(B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2\right) \cap Q_k \to \mathbb{R}^2 \times \mathbb{R}^2$ be the function defined by

$$\Phi_{k,j,i}(x, y) = \left(y - \varphi\left(x - x^{(1)}\right), y - \varphi\left(x - x^{(2)}\right)\right)$$

and let $W_{k,j,i}$ its image. Also

$$\det\left(\Phi'_{k,j,i}\right)(x, y) = \det\left(\varphi'\left(x - x^{(2)}\right) - \varphi'\left(x - x^{(1)}\right)\right).$$

Thus

$$\left|\det\left(\Phi'_{k,j,i}\right)(x, y)\right| \geq J_{U_{a,j,i}}\left(x^{(2)} - x^{(1)}\right) \tag{2.4}$$

for $(x,y) \in \left(B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2\right) \cap Q_k$.

Since $\Phi_{k,j,i}(x,y) = \Phi_{k,j,i}(\overline{x},\overline{y})$ implies that $\varphi\left(x - x^{(1)}\right) - \varphi\left(\overline{x} - x^{(1)}\right) = \varphi\left(x - x^{(2)}\right) - \varphi\left(\overline{x} - x^{(2)}\right)$, taking into account Lemma 2.1, from Lemma 2.3 it follows the existence of $\widetilde{j} \in N$ with $\widetilde{j}$ independent of $x^{(1)}, x^{(2)}$ such that for $j \geq \widetilde{j}$ there exists $\widetilde{\eta} = \widetilde{\eta}(j) > 0$ satisfying that for $0 < \eta < \widetilde{\eta}(j)$ the map $\Phi_{k,j,i}$ is injective for all $k \in Z^4$. Let $\Psi_{k,j,i} : W_{k,j,i} \to \left(B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2\right) \cap Q_k$ its inverse. Lemma 2.5 says that $\left|\det\left(\Phi'_{k,j,i}\right)\right| \geq c2^{-j\beta}|h|^2$ on $\left(B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2\right) \cap Q_k$. We have

$$\begin{aligned}
&\int_{B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2} f\left(y - \varphi\left(x - x^{(1)}\right), y - \varphi\left(x - x^{(2)}\right)\right) dxdy \\
&= \sum_{k \in Z^4} \int_{\left(B_{a,j,i}^{x^{(1)},x^{(2)}} \times \mathbb{R}^2\right) \cap Q_k} f\left(y - \varphi\left(x - x^{(1)}\right), y - \varphi\left(x - x^{(2)}\right)\right) dxdy \\
&= \sum_{k \in Z^4} \int_{W_{k,j,i}} f(z,w) \frac{1}{\left|\det\left(\Phi'_{k,j,i}\right)\left(\Psi_{k,j,i}(z,w)\right)\right|} dzdw \\
&\leq \frac{1}{J_{U_{a,j,i}}\left(x^{(2)} - x^{(1)}\right)} \int_{\mathbb{R}^4} \sum_{k \in Z^4} \chi_{W_{k,j,i}}(v) f(v)\, dv \\
&\leq \frac{m^2}{J_{U_{a,j,i}}\left(x^{(2)} - x^{(1)}\right)} \int_{\mathbb{R}^4} f
\end{aligned}$$

where we have used (2.4). □

Proposition 2.7. *Let $0 < a < 1$. Suppose that there exist $\beta \in \mathbb{N}$, $j_0 \in \mathbb{N}$ and a positive constant c such that $\left|\det\left(\varphi_1''(x)h, \varphi_2''(y)h\right)\right| \geq c2^{-j\beta}|h|^2$ for all*

$h \in \mathbb{R}^2$, $x, y \in U_{a,j,i}$, $j \geq j_0$, $i = 1, 2, 3, 4$. *Then, there exist* $j_1 \in N$, $c' > 0$ *such that for all* $j \geq j_1$, $f \in S(\mathbb{R}^4)$

$$\left\| T_{\mu_{U_{a,j}}} f \right\|_3 \leq c' 2^{\frac{j\beta}{3}} \|f\|_{\frac{3}{2}} .$$

Proof. For $i = 1, 2, 3, 4$, let

$$K_{a,j,i} = \left\{ \left(x, y, x^{(1)}, x^{(2)}, x^{(3)}\right) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 : x - x^{(s)} \in U_{a,j,i},\ s = 1, 2, 3 \right\}.$$

We can proceed as in Theorem 0 in [3] to obtain, as there, that

$$\left\| \mu_{U_{a,j,i}} * f \right\|_3^3 = \int_{K_{a,j,i}} \prod_{1 \leq j \leq 3} f\left(x_j, y - \varphi\left(x - x_j\right)\right) dx dy dx^{(1)} dx^{(2)} dx^{(3)}$$

taking into account of Lemma 2.6 and reasoning, with the obvious changes, as in [3], Theorem 0, we obtain that

$$\left\| \mu_{U_{a,j,i}} * f \right\|_3^3 \leq m^2 \left(A_1 A_2 A_3\right)^{\frac{1}{3}}$$

with

$$A_j = \int_{\mathbb{R}^2} F_j(x) \prod_{1 \leq m \leq 3, m \neq j} R_{\frac{1}{2}}^{U_{a,j,i}} F_m(x)\, dx$$

and $F_j(x) = \|f(x, .)\|_{\frac{3}{2}}$. Now the proof follows as in Proposition 2.4. □

3. About the Type Set

Proposition 3.1. *For $\delta > 0$ let V_δ be defined by (2.1). Suppose that the set of the non elliptic points for φ in V_δ are those lying in the x_1 axis and let α be defined by (2.3).Then $E_{\mu_{V_\delta}}$ contains the closed trapezoidal region with vertices $(0,0)$, $(1,1)$, $\left(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha}\right)$, $\left(\frac{2}{7\alpha}, \frac{1}{7\alpha}\right)$, except perhaps the closed edge parallel to the principal diagonal.*

Proof. We first show that $(1-\theta)(1,1) + \theta\left(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha}\right) \in E_{\mu_{V_\delta}}$ if $0 \le \theta < 1$.

If $w = (w_1, w_2) \in U_{\frac{1}{2},j}$ then $2^{-j-1} \le |w_2| \le 2^{-j+1}$. Thus, from Lemmas 2.2, 2.3 and Proposition 2.7, follows the existence of $j_0 \in N$ and of a positive constant $c = c\left(\|\varphi_1\|_{C^3(B)}, \|\varphi_2\|_{C^3(B)}\right)$ such that if $r_j = c2^{-j\alpha}$, then

$$\left\|T_{\mu_{B_{r_j(w)}}} f\right\|_3 \le c' 2^{\frac{j\alpha}{3}} \|f\|_{\frac{3}{2}}$$

for some $c' > 0$ and all $j \ge j_0$, $w \in U_{\frac{1}{2},j}$, $f \in S(\mathbb{R}^4)$. For $0 \le t \le 1$ let p_t, q_t be defined by $\left(\frac{1}{p_t}, \frac{1}{q_t}\right) = t\left(\frac{2}{3}, \frac{1}{3}\right) + (1-t)(1,1)$. We have also $\left\|T_{\mu_{B_{r_j(w)}}} f\right\|_1 \le \pi c^2 2^{-2j\alpha} \|f\|_1$, thus, the Riesz-Thorin theorem gives

$$\left\|T_{\mu_{B_r(w)}} f\right\|_{q_t} \le c 2^{j\left(\frac{t\alpha}{3} - (1-t)2\alpha\right)} \|f\|_{p_t}.$$

Since $U_{\frac{1}{2},j}$ can be covered with N of such balls $B_r(w)$ with $N \simeq 2^{j(2\alpha-1)}$ we get that

$$\left\|T_{\mu_{U_{\frac{1}{2},j}}}\right\|_{p_t,q_t} \le c 2^{j\left(\frac{7}{3}\alpha t - 1\right)}.$$

Let $U = \cup_{j \geq j_0} U_{\frac{1}{2},j}$. We have that $\|T_{\mu_U}\|_{p_t,q_t} \leq \sum_{j \geq j_0} \left\| T_{\mu_{U_{\frac{1}{2},j}}} \right\|_{p_t,q_t} < \infty$, for $t < \frac{3}{7\alpha}$. Since for $t = \frac{3}{7\alpha}$ we have $\frac{1}{p_t} = 1 - \frac{1}{7\alpha}$ and $\frac{1}{q_t} = 1 - \frac{2}{7\alpha}$ and since every point in $V_\delta \backslash \overset{\circ}{U}$ is an elliptic point (and so, from Theorem 3 in [3], $\left\| T_{\mu_{V_\delta \backslash U}} \right\|_{\frac{3}{2},3} < \infty$), we get that $(1-\theta)(1,1) + \theta\left(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha}\right) \in E_{\mu_{V_\delta}}$ for $0 \leq \theta < 1$. On the other hand, a standard computation shows that the adjoint operator $T^*_{\mu_{V_\delta}}$ is given by $T^*_{\mu_{V_\delta}} f = \left(T_{\mu_{V_\delta}}(f^\vee)\right)^\vee$, where we write, for $g : \mathbb{R}^4 \to C$, $g^\vee(x) = g(-x)$. Thus $E_{\mu_{V_\delta}}$ is symmetric with respect to the nonprincipal diagonal. Finally, after an application of the Riesz-Thorin interpolation theorem, the proposition follows. □

For $\delta > 0$, let $A_\delta = \{(x_1, x_2) \in B : |x_2| \leq \delta |x_1|\}$.

Remark 3.1. For $s > 0$, $x = (x_1, ..., x_4) \in \mathbb{R}^4$ we set $s \bullet x = (sx_1, sx_2, s^m x_3, s^m x_4)$. If $E \subset \mathbb{R}^2$, $F \subset \mathbb{R}^4$ we set $sE = \{sx : x \in E\}$ and $s \bullet F = \{s \bullet x : x \in F\}$. For $f : \mathbb{R}^4 \to C$, $s > 0$, let f_s denotes the function given by $f_s(x) = f(s \bullet x)$. A computation shows that

$$\left(T_{\mu_{2^{-j}V_\delta}} f\right)(2^{-j} \bullet x) = 2^{-2j} \left(T_{\mu_{V_\delta}} f_{2^{-j}}\right)(x) \tag{3.1}$$

for all $f \in S(\mathbb{R}^4)$, $x \in \mathbb{R}^4$.

From this it follows easily that

$$\left\| T_{\mu_{2^{-j}V_\delta}} \right\|_{p,q} = 2^{-j\left(\frac{2(m+1)}{q} - \frac{2(m+1)}{p} + 2\right)} \left\| T_{\mu_{V_\delta}} \right\|_{p,q}.$$

This fact implies that

$$E_\mu \subset \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) : \frac{1}{q} \geq \frac{1}{p} - \frac{1}{m+1} \right\} \tag{3.2}$$

and that if $\frac{1}{q} > \frac{1}{p} - \frac{1}{m+1}$ then $\left(\frac{1}{p}, \frac{1}{q} \right) \in E_{\mu_{A_\delta}}$ if and only if $\left(\frac{1}{p}, \frac{1}{q} \right) \in E_{\mu_{V_\delta}}$.

Theorem 3.2. *Suppose that for some* $\delta > 0$ *the set of the non elliptic points for* φ *in* A_δ *are those lying on the* x_1 *axis and let* α *be defined by (2.3). Then* $E_{\mu_{A_\delta}}$ *contains the intersection of the two closed trapezoidal regions with vertices* $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$, $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$ *and* $(0,0)$, $(1,1)$, $\left(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha}\right)$, $\left(\frac{2}{7\alpha}, \frac{1}{7\alpha}\right)$ *respectively, except perhaps the closed edge parallel to the diagonal.*

Moreover, if $7\alpha \leq m+1$ *then the interior of* $E_{\mu_{A_\delta}}$ *is the open trapezoidal region with vertices* $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$ *and* $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$.

Proof. Taking into account Proposition 3.1, the theorem follows from the facts of Remark 3.1. □

For $0 < a < 1$ and $\delta > 0$ we set $V_{a,\delta} = \{(x_1, x_2) \in B : a \leq |x_1| \leq 1$ and $|x_2| \leq \delta |x_1|\}$. We have

Proposition 3.3. *Let* $0 < a < 1$. *Suppose that for some* $0 < a < 1$, $j_0, \beta \in N$ *and some positive constant* c *we have* $|\det(\varphi_1''(x) h, \varphi_1''(y) h)| \geq c 2^{-j\beta} |h|^2$ *for all* $h \in \mathbb{R}^2$, $x, y \in U_{a,j,i}$, $j \geq j_0$ *and* $i = 1, 2, 3, 4$. *Then, for* δ *positive and small enough,* $E_{\mu_{V_{a,\delta}}}$ *contains the closed trapezoidal region with vertices* $(0,0)$, $(1,1)$, $\left(\frac{\beta+2}{\beta+3}, \frac{\beta+1}{\beta+3}\right)$, $\left(\frac{2}{\beta+3}, \frac{1}{\beta+3}\right)$, *except perhaps the closed edge parallel to the principal diagonal.*

Proof. Proposition 2.7 says that there exist $j_1 \in N$ and a positive constant c such that for $j \geq j_1$ and $f \in S(\mathbb{R}^4)$

$$\left\| T_{\mu_{U_{a,j,i}}} f \right\|_3 \leq c 2^{\frac{j\beta}{3}} \|f\|_{\frac{3}{2}}.$$

Also, for some $c > 0$ and all $f \in S(\mathbb{R}^4)$ we have $\left\| T_{\mu_{U_{a,j,i}}} f \right\|_1 \leq c 2^{-j} \|f\|_1$. Then $\left\| T_{\mu_{U_{a,j,i}}} f \right\|_{q_t} \leq c 2^{j\left(t\frac{\beta}{3}-(1-t)\right)} \|f\|_{p_t}$ where p_t, q_t are defined as in the proof of Proposition 3.1. Let $U = \cup_{j \geq j_1} U_{a,j}$. Then $\|T_{\mu_U} f\|_{p_t,q_t} < \infty$ if $t < \frac{3}{\beta+3}$. Now, the proof follows as in Proposition 3.1. □

Theorem 3.4. *Suppose that for some $0 < a < 1$, $j_0, \beta \in N$ and for some positive constant c we have $|\det(\varphi_1''(x)h, \varphi_1''(y)h)| \geq c2^{-j\beta}|h|^2$ for all $x, y \in U_{a,j,i}$, $j \geq j_0$ and $i = 1,2,3,4$. Then for δ positive and small enough, $E_{\mu_{A_\delta}}$ contains the intersection of the two closed trapezoidal regions with vertices $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$, $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$ and $(0,0)$, $(1,1)$, $\left(\frac{\beta+2}{\beta+3}, \frac{\beta+1}{\beta+3}\right)$, $\left(\frac{2}{\beta+3}, \frac{1}{\beta+3}\right)$, respectively, except perhaps the closed edge parallel to the diagonal.*

Moreover, if $\beta \leq m-2$ then the interior of E_μ is the open trapezoidal region with vertices $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$ and $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$.

Proof. Follows as in Theorem 3.2 using now Proposition 3.3 instead of Proposition 3.1. □

Remark 3.2. We now turn out to the case when φ is a homogeneous polynomial function whose set of non elliptic points is a finite union of lines through the origin, L_1,...,L_k.

For each l, $1 \leq l \leq k$, let $A_\delta^l = \left\{x \in \mathbb{R}^2 : \left|\pi_{L_l}^\perp x\right| \leq \delta \left|\pi_{L_l} x\right|\right\}$ where π_{L_l} and $\pi_{L_l}^\perp$ denote the orthogonal projections from $\mathbb{R}^2$ into L_l and $L_l^\perp$ respectively. Thus each A_δ^l is a closed conical sector around L_l. We choose δ small enough such that $A_\delta^l \cap A_\delta^i = \emptyset$ for $l \neq i$.

It is easy to see that there exists (a unique) $\alpha_l \in N$ and positive constants c_l', c_l'' such that

$$c_l' \left|\pi_{L_l}^\perp w\right|^{\alpha_l} \leq \inf_{|h|=1} \left|\det\left(\varphi''(w)\,h\right)\right| \leq c_l'' \left|\pi_{L_l}^\perp x\right|^{\alpha_l} \tag{3.3}$$

for all $w \in A_\delta^l$. Indeed, after a rotation the situation reduces to that considered in Remark 3.1.

Theorem 3.5. *Suppose that the set of non elliptic points is a finite union of lines through the origin, L_1,...,L_k. For $l = 1, 2, \dots, k$, let α_l be defined by (3.3), and let $\alpha = \max_{1\leq l\leq k} \alpha_l$. Then E_μ contains the intersection of the two closed trapezoidal regions with vertices $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$, $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$ and $(0,0)$, $(1,1)$, $\left(\frac{7\alpha-1}{7\alpha}, \frac{7\alpha-2}{7\alpha}\right)$, $\left(\frac{2}{7\alpha}, \frac{1}{7\alpha}\right)$, respectively, except perhaps the closed edge parallel to the diagonal.*

Moreover, if $7\alpha \leq m+1$ then the interior of E_μ is the interior of the trapezoidal regions with vertices $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$, $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$.

Proof. For $l = 1, 2, \dots, k$, let A_δ^l be as above. From Theorem 3.2, we obtain that $E_{\mu_{A_\delta^l}}$ contains the intersection of the two closed trapezoidal regions with vertices $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1}, \frac{m-1}{m+1}\right)$, $\left(\frac{2}{m+1}, \frac{1}{m+1}\right)$ and $(0,0)$, $(1,1)$, $\left(\frac{7\alpha_l-1}{7\alpha_l}, \frac{7\alpha_l-2}{7\alpha_l}\right)$, $\left(\frac{2}{7\alpha_l}, \frac{1}{7\alpha_l}\right)$ respectively, except perhaps the closed edge parallel to the diagonal.

Since every $x \in B \backslash \cup_l A_\delta^l$ is an elliptic point for φ, Theorem 0 in [3] and a compactness argument give that $\|T_{\mu_D}\|_{\frac{3}{2},3} < \infty$ where $D = \left\{x \in B \backslash \cup_l A_\delta^l : \frac{1}{2} \leq |x|\right\}$. Then (using the symmetry of E_{μ_D}, the fact of that μ_D is a finite measure and the Riesz-Thorin theorem) E_{μ_D} is the closed triangle with vertices $(0,0)$, $(1,1)$, $\left(\frac{2}{3},\frac{1}{3}\right)$. Now, proceeding as in the proof of Theorem 3.2 we get that $\left\|T_{\mu_{B\backslash\cup_l A_\delta^l}}\right\|_{p,q} < \infty$ if $\frac{1}{q} > \frac{1}{p} - \frac{1}{m+1}$. Then the first assertion of the theorem is true. The second one follows also using the facts of Remark 3.1. □

For $0 < a < 1$, we set

$$U_{a,j}^l = \{x \in \mathbb{R}^2 : a \leq |\pi_{L^l}(x)| \leq 1 \text{ and } 2^{-j}|\pi_{L^l}(x)| \leq \left|\pi_{L^l}^{\perp}(x)\right| \leq 2^{-j+1}|\pi_{L^l}(x)|\}$$

let $U_{a,j,i}^l$, $i = 1,2,3,4$ be the connected components of $U_{a,j}^l$.

Theorem 3.6. *Suppose that the set of non elliptic points for φ is a finite union of lines through the origin, L_1,...,L_k. Let $0 < a < 1$ and let $j_0 \in N$ such that*

For $l = 1,2,...,k$, there exists $\beta_l \in N$ satisfying $|\det(\varphi_1''(x)h, \varphi_1''(y)h)| \geq c2^{-j\beta_j}|h|^2$ for all $x, y \in U_{a,j,i}^l$, $j \geq j_0$ and $i = 1,2,3,4$. Let $\beta = \max_{1\leq j\leq k}\beta_j$. Then E_μ contains the intersection of the two closed trapezoidal regions with vertices $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1},\frac{m-1}{m+1}\right)$, $\left(\frac{2}{m+1},\frac{1}{m+1}\right)$ and $(0,0)$, $(1,1)$, $\left(\frac{\beta+2}{\beta+3},\frac{\beta+1}{\beta+3}\right)$, $\left(\frac{2}{\beta+3},\frac{1}{\beta+3}\right)$, respectively, except perhaps the closed edge parallel to the diagonal.

Moreover, if $\beta \leq m-2$ then the interior of E_μ is the interior of the trapezoidal region with vertices $(0,0)$, $(1,1)$, $\left(\frac{m}{m+1},\frac{m-1}{m+1}\right)$, $\left(\frac{2}{m+1},\frac{1}{m+1}\right)$.

Proof. Follows as in Theorem 3.5, using now Theorem 3.4 instead of Theorem 3.2. □

Example 3.1. $\varphi(x_1, x_2) = (x_1^2 x_2 - x_1 x_2^2, x_1^2 x_2 + x_1 x_2^2)$

It is easy to check that the set of non elliptic points is the union of the coordinate axes. Indeed, for $h = (h_1, h_2)$ we have $\det \varphi''(x_1, x_2) h = 8x_2^2 h_1^2 + 8x_1 x_2 h_1 h_2 + 8x_1^2 h_2^2$ and this quadratic form in (h_1, h_2) has non trivial zeros only if $x_1 = 0$ or $x_2 = 0$. The associated symmetric matrix to the quadratic form is

$$\begin{bmatrix} 8x_2^2 & 4x_1 x_2 \\ 4x_1 x_2 & 8x_1^2 \end{bmatrix}$$

and for $x_1 \neq 0$ and $|x_2| \leq \delta |x_1|$ with δ small enough, its eigenvalue of lower absolute value is $\lambda_1(x_1, x_2) = 4x_1^2 + 4x_2^2 - 4\sqrt{(x_2^4 - x_1^2 x_2^2 + x_1^4)}$. Thus $\lambda_1(x_1, x_2) \simeq 6x_2^2$ for such (x_1, x_2). Similarly, for $x_2 \neq 0$ and $|x_1| \leq \delta |x_2|$ with δ small enough, the eigenvalue of lower absolute value is comparable with $6x_1^2$. Then, in the notation of Theorem 3.5, we obtain $\alpha = 2$ and so E_μ contains the closed trapezoidal region with vertices $(0, 0), (1, 1), \left(\frac{13}{14}, \frac{6}{7}\right)$ and $\left(\frac{1}{7}, \frac{1}{14}\right)$ except perhaps the closed edge parallel to the principal diagonal. Observe that, in this case, Theorem 3.6 does not apply. In fact, for $x = (x_1, x_2)$, $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$ and $h = (h_1, h_2)$ we have

$$\begin{aligned} \det\left(\varphi_1''(x) h, \varphi_2''(\widetilde{x}) h\right) &= 4h_1^2 (x_2 \widetilde{x}_1 - \widetilde{x}_2 x_1 + 2x_2 \widetilde{x}_2) \\ &\quad + 4h_1 h_2 (x_1 \widetilde{x}_2 + \widetilde{x}_1 x_2) + 4h_2^2 (x_1 \widetilde{x}_2 - x_2 \widetilde{x}_1 + 2x_2 \widetilde{x}_1). \end{aligned}$$

Take $x_1 = \widetilde{x}_1 = 1$ and let $A = A(x_2, \widetilde{x}_2)$ the matrix of the above quadratic form in (h_1, h_2). For $x_2 = 2^{-j}$, $\widetilde{x}_2 = 2^{-j+1}$ we have $\det A < 0$ for j large enough but if we take $x_2 = 2^{-j+1}$ and $\widetilde{x}_2 = 2^{-j}$, we get $\det A > 0$ for j large enough, so, for all j large enough, $\det A = 0$ for some $2^{-j} \leq x_2, \widetilde{x}_2 \leq 2^{-j+1}$. Thus, for such $x_2, \widetilde{x}_2$,

$$\inf_{|(h_1,h_2)|=1} \det\left(\varphi_1''(1,x_2)(h_1,h_2), \varphi_2''(1,\widetilde{x}_2)(h_1,h_2)\right) = 0.$$

Example 3.2. Let us show an example where Theorem 3.6 characterizes $\overset{\circ}{E}_\mu$. Let

$$\varphi(x_1,x_2) = \left(x_1^3 x_2 - 3x_1 x_2^3, 3x_1^2 x_2^2 - x_2^4\right).$$

In this case the set of non elliptic points for φ is the x_1 axis. Indeed,

$$\det\left(\varphi''(x_1,x_2)(h_1,h_2)\right) = 18\left(x_1^2 + x_2^2\right)\left((h_2 x_1 + x_2 h_1)^2 + 2x_2^2 h_1^2 + 6h_2^2 x_2^2\right).$$

In order to apply Theorem 3.6, we consider the quadratic form in $h = (h_1, h_2)$

$$\det\left(\varphi_1''(x_1,x_2)h, \varphi_2''(\widetilde{x}_1,\widetilde{x}_2)h\right).$$

If $x = (x_1,x_2)$ and $\widetilde{x} = (\widetilde{x}_1,\widetilde{x}_2)$, let $A = A(x,\widetilde{x})$ its associated symmetric matrix. An explicit computation of A shows that, for a given $0 < a < 1$ and for all j large enough and $i = 1,2,3,4$, if x and $\widetilde{x}$ belong to $U_{a,j,i}$, then

$$a^2 \leq tr(A) \leq 20$$

thus, if $\lambda_1(x,\widetilde{x})$ denotes the eigenvalue of lower absolute value of $A(x,\widetilde{x})$, we have, for $x, \widetilde{x} \in W_a$ that

$$c_1 |\det A| \leq |\lambda_1(x,\widetilde{x})| \leq c_2 |\det A|$$

where c_1, c_2 are positive constants independent of j. Now, a computation gives

$$\det A = 324 \left(-x_1^2\widetilde{x}_1^2 - 9x_2^2\widetilde{x}_2^2 - 12x_1x_2\widetilde{x}_1\widetilde{x}_2 + 2x_1^2\widetilde{x}_2^2\right) \times \left(x_2^2\widetilde{x}_1^2 - 2x_2^2\widetilde{x}_2^2 - 4x_1x_2\widetilde{x}_1\widetilde{x}_2 + x_1^2\widetilde{x}_2^2\right).$$

Now we write $\widetilde{x}_2 = tx_2$, with $\frac{1}{2} \leq t \leq 2$. Then

$$\det A = 324x_2^2 \left[-x_1^2\widetilde{x}_1^2 - 9t^2x_2^4 - 12tx_1x_2^2\widetilde{x}_1 + 2t^2x_2^2x_1^2\right] \times \left[\widetilde{x}_1^2 - 2t^2x_2^2 - 4tx_1\widetilde{x}_1 + t^2x_1^2\right].$$

Note that the the first bracket is negative for $x, \widetilde{x} \in W_a$ if j is large enough. To study the sign of the second one, we consider the function $F(t, x_1, \widetilde{x}_1) = \widetilde{x}_1^2 - 4tx_1\widetilde{x}_1 + t^2x_1^2$. Since F has a negative maximum on $\{1\} \times \{1\} \times \left[\frac{1}{2}, 2\right]$, it follows easily that we can choose a such that for $x, \widetilde{x} \in W_a$ and j large enough, the same assertion holds for the second bracket. So $\det A$ is comparable with 2^{-2j}, thus the hypothesis of the Theorem 3.6 are satisfied with $\beta = 2$ and such a. Moreover, we have $\beta = m - 2$, then we conclude that the interior of E_μ is the open trapezoidal region with vertices $(0,0), (1,1), \left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{2}{5}, \frac{1}{5}\right)$.

On the other hand, in a similar way than in Example 3.1 we can see that $\alpha = 2$ (in fact $\det A(x,x) = 648\left(x_1^2 + 9x_2^2\right)\left(x_1^2 + x_2^2\right)^2 x_2^2$), so in this case Theorem 3.6 gives a better result (a precise description of $\overset{\circ}{E}_\mu$) than that given by Theorem 3.5, that asserts only that $\overset{\circ}{E}_\mu$ contains the trapezoidal region with vertices $(0,0), (1,1), \left(\frac{13}{14}, \frac{6}{7}\right)$ and $\left(\frac{1}{7}, \frac{1}{14}\right)$.

Example 3.3. The following is an example where Theorem 3.5 characterizes $\overset{\circ}{E}_\mu$. Let

$$\varphi(x_1, x_2) = \left(x_2 \operatorname{Re}(x_1 + ix_2)^{12}, x_2 \operatorname{Im}(x_1 + ix_2)^{12}\right).$$

A computation gives that for $x = (x_1, x_2)$ and $h = (h_1, h_2)$

$$\det(\varphi''(x)h) = 288\left(x_1^2 + x_2^2\right)^{10}\left(66x_2^2h_1^2 + 11x_1x_2h_1h_2 + \left(x_1^2 + 78x_2^2\right)h_2^2\right)$$

and this quadratic form in (h_1, h_2) does not vanish for $h \neq 0$ unless $x_2 = 0$. So the set of non elliptic points for φ is the x_1 axis. Moreover, its associate symmetric matrix

$$A = A(x) = 288\left(x_1^2 + x_2^2\right)^{10}\begin{bmatrix} 66x_2^2 & \frac{11}{2}x_1x_2 \\ \frac{11}{2}x_1x_2 & x_1^2 + 78x_2^2 \end{bmatrix}$$

satisfies $c_1 \leq trA(x) \leq c_2$ for $x \in B$, $\frac{1}{2} \leq |x_1|$, and $|x_2| \leq \delta|x_1|$, $\delta > 0$ small enough.

Thus if $\lambda_1 = \lambda_1(x)$ denotes the eigenvalue of lower absolute value of $A(x)$, we have, for x in this region, that

$$k_1|\det A| \leq |\lambda_1| \leq k_2|\det A|,$$

where k_1 and k_2 are positive constants.

Since $\det A(1, x_2) = (288)^2\left(1 + x_2^2\right)^{20}\left(\frac{143}{4}x_2^2 + 5148x_2^4\right)$, we have that $\alpha = 2$. So $7\alpha = m + 1$ and, from Theorem 3.5, we conclude that the interior of E_μ is the open trapezoidal region with vertices $(0,0)$, $(1,1)$, $\left(\frac{13}{14}, \frac{6}{7}\right)$, $\left(\frac{1}{7}, \frac{1}{14}\right)$.

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