

Journal of Inequalities in Pure and Applied Mathematics

ON A REVERSE OF JESSEN'S INEQUALITY FOR ISOTONIC LINEAR FUNCTIONALS

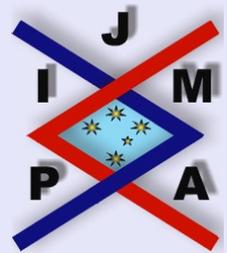
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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

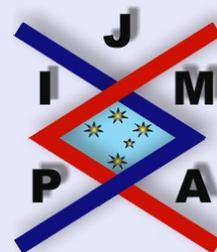
A reverse of Jessen's inequality and its version for $m - \Psi$ -convex and $M - \Psi$ -convex functions are obtained. Some applications for particular cases are also pointed out.

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Key words: Jessen's Inequality, Isotonic Linear Functionals.

Contents

1	Introduction	3
2	The Concepts of $m - \Psi$ -Convex and $M - \Psi$ -Convex Functions	6
3	A Reverse Inequality	12
4	A Further Result for $m - \Psi$ -Convex and $M - \Psi$ -Convex Functions	15
5	Some Applications For Bullen's Inequality	25
	References	



On a Reverse of Jessen's Inequality for Isotonic Linear Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 2 of 30

1. Introduction

Let L be a linear class of real-valued functions $g : E \rightarrow \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1, t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \rightarrow \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.

(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.

The mapping A is said to be *normalised* if

(A3) $A(\mathbf{1}) = 1$.

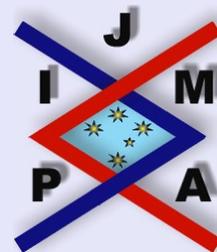
Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2] and [10]).

We recall Jessen's inequality (see also [8]).

Theorem 1.1. *Let $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ (I is an interval), be a convex function and $f : E \rightarrow I$ such that $\phi \circ f, f \in L$. If $A : L \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$(1.1) \quad \phi(A(f)) \leq A(\phi \circ f).$$

A counterpart of this result was proved by Beesack and Pečarić in [2] for compact intervals $I = [\alpha, \beta]$.



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 3 of 30

Theorem 1.2. Let $\phi : [\alpha, \beta] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $f : E \rightarrow [\alpha, \beta]$ such that $\phi \circ f, f \in L$. If $A : L \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional, then

$$(1.2) \quad A(\phi \circ f) \leq \frac{\beta - A(f)}{\beta - \alpha} \phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \phi(\beta).$$

Remark 1.1. Note that (1.2) is a generalisation of the inequality

$$(1.3) \quad A(\phi) \leq \frac{b - A(e_1)}{b - a} \phi(a) + \frac{A(e_1) - a}{b - a} \phi(b)$$

due to Lupaş [9] (see for example [2, Theorem A]), which assumed $E = [a, b]$, L satisfies (L1), (L2), $A : L \rightarrow \mathbb{R}$ satisfies (A1), (A2), $A(1) = 1$, ϕ is convex on E and $\phi \in L$, $e_1 \in L$, where $e_1(x) = x$, $x \in [a, b]$.

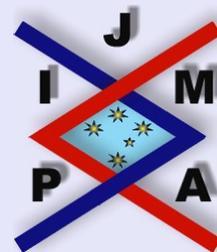
The following inequality is well known in the literature as the Hermite-Hadamard inequality

$$(1.4) \quad \varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \varphi(t) dt \leq \frac{\varphi(a) + \varphi(b)}{2},$$

provided that $\varphi : [a, b] \rightarrow \mathbb{R}$ is a convex function.

Using Theorem 1.1 and Theorem 1.2, we may state the following generalisation of the Hermite-Hadamard inequality for isotonic linear functionals ([11] and [12]).

Theorem 1.3. Let $\phi : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $e : E \rightarrow [a, b]$ with $e, \phi \circ e \in L$. If $A : \mathbb{R} \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional,



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

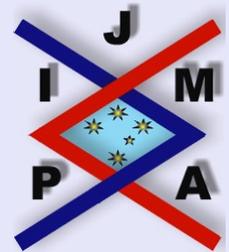
Quit

Page 4 of 30

with $A(e) = \frac{a+b}{2}$, then

$$(1.5) \quad \varphi\left(\frac{a+b}{2}\right) \leq A(\phi \circ e) \leq \frac{\varphi(a) + \varphi(b)}{2}.$$

For other results concerning convex functions and isotonic linear functionals, see [3] – [6], [12] – [14] and the recent monograph [7].



**On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 5 of 30

2. The Concepts of $m - \Psi$ -Convex and $M - \Psi$ -Convex Functions

Assume that the mapping $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ (I is an interval) is convex on I and $m \in \mathbb{R}$. We shall say that the mapping $\phi : I \rightarrow \mathbb{R}$ is $m - \Psi$ -lower convex if $\phi - m\Psi$ is a convex mapping on I (see [4]). We may introduce the class of functions

$$(2.1) \quad \mathcal{L}(I, m, \Psi) := \{\phi : I \rightarrow \mathbb{R} \mid \phi - m\Psi \text{ is convex on } I\}.$$

Similarly, for $M \in \mathbb{R}$ and Ψ as above, we can introduce the class of $M - \Psi$ -upper convex functions by (see [4])

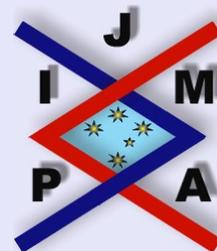
$$(2.2) \quad \mathcal{U}(I, M, \Psi) := \{\phi : I \rightarrow \mathbb{R} \mid M\Psi - \phi \text{ is convex on } I\}.$$

The intersection of these two classes will be called the class of $(m, M) - \Psi$ -convex functions and will be denoted by

$$(2.3) \quad \mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi).$$

Remark 2.1. If $\Psi \in \mathcal{B}(I, m, M, \Psi)$, then $\phi - m\Psi$ and $M\Psi - \phi$ are convex and then $(\phi - m\Psi) + (M\Psi - \phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t) = 0, t \in I$).

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 6 of 30

In [6], S.S. Dragomir and N.M. Ionescu introduced the concept of g -convex dominated mappings, for a mapping $f : I \rightarrow \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \rightarrow \mathbb{R}$, the function $f : I \rightarrow \mathbb{R}$ is g -convex dominated iff $g + f$ and $g - f$ are convex mappings on I . In [6], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Marshall-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of g -convex dominated functions can be obtained as a particular case from $(m, M) - \Psi$ -convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds (see also [4]).

Lemma 2.1. *Let $\Psi, \phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable functions on $\overset{\circ}{I}$ and Ψ is a convex function on $\overset{\circ}{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}(\overset{\circ}{I}, m, \Psi)$ iff*

$$(2.4) \quad m [\Psi(x) - \Psi(y) - \Psi'(y)(x - y)] \leq \phi(x) - \phi(y) - \phi'(y)(x - y)$$

for all $x, y \in \overset{\circ}{I}$.

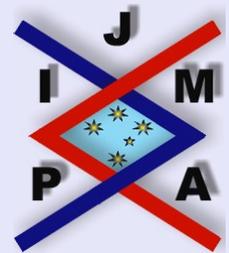
(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}(\overset{\circ}{I}, M, \Psi)$ iff*

(2.5)

$$\phi(x) - \phi(y) - \phi'(y)(x - y) \leq M [\Psi(x) - \Psi(y) - \Psi'(y)(x - y)]$$

for all $x, y \in \overset{\circ}{I}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}(\overset{\circ}{I}, m, M, \Psi)$ iff both (2.4) and (2.5) hold.*



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 7 of 30

Proof. Follows by the fact that a differentiable mapping $h : I \rightarrow \mathbb{R}$ is convex on \mathring{I} iff $h(x) - h(y) \geq h'(y)(x - y)$ for all $x, y \in \mathring{I}$. \square

Another elementary fact for twice differentiable functions also holds (see also [4]).

Lemma 2.2. Let $\Psi, \phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable on \mathring{I} and Ψ is convex on \mathring{I} .

(i) For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}(\mathring{I}, m, \Psi)$ iff

$$(2.6) \quad m\Psi''(t) \leq \phi''(t) \text{ for all } t \in \mathring{I}.$$

(ii) For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}(\mathring{I}, M, \Psi)$ iff

$$(2.7) \quad \phi''(t) \leq M\Psi''(t) \text{ for all } t \in \mathring{I}.$$

(iii) For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}(\mathring{I}, m, M, \Psi)$ iff both (2.6) and (2.7) hold.

Proof. Follows by the fact that a twice differentiable function $h : I \rightarrow \mathbb{R}$ is convex on \mathring{I} iff $h''(t) \geq 0$ for all $t \in \mathring{I}$. \square

We consider the p -logarithmic mean of two positive numbers given by

$$L_p(a, b) := \begin{cases} a & \text{if } b = a, \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases}$$

Proposition 2.3.



On a Reverse of Jensen's Inequality for Isotonic Linear Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 8 of 30

and $p \in \mathbb{R} \setminus \{-1, 0\}$.

The following proposition holds (see also [4]).

Let $\phi : (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping.

(i) For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}((0, \infty), m, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff

$$(2.8) \quad mp(x-y) [L_{p-1}^{p-1}(x, y) - y^{p-1}] \leq \phi(x) - \phi(y) - \phi'(y)(x-y)$$

for all $x, y \in (0, \infty)$.

(ii) For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}((0, \infty), M, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff

$$(2.9) \quad \phi(x) - \phi(y) - \phi'(y)(x-y) \leq Mp(x-y) [L_{p-1}^{p-1}(x, y) - y^{p-1}]$$

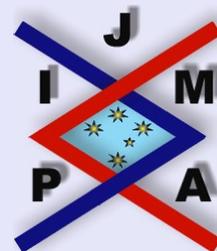
for all $x, y \in (0, \infty)$.

(iii) For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}((0, \infty), M, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff both (2.8) and (2.9) hold.

The proof follows by Lemma 2.1 applied for the convex mapping $\Psi(t) = t^p$, $p \in (-\infty, 0) \cup (4, \infty)$ and via some elementary computation. We omit the details.

The following corollary is useful in practice.

Corollary 2.4. Let $\phi : (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function.



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 9 of 30

(i) For $m \in \mathbb{R}$, the function ϕ is m -quadratic-lower convex (i.e., for $p = 2$) iff

$$(2.10) \quad m(x - y)^2 \leq \phi(x) - \phi(y) - \phi'(y)(x - y)$$

for all $x, y \in (0, \infty)$.

(ii) For $M \in \mathbb{R}$, the function ϕ is M -quadratic-upper convex iff

$$(2.11) \quad \phi(x) - \phi(y) - \phi'(y)(x - y) \leq M(x - y)^2$$

for all $x, y \in (0, \infty)$.

(iii) For $m, M \in \mathbb{R}$ with $M \geq m$, the function ϕ is (m, M) -quadratic convex if both (2.10) and (2.11) hold.

The following proposition holds (see also [4]).

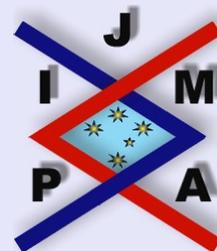
Proposition 2.5. Let $\phi : (0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function.

(i) For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}((0, \infty), m, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff

$$(2.12) \quad p(p - 1)mt^{p-2} \leq \phi''(t) \quad \text{for all } t \in (0, \infty).$$

(ii) For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}((0, \infty), M, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff

$$(2.13) \quad \phi''(t) \leq p(p - 1)Mt^{p-2} \quad \text{for all } t \in (0, \infty).$$



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 10 of 30

(iii) For $m, M \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}((0, \infty), m, M, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff both (2.12) and (2.13) hold.

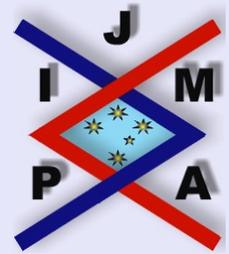
As can be easily seen, the above proposition offers the practical criterion of deciding when a twice differentiable mapping is $(\cdot)^p$ -lower or $(\cdot)^p$ -upper convex and which weights the constant m and M are.

The following corollary is useful in practice.

Corollary 2.6. Assume that the mapping $\phi : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is twice differentiable.

(i) If $\inf_{t \in (a,b)} \phi''(t) = k > -\infty$, then ϕ is $\frac{k}{2}$ -quadratic lower convex on (a, b) ;

(ii) If $\sup_{t \in (a,b)} \phi''(t) = K < \infty$, then ϕ is $\frac{K}{2}$ -quadratic upper convex on (a, b) .



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 11 of 30

3. A Reverse Inequality

We start with the following result which gives another counterpart for $A(\phi \circ f)$, as did the Lupaş-Beesack-Pečarić result (1.2).

Theorem 3.1. *Let $\phi : (\alpha, \beta) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable convex function on (α, β) , $f : E \rightarrow (\alpha, \beta)$ such that $\phi \circ f, f, \phi' \circ f, \phi' \circ f \cdot f \in L$. If $A : L \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\begin{aligned}
 (3.1) \quad 0 &\leq A(\phi \circ f) - \phi(A(f)) \\
 &\leq A(\phi' \circ f \cdot f) - A(f) \cdot A(\phi' \circ f) \\
 &\leq \frac{1}{4} [\phi'(\beta) - \phi'(\alpha)] (\beta - \alpha) \quad (\text{if } \alpha, \beta \text{ are finite}).
 \end{aligned}$$

Proof. As ϕ is differentiable convex on (α, β) , we may write that

$$(3.2) \quad \phi(x) - \phi(y) \geq \phi'(y)(x - y), \text{ for all } x, y \in (\alpha, \beta),$$

from where we obtain

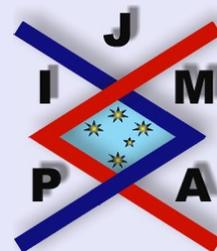
$$(3.3) \quad \phi(A(f)) - (\phi \circ f)(t) \geq (\phi' \circ f)(t)(A(f) - f(t))$$

for all $t \in E$, as, obviously, $A(f) \in (\alpha, \beta)$.

If we apply to (3.3) the functional A , we may write

$$\phi(A(f)) - A(\phi \circ f) \geq A(f) \cdot A(\phi' \circ f) - A(\phi' \circ f \cdot f),$$

which is clearly equivalent to the first inequality in (3.1).



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 12 of 30

It is well known that the following Grüss inequality for isotonic linear and normalised functionals holds (see [1])

$$(3.4) \quad |A(hk) - A(h)A(k)| \leq \frac{1}{4}(M - m)(N - n),$$

provided that $h, k \in L$, $hk \in L$ and $-\infty < m \leq h(t) \leq M < \infty$, $-\infty < n \leq k(t) \leq N < \infty$, for all $t \in E$.

Taking into account that for finite α, β we have $\alpha < f(t) < \beta$ with ϕ' being monotonic on (α, β) , we have $\phi'(\alpha) \leq \phi' \circ f \leq \phi'(\beta)$, and then by the Grüss inequality, we may state that

$$A(\phi' \circ f \cdot f) - A(f) \cdot A(\phi' \circ f) \leq \frac{1}{4}[\phi'(\beta) - \phi'(\alpha)](\beta - \alpha)$$

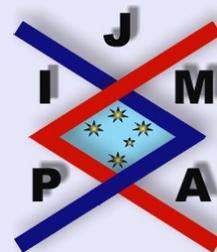
and the theorem is completely proved. □

The following corollary holds.

Corollary 3.2. *Let $\phi : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable convex function on \mathbb{I} . If $\phi, e_1, \phi', \phi' \cdot e_1 \in L$ ($e_1(x) = x, x \in [a, b]$) and $A : L \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional, then:*

$$(3.5) \quad \begin{aligned} 0 &\leq A(\phi) - \phi(A(e_1)) \\ &\leq A(\phi' \cdot e_1) - A(e_1) \cdot A(\phi') \\ &\leq \frac{1}{4}[\phi'(b) - \phi'(a)](b - a). \end{aligned}$$

There are some particular cases which can naturally be considered.



**On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 13 of 30

1. Let $\phi(x) = \ln x$, $x > 0$. If $\ln f$, f , $\frac{1}{f} \in L$ and $A : L \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional, then:

$$(3.6) \quad 0 \leq \ln [A(f)] - A[\ln(f)] \leq A(f) A\left(\frac{1}{f}\right) - 1,$$

provided that $f(t) > 0$ for all $t \in E$ and $A(f) > 0$.

If $0 < m \leq f(t) \leq M < \infty$, $t \in E$, then, by the second part of (3.1) we have:

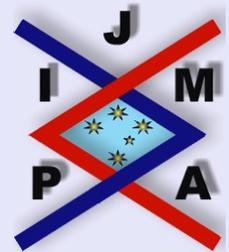
$$(3.7) \quad A(f) A\left(\frac{1}{f}\right) - 1 \leq \frac{(M - m)^2}{4mM} \quad (\text{which is a known result}).$$

Note that the inequality (3.6) is equivalent to

$$(3.8) \quad 1 \leq \frac{A(f)}{\exp[A[\ln(f)]]} \leq \exp\left[A(f) A\left(\frac{1}{f}\right) - 1\right].$$

2. Let $\phi(x) = \exp(x)$, $x \in \mathbb{R}$. If $\exp(f)$, f , $f \cdot \exp(f) \in L$ and $A : L \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional, then

$$(3.9) \quad \begin{aligned} 0 &\leq A[\exp(f)] - \exp[A(f)] \\ &\leq A[f \exp(f)] - A(f) \exp[A(f)] \\ &\leq \frac{1}{4} [\exp(M) - \exp(m)] (M - m) \quad (\text{if } m \leq f \leq M \text{ on } E). \end{aligned}$$



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 14 of 30

4. A Further Result for $m - \Psi$ -Convex and $M - \Psi$ -Convex Functions

In [4], S.S. Dragomir proved the following inequality of Jessen's type for $m - \Psi$ -convex and $M - \Psi$ -convex functions.

Theorem 4.1. *Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $f : E \rightarrow I$ such that $\Psi \circ f, f \in L$ and $A : L \rightarrow \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If $\phi \in \mathcal{L}(I, m, \Psi)$ and $\phi \circ f \in L$, then we have the inequality*

$$(4.1) \quad m[A(\Psi \circ f) - \Psi(A(f))] \leq A(\phi \circ f) - \phi(A(f)).$$

(ii) *If $\phi \in \mathcal{U}(I, M, \Psi)$ and $\phi \circ f \in L$, then we have the inequality*

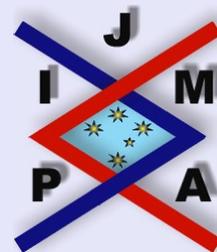
$$(4.2) \quad A(\phi \circ f) - \phi(A(f)) \leq M[A(\Psi \circ f) - \Psi(A(f))].$$

(iii) *If $\phi \in \mathcal{B}(I, m, M, \Psi)$ and $\phi \circ f \in L$, then both (4.1) and (4.2) hold.*

The following corollary is useful in practice.

Corollary 4.2. *Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable convex function on $\overset{\circ}{I}$, $f : E \rightarrow I$ such that $\Psi \circ f, f \in L$ and $A : L \rightarrow \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If $\phi : I \rightarrow \mathbb{R}$ is twice differentiable and $\phi''(t) \geq m\Psi''(t)$, $t \in \overset{\circ}{I}$ (where m is a given real number), then (4.1) holds, provided that $\phi \circ f \in L$.*



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents

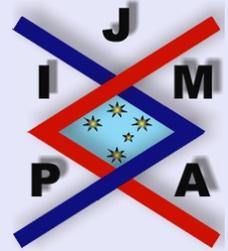


Go Back

Close

Quit

Page 15 of 30



On a Reverse of Jensen's Inequality for Isotonic Linear Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 16 of 30

(ii) If $\phi : I \rightarrow \mathbb{R}$ is twice differentiable and $\phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$ (where M is a given real number), then (4.2) holds, provided that $\phi \circ f \in L$.

(iii) If $\phi : I \rightarrow \mathbb{R}$ is twice differentiable and $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, then both (4.1) and (4.2) hold, provided $\phi \circ f \in L$.

In [5], S.S. Dragomir obtained the following result of Lupaş-Beesack-Pečarić type for $m - \Psi$ -convex and $M - \Psi$ -convex functions.

Theorem 4.3. Let $\Psi : [\alpha, \beta] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $f : I \rightarrow [\alpha, \beta]$ such that $\Psi \circ f$, $f \in L$ and $A : L \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional.

(i) If $\phi \in \mathcal{L}(I, m, \Psi)$ and $\phi \circ f \in L$, then we have the inequality

$$(4.3) \quad m \left[\frac{\beta - A(f)}{\beta - \alpha} \Psi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \Psi(\beta) - A(\Psi \circ f) \right] \leq \frac{\beta - A(f)}{\beta - \alpha} \phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \phi(\beta) - A(\phi \circ f).$$

(ii) If $\phi \in \mathcal{U}(I, M, \Psi)$ and $\phi \circ f \in L$, then

$$(4.4) \quad \frac{\beta - A(f)}{\beta - \alpha} \phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \phi(\beta) - A(\phi \circ f) \leq M \left[\frac{\beta - A(f)}{\beta - \alpha} \Psi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \Psi(\beta) - A(\Psi \circ f) \right].$$

(iii) If $\phi \in \mathcal{B}(I, m, M, \Psi)$ and $\phi \circ f \in L$, then both (4.3) and (4.4) hold.

The following corollary is useful in practice.

Corollary 4.4. *Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable convex function on $\overset{\circ}{I}$, $f : E \rightarrow I$ such that $\Psi \circ f, f \in L$ and $A : L \rightarrow \mathbb{R}$ is an isotonic linear and normalised functional.*

- (i) *If $\phi : I \rightarrow \mathbb{R}$ is twice differentiable, $\phi \circ f \in L$ and $\phi''(t) \geq m\Psi''(t)$, $t \in \overset{\circ}{I}$ (where m is a given real number), then (4.3) holds.*
- (ii) *If $\phi : I \rightarrow \mathbb{R}$ is twice differentiable, $\phi \circ f \in L$ and $\phi''(t) \leq M\Psi''(t)$, $t \in \overset{\circ}{I}$ (where m is a given real number), then (4.4) holds.*
- (iii) *If $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \overset{\circ}{I}$, then both (4.3) and (4.4) hold.*

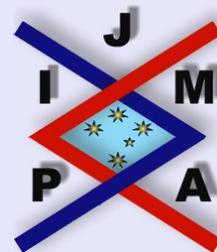
We now prove the following new result.

Theorem 4.5. *Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable convex function and $f : E \rightarrow I$ such that $\Psi \circ f, \Psi' \circ f, \Psi' \circ f \cdot f, f \in L$ and $A : L \rightarrow \mathbb{R}$ be an isotonic linear and normalised functional.*

- (i) *If ϕ is differentiable, $\phi \in \mathcal{L}(\overset{\circ}{I}, m, \Psi)$ and $\phi \circ f, \phi' \circ f, \phi' \circ f \cdot f \in L$, then we have the inequality*

$$(4.5) \quad m[A(\Psi' \circ f \cdot f) + \Psi(A(f)) - A(f) \cdot A(\Psi' \circ f) - A(\Psi \circ f)] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f).$$

- (ii) *If ϕ is differentiable, $\phi \in \mathcal{U}(\overset{\circ}{I}, M, \Psi)$ and $\phi \circ f, \phi' \circ f, \phi' \circ f \cdot f \in L$, then*



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents

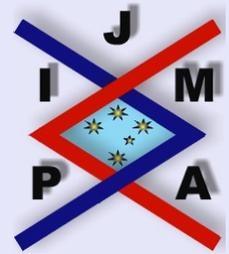


Go Back

Close

Quit

Page 17 of 30



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 18 of 30

we have the inequality

$$(4.6) \quad A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \\ \leq M [A(\Psi' \circ f \cdot f) + \Psi(A(f)) \\ - A(f) \cdot A(\Psi' \circ f) - A(\Psi \circ f)].$$

(iii) If ϕ is differentiable, $\phi \in \mathcal{B}(I, m, M, \Psi)$ and $\phi \circ f, \phi' \circ f, \phi' \circ f \cdot f \in L$, then both (4.5) and (4.6) hold.

Proof. The proof is as follows.

(i) As $\phi \in \mathcal{L}(I, m, \Psi)$, then $\phi - m\Psi$ is convex and we can apply the first part of the inequality (3.1) for $\phi - m\Psi$ getting

$$(4.7) \quad A[(\phi - m\Psi) \circ f] - (\phi - m\Psi)(A(f)) \\ \leq A[(\phi - m\Psi)' \circ f \cdot f] - A(f) A((\phi - m\Psi)' \circ f).$$

However,

$$\begin{aligned} A[(\phi - m\Psi) \circ f] &= A(\phi \circ f) - mA(\Psi \circ f), \\ (\phi - m\Psi)(A(f)) &= \phi(A(f)) - m\Psi(A(f)), \\ A[(\phi - m\Psi)' \circ f \cdot f] &= A(\phi' \circ f \cdot f) - mA(\Psi' \circ f \cdot f) \end{aligned}$$

and

$$A((\phi - m\Psi)' \circ f) = A(\phi' \circ f) - mA(\Psi' \circ f)$$

and then, by (4.7), we deduce the desired inequality (4.5).

(ii) Goes likewise and we omit the details.

(iii) Follows by (i) and (ii).

□

The following corollary is useful in practice,

Corollary 4.6. *Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable convex function on \mathring{I} , $f : E \rightarrow I$ such that $\Psi \circ f, \Psi' \circ f, \Psi' \circ f \cdot f, f \in L$ and $A : L \rightarrow \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If $\phi : I \rightarrow \mathbb{R}$ is twice differentiable, $\phi \circ f, \phi' \circ f, \phi' \circ f \cdot f \in L$ and $\phi''(t) \geq m\Psi''(t)$, $t \in \mathring{I}$, (where m is a given real number), then the inequality (4.5) holds.*

(ii) *With the same assumptions, but if $\phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, (where M is a given real number), then the inequality (4.6) holds.*

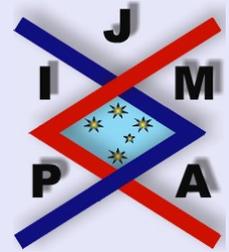
(iii) *If $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, then both (4.5) and (4.6) hold.*

Some particular important cases of the above corollary are embodied in the following proposition.

Proposition 4.7. *Assume that the mapping $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is twice differentiable on \mathring{I} .*

(i) *If $\inf_{t \in \mathring{I}} \phi''(t) = k > -\infty$, then we have the inequality*

$$(4.8) \quad \frac{1}{2}k [A(f^2) - [A(f)]^2] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f),$$



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents

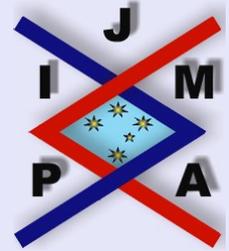


Go Back

Close

Quit

Page 19 of 30



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 20 of 30

provided that $\phi \circ f, \phi' \circ f, \phi' \circ f \cdot f, f^2 \in L$.

(ii) If $\sup_{t \in \mathring{I}} \phi''(t) = K < \infty$, then we have the inequality

$$(4.9) \quad A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \\ \leq \frac{1}{2}K [A(f^2) - [A(f)]^2].$$

(iii) If $-\infty < k \leq \phi''(t) \leq K < \infty, t \in \mathring{I}$, then both (4.8) and (4.9) hold.

The proof follows by Corollary 4.6 applied for $\Psi(t) = \frac{1}{2}t^2$ and $m = k, M = K$.

Another result is the following one.

Proposition 4.8. Assume that the mapping $\phi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is twice differentiable on \mathring{I} . Let $p \in (-\infty, 0) \cup (1, \infty)$ and define $g_p : I \rightarrow \mathbb{R}, g_p(t) = \phi''(t)t^{2-p}$.

(i) If $\inf_{t \in \mathring{I}} g_p(t) = \gamma > -\infty$, then we have the inequality

$$(4.10) \quad \frac{\gamma}{p(p-1)} [(p-1)[A(f^p) - [A(f)]^p] \\ - pA(f)[A(f^{p-1}) - [A(f)]^{p-1}] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f),$$

provided that $\phi \circ f, \phi' \circ f, \phi' \circ f \cdot f, f^p, f^{p-1} \in L$.

(ii) If $\sup_{t \in I} g_p(t) = \Gamma < \infty$, then we have the inequality

$$(4.11) \quad A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \\ \leq \frac{\Gamma}{p(p-1)} [(p-1)[A(f^p) - [A(f)]^p] \\ - pA(f)[A(f^{p-1}) - [A(f)]^{p-1}].$$

(iii) If $-\infty < \gamma \leq g_p(t) \leq \Gamma < \infty$, $t \in I$, then both (4.10) and (4.11) hold.

Proof. The proof is as follows.

(i) We have for the auxiliary mapping $h_p(t) = \phi(t) - \frac{\gamma}{p(p-1)}t^p$ that

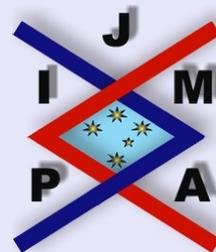
$$h_p''(t) = \phi''(t) - \gamma t^{p-2} = t^{p-2}(t^{2-p}\phi''(t) - \gamma) \\ = t^{p-2}(g_p(t) - \gamma) \geq 0.$$

That is, h_p is convex or, equivalently, $\phi \in \mathcal{L}\left(I, \frac{\gamma}{p(p-1)}, (\cdot)^p\right)$. Applying Corollary 4.6, we get

$$\frac{\gamma}{p(p-1)} [pA(f^p) + [A(f)]^p - pA(f)A(f^{p-1}) - A(f^p)] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f),$$

which is clearly equivalent to (4.10).

(ii) Goes similarly.



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 21 of 30

(iii) Follows by (i) and (ii). □

The following proposition also holds.

Proposition 4.9. Assume that the mapping $\phi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is twice differentiable on $\overset{\circ}{I}$. Define $l(t) = t^2 \phi''(t)$, $t \in I$.

(i) If $\inf_{t \in \overset{\circ}{I}} l(t) = s > -\infty$, then we have the inequality

$$(4.12) \quad s \left[A(f) A\left(\frac{1}{f}\right) - 1 - (\ln[A(f)] - A[\ln(f)]) \right] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f),$$

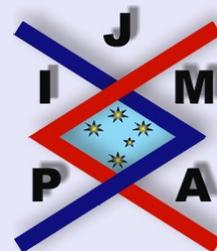
provided that $\phi \circ f, \phi^{-1} \circ f, \phi^{-1} \circ f \cdot f, \frac{1}{f}, \ln f \in L$ and $A(f) > 0$.

(ii) If $\sup_{t \in \overset{\circ}{I}} l(t) = S < \infty$, then we have the inequality

$$(4.13) \quad A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \\ \leq S \left[A(f) A\left(\frac{1}{f}\right) - 1 - (\ln[A(f)] - A[\ln(f)]) \right].$$

(iii) If $-\infty < s \leq l(t) \leq S < \infty$ for $t \in \overset{\circ}{I}$, then both (4.12) and (4.13) hold.

Proof. The proof is as follows.



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 22 of 30

(i) Define the auxiliary function $h(t) = \phi(t) + s \ln t$. Then

$$h''(t) = \phi''(t) - \frac{s}{t^2} = \frac{1}{t^2} (\phi''(t)t^2 - s) \geq 0$$

which shows that h is convex, or, equivalently, $\phi \in \mathcal{L}(I, s, -\ln(\cdot))$. Applying Corollary 4.6, we may write

$$\begin{aligned} & s \left[-A(\mathbf{1}) - \ln A(f) + A(f) A\left(\frac{1}{f}\right) + A(\ln(f)) \right] \\ & \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f), \end{aligned}$$

which is clearly equivalent to (4.12).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

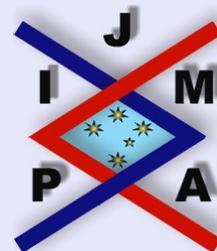
Finally, the following result also holds.

Proposition 4.10. *Assume that the mapping $\phi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is twice differentiable on \mathring{I} . Define $\tilde{I}(t) = t\phi''(t)$, $t \in I$.*

(i) *If $\inf_{t \in \mathring{I}} \tilde{I}(t) = \delta > -\infty$, then we have the inequality*

$$\begin{aligned} (4.14) \quad & \delta A(f) [\ln[A(f)] - A(\ln(f))] \\ & \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f), \end{aligned}$$

provided that $\phi \circ f, \phi' \circ f, \phi' \circ f \cdot f, \ln f, f \in L$ and $A(f) > 0$.



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 23 of 30

(ii) If $\sup_{t \in \tilde{I}} \tilde{I}(t) = \Delta < \infty$, then we have the inequality

$$(4.15) \quad A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \leq \Delta A(f) [\ln[A(f)] - A(\ln(f))].$$

(iii) If $-\infty < \delta \leq \tilde{I}(t) \leq \Delta < \infty$ for $t \in \tilde{I}$, then both (4.14) and (4.15) hold.

Proof. The proof is as follows.

(i) Define the auxiliary mapping $h(t) = \phi(t) - \delta t \ln t$, $t \in I$. Then

$$h''(t) = \phi''(t) - \frac{\delta}{t} = \frac{1}{t^2} [\phi''(t)t - \delta] = \frac{1}{t} [\tilde{I}(t) - \delta] \geq 0$$

which shows that h is convex or equivalently, $\phi \in \mathcal{L}(I, \delta, (\cdot) \ln(\cdot))$. Applying Corollary 4.6, we get

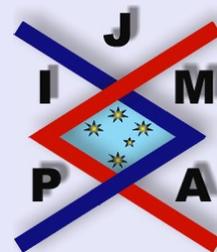
$$\begin{aligned} \delta [A[(\ln f + 1)f] + A(f) \ln A(f) - A(f)A(\ln f + 1) - A(f \ln f)] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \end{aligned}$$

which is equivalent with (4.14).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 24 of 30

5. Some Applications For Bullen's Inequality

The following inequality is well known in the literature as Bullen's inequality (see for example [7, p. 10])

$$(5.1) \quad \frac{1}{b-a} \int_a^b \phi(t) dt \leq \frac{1}{2} \left[\frac{\phi(a) + \phi(b)}{2} + \phi\left(\frac{a+b}{2}\right) \right],$$

provided that $\phi : [a, b] \rightarrow \mathbb{R}$ is a convex function on $[a, b]$. In other words, as (5.1) is equivalent to:

$$(5.2) \quad 0 \leq \frac{1}{b-a} \int_a^b \phi(t) dt - \phi\left(\frac{a+b}{2}\right) \leq \frac{\phi(a) + \phi(b)}{2} - \frac{1}{b-a} \int_a^b \phi(t) dt$$

we can conclude that in the Hermite-Hadamard inequality

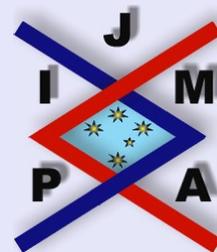
$$(5.3) \quad \frac{\phi(a) + \phi(b)}{2} \geq \frac{1}{b-a} \int_a^b \phi(t) dt \geq \phi\left(\frac{a+b}{2}\right)$$

the integral mean $\frac{1}{b-a} \int_a^b \phi(t) dt$ is closer to $\phi\left(\frac{a+b}{2}\right)$ than to $\frac{\phi(a) + \phi(b)}{2}$.

Using some of the results pointed out in the previous sections, we may upper and lower bound the *Bullen difference*:

$$B(\phi; a, b) := \frac{1}{2} \left[\frac{\phi(a) + \phi(b)}{2} + \phi\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b \phi(t) dt$$

(which is positive for convex functions) for different classes of twice differentiable functions ϕ .



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 25 of 30

Now, if we assume that $A(f) := \frac{1}{b-a} \int_a^b f(t) dt$, then for $f = e$, $e(x) = x$, $x \in [a, b]$, we have, for a differentiable function ϕ , that

$$\begin{aligned} & A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \\ &= \frac{1}{b-a} \int_a^b x\phi'(x) dx + \phi\left(\frac{a+b}{2}\right) \\ &\quad - \frac{a+b}{2} \cdot \frac{1}{b-a} \int_a^b \phi'(x) dx - \frac{1}{b-a} \int_a^b \phi(x) dx \\ &= \frac{1}{b-a} \left[b\phi(b) - a\phi(a) - \int_a^b \phi(x) dx \right] + \phi\left(\frac{a+b}{2}\right) \\ &\quad - \frac{a+b}{2} \cdot \frac{\phi(b) - \phi(a)}{b-a} - \frac{1}{b-a} \int_a^b \phi(x) dx \\ &= \frac{\phi(a) + \phi(b)}{2} + \phi\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^b \phi(x) dx \\ &= 2B(\phi; a, b). \end{aligned}$$

a) Assume that $\phi : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a twice differentiable function satisfying the property that $-\infty < k \leq \phi''(t) \leq K < \infty$. Then by Proposition 4.7, we may state the inequality

$$(5.4) \quad \frac{1}{48} (b-a)^2 k \leq B(\phi; a, b) \leq \frac{1}{48} (b-a)^2 K.$$

This follows by Proposition 4.7 on taking into account that

$$\frac{1}{b-a} \int_a^b x^2 dx - \left(\frac{1}{b-a} \int_a^b x dx \right)^2 = \frac{(b-a)^2}{12}.$$



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 26 of 30

b) Now, assume that the twice differentiable function $\phi : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ satisfies the property that $-\infty < \gamma \leq t^{2-p}\phi''(t) \leq \Gamma < \infty, t \in (a, b), p \in (-\infty, 0) \cup (1, \infty)$. Then by Proposition 4.8 and taking into account that

$$\begin{aligned} A(f^p) - (A(f))^p &= \frac{1}{b-a} \int_a^b x^p dx - \left(\frac{1}{b-a} \int_a^b x dx \right)^p \\ &= L_p^p(a, b) - A^p(a, b), \end{aligned}$$

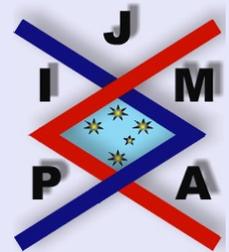
and

$$A(f^{p-1}) - (A(f))^{p-1} = L_{p-1}^{p-1}(a, b) - A^{p-1}(a, b),$$

we may state the inequality

$$\begin{aligned} (5.5) \quad \frac{\gamma}{p(p-1)} [(p-1) [L_p^p(a, b) - A^p(a, b)] \\ - pA(a, b) [L_{p-1}^{p-1}(a, b) - A^{p-1}(a, b)]] \\ \leq B(\phi; a, b) \\ \leq \frac{\Gamma}{p(p-1)} [(p-1) [L_p^p(a, b) - A^p(a, b)] \\ - pA(a, b) [L_{p-1}^{p-1}(a, b) - A^{p-1}(a, b)]] . \end{aligned}$$

c) Assume that the twice differentiable function $\phi : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ satisfies the property that $-\infty < s \leq t^2\phi''(t) \leq S < \infty, t \in (a, b)$, then by



On a Reverse of Jensen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 27 of 30

Proposition 4.9, and taking into account that

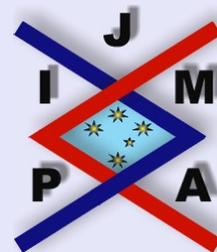
$$\begin{aligned} & A(f) A(f^{-1}) - 1 - \ln[A(f)] + A \ln(f) \\ &= \frac{A(a, b)}{L(a, b)} - 1 - \ln A(a, b) + I(a, b) \\ &= \ln \left[\frac{I(a, b)}{A(a, b)} \cdot \exp \left(\frac{A(a, b) - L(a, b)}{L(a, b)} \right) \right], \end{aligned}$$

we get the inequality

$$\begin{aligned} (5.6) \quad \frac{s}{2} \ln \left[\frac{I(a, b)}{A(a, b)} \cdot \exp \left(\frac{A(a, b) - L(a, b)}{L(a, b)} \right) \right] \\ \leq B(\phi; a, b) \\ \leq \frac{S}{2} \ln \left[\frac{I(a, b)}{A(a, b)} \cdot \exp \left(\frac{A(a, b) - L(a, b)}{L(a, b)} \right) \right]. \end{aligned}$$

d) Finally, if ϕ satisfies the condition $-\infty < \delta \leq t\phi''(t) \leq \Delta < \infty$, then by Proposition 4.10, we may state the inequality

$$(5.7) \quad \delta A(a, b) \ln \left[\frac{A(a, b)}{I(a, b)} \right] \leq B(\phi; a, b) \leq \Delta A(a, b) \ln \left[\frac{A(a, b)}{I(a, b)} \right].$$



On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

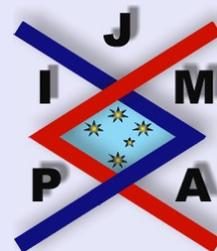
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Page 28 of 30

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On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



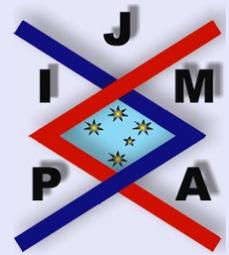
Go Back

Close

Quit

Page 29 of 30

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On a Reverse of Jessen's
Inequality for Isotonic Linear
Functionals

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 30 of 30