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CONSEQUENCES OF A THEOREM OF ERDÖS-PRACHAR

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ABSTRACT. In the present article we study the asymptotic behavior of the sums $\sum_{n\le x}\left|\frac{c_{n+1}}{p_{n+1}}-\frac{c_n}{p_n}\right|$ and $\sum_{n\le x}\left|\frac{p_{n+1}}{c_{n+1}}-\frac{p_n}{c_n}\right|$, and of the series $\sum_{n=1}^{\infty}\left|\frac{c_{n+1}}{p_{n+1}}-\frac{c_n}{p_n}\right|^{\alpha}$, where p_n denotes the n-th prime number while c_n stands for the n-th composed number.

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1. INTRODUCTION

We are going to use the following notation

$$\begin{aligned} \pi(x) &\text{ the number of prime numbers } \le x, \\ C(x) &\text{ the number of composed numbers } \le x, \\ p_n &\text{ the } n\text{-th prime number}, \\ c_n &\text{ the } n\text{-th composed number; } c_1=4, c_2=6,\dots, \\ \log_2 n &= \log(\log n). \end{aligned}$$

The present work originates in a result due to Erdös and Prachar [2]: they proved that there exist $c', c''>0$ such that

$$c'\log^2 x > \sum_{p_k\le x}\left|\frac{p_{k+1}}{k+1}-\frac{p_k}{k}\right| > c''\log^2 x \text{ for } x\ge 2,$$

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that is

$$\sum_{p_k \le x} \left| \frac{p_{k+1}}{k+1} - \frac{p_k}{k} \right| \asymp \log^2 x. \tag{1.1}$$

In a recent paper [3], Panaitopol proved that

$$\sum_{p_k \le x} \left| \frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right| \asymp \log\log x. \tag{1.2}$$

The proofs of these results rely on the following result due to Schnirelmann: if for x positive and n a positive integer one denotes by $M(n,x)$ the number of the indices k such that $p_k \le x$ and $p_{k+1} - p_k = n$, then

$$M(n,x) < c''' \frac{x}{\log^2 x} \sum_{d|n} \frac{1}{d},$$

where c''' is a positive constant.

In the present paper, several well known results will be used:

$$\pi(x) \sim \frac{x}{\log x}; \tag{1.3}$$

$$p_n \sim n \log n; \tag{1.4}$$

(1.5) the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \log n (\log\log n)^{\alpha}}$ is convergent if and only if $\alpha > 1$;

$$\sum_{2 \le n \le x} \frac{1}{n \log n} = \log\log x + O(1); \tag{1.6}$$

$$\sum_{2 \le n \le x} \frac{1}{n} = \log x + O(1); \tag{1.7}$$

$$\sum_{p\ prime \le x} \frac{\log p}{p} \sim \log x. \tag{1.8}$$

We also need the following result of Bojarincev [1]:

$$c_n = n + \frac{n}{\log n} \cdot u_n, \text{ where } \lim_{n\to\infty} u_n = 1. \tag{1.9}$$

2. PROPERTIES OF THE SEQUENCE $\left(\frac{n}{p_n}\right)_{n \ge 1}$

The series $\sum_{n=1}^{\infty} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|$ is divergent by (1.2). In connection with this fact we prove the following result.

Theorem 2.1. *The series*

$$\sum_{n=3}^{\infty} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right| \cdot \frac{1}{(\log\log n)^{\alpha}}$$

is convergent if and only if $\alpha > 1$.

Let us first prove the following auxiliary result.

Lemma 2.2. *Consider the sequences* $(a_n)_{n\geq 1}$, $(x_n)_{n\geq 1}$ *and* $(s_n)_{n\geq 1}$, *where* $s_n = \sum_{i=1}^n a_i$. *If the sequence* $(s_n x_n)_{n\geq 1}$ *is convergent, then one of the series*

$$\sum_{n=1}^{\infty} s_n(x_{n+1} - x_n) \text{ and } \sum_{n=1}^{\infty} a_n x_n$$

is convergent if and only if the other one is convergent.

Proof. If $\sum_{n=1}^{\infty} s_n(x_{n+1} - x_n)$ is convergent, then $\lim_{n\to\infty} s_n(x_{n+1} - x_n) = 0$. But $\lim_{n\to\infty} s_n x_n = k$ for some $k \in \mathbb{R}$, hence $\lim_{n\to\infty} s_n x_{n+1} = k$ and $\lim_{n\to\infty}(s_{n+1} - s_n)x_{n+1} = 0$.

On the other hand, if $\sum_{n=1}^{\infty} a_n x_n$ is convergent, then $\lim_{n\to\infty} a_{n+1}x_{n+1} = 0$, hence $\lim_{n\to\infty}(s_{n+1} - s_n)x_{n+1} = 0$.

Now let us denote $S_n = \sum_{i=1}^n a_i x_i$ and $\sigma_n = \sum_{i=1}^n s_i(x_{i+1} - x_i)$. Then for each p we have

$$S_{n+p} - S_n = \sigma_{n+p} - \sigma_n + s_{n+p}x_{n+p} - s_{n+1}x_{n+1} + (s_{n+1} - s_n)x_{n+1}. \tag{2.1}$$

Since we have just seen that in either case we have $\lim\limits_{n\to\infty}(s_{n+1} - s_n)x_{n+1} = 0$, relation (2.1) implies that in either case one of the sequences $(S_n)_{n\geq 1}$ and $(\sigma_n)_{n\geq 1}$ is Cauchy if and only if the other one is Cauchy. Now, by Cauchy's criterion, one of the two series is convergent if and only if the other one is convergent. □

Proof of Theorem 2.1. If $\alpha \leq 0$, then the series is divergent by (1.2). Next assume $\alpha > 0$ and choose $a_n = \left|\frac{n+1}{p_{n+1}} - \frac{n}{p_n}\right|$ and $x_n = \frac{1}{(\log\log n)^\alpha}$.

If we consider the function $f : [n, n+1] \to \mathbb{R}$, $f(x) = (\log\log x)^{-\alpha}$, then Lagrange's theorem implies

$$(\log\log(n+1))^{-\alpha} - (\log\log n)^{-\alpha} = -\frac{\alpha(\log\log\theta_n)^{-\alpha-1}}{\theta_n \log\theta_n},$$

where $n < \theta_n < n+1$. Since $\theta_n \sim n$, it follows that

$$x_{n+1} - x_n \sim \frac{-\alpha}{n\log n(\log\log n)^{\alpha+1}}. \tag{2.2}$$

By (1.2) and (1.4) it follows that $s_n \asymp \log\log n$ and (2.2) implies

$$s_n(x_{n+1} - x_n) \asymp -\frac{1}{n\log n(\log\log n)^\alpha}. \tag{2.3}$$

If $\alpha > 1$, then we have

$$\lim_{n\to\infty} s_n x_n = \lim_{n\to\infty} \frac{\log\log n}{(\log\log n)^\alpha} = 0,$$

while for $\alpha = 1$ we get $\lim_{n\to\infty} s_n x_n = 1$. Thus, for $\alpha \geq 1$, the above lemma implies that one of the series $\sum_{n=3}^{\infty} s_n(x_{n+1} - x_n)$ and $\sum_{n=3}^{\infty} a_n x_n$ is convergent if and only if the other one is convergent. In view of (1.5) and (2.3), the series $\sum_{n=3}^{\infty} a_n x_n$ is convergent for $\alpha > 1$ and divergent for $\alpha = 1$.

Finally, if $0 < \alpha < 1$, then

$$x_n \left|\frac{n+1}{p_{n+1}} - \frac{n}{p_n}\right| > \frac{1}{\log\log n}\left|\frac{n+1}{p_{n+1}} - \frac{n}{p_n}\right|$$

and the desired conclusion follows. □

Consequence 1. If $\alpha > 1$, then the series

$$\sum_{n=1}^{\infty} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|^{\alpha}$$

is convergent.

Proof. We have

$$\left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|^{\alpha-1} \leq \max\left(\frac{n+1}{p_{n+1}}, \frac{n}{p_n} \right)^{\alpha-1}.$$

For $K > 0$ and $n \geq 3$, we have $\frac{1}{(\log n)^{\alpha-1}} < \frac{K}{(\log\log n)^{\alpha}}$ and (1.4) implies that $\frac{n+1}{p_{n+1}} \sim \frac{n}{p_n} \sim \frac{1}{\log n}$. There exists K' such that

$$\left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|^{\alpha} < K' \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right| \frac{1}{(\log\log n)^{\alpha}}$$

and the convergence of the series $\sum_{n=1}^{\infty} \left| \frac{n+1}{p_{n+1}} - \frac{n}{p_n} \right|^{\alpha}$ follows by Theorem 2.1. □

3. PROPERTIES OF THE SEQUENCE $\left(\frac{c_n}{p_n}\right)_{n\geq 1}$

Since $c_n \sim n$ (see (1.9)), for the sequence $\left(\frac{c_n}{p_n}\right)_{n\geq 1}$ we obtain properties which are similar to those of the sequence $\left(\frac{n}{p_n}\right)_{n\geq 1}$.

In connection with (1.2) we have the following fact.

Theorem 3.1. *We have*

$$\sum_{p_k \leq x} \left| \frac{c_{k+1}}{p_{k+1}} - \frac{c_k}{p_k} \right| \asymp \log\log x$$

for every $x > e$.

Proof. If we denote $\alpha_k = c_{k+1} - c_k - 1$, then it follows that $\alpha_k = 0$ if $c_k + 1$ is a composed number, and $\alpha_k = 1$ if $c_k + 1$ is prime. In the last case $c_k + 1 = p_m$. Setting $k = k(m)$, we deduce by (1.9) that $c_k - k \sim \frac{k}{\log k}$ and $\log k \sim \log c_k \sim \log p_m \sim \log m$. It then follows that $k(m) - p_m \sim -\frac{p_m}{\log m}$ and (1.4) implies that

$$k(m) = p_m - my_m \text{ with } \lim_{m\to\infty} y_m = 1. \tag{3.1}$$

We have by (1.9)

$$\begin{aligned} \frac{c_{k+1}}{p_{k+1}} - \frac{c_k}{p_k} &= \frac{c_k + 1 + \alpha_k}{p_{k+1}} - \frac{c_k}{p_k} = \frac{\alpha_k + 1}{p_{k+1}} - \frac{c_k(p_{k+1} - p_k)}{p_k p_{k+1}} \\ &= \frac{\alpha_k + 1}{p_{k+1}} - \frac{k(p_{k+1} - p_k)}{p_k p_{k+1}} - \frac{ku_k}{(\log k) p_k p_{k+1}} \\ &= \left(\frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right) + \frac{\alpha_k}{p_{k+1}} - \frac{ku_k(p_{k+1} - p_k)}{(\log k) p_k p_{k+1}}. \end{aligned}$$

We have the inequality

$$\begin{aligned} &\left| \frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right| - \left| \frac{\alpha_k}{p_{k+1}} \right| - \left| \frac{ku_k(p_{k+1} - p_k)}{(\log k) p_k p_{k+1}} \right| \\ &\qquad \leq \left| \frac{c_{k+1}}{p_{k+1}} - \frac{c_k}{p_k} \right| \leq \left| \frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right| + \left| \frac{\alpha_k}{p_{k+1}} \right| + \left| \frac{ku_k(p_{k+1} - p_k)}{(\log k) p_k p_{k+1}} \right|. \end{aligned} \tag{3.2}$$

We have

$$\frac{ku_k(p_{k+1}-p_k)}{(\log k)p_kp_{k+1}} \sim \frac{k(p_{k+1}-p_k)}{k^2\log^3 k} = \frac{p_{k+1}-p_k}{k\log^3 k}.$$

Panaitopol proves in [4] that for $\beta > 2$ the series $\sum_{n=2}^{\infty} \frac{p_{n+1}-p_n}{n\log^\beta n}$ is convergent, hence the series $\sum_{k=2}^{\infty} \frac{ku_k(p_{k+1}-p_k)}{(\log k)p_kp_{k+1}}$ is also convergent.

We have furthermore

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{p_{k+1}} = \sum{}' \frac{1}{p_{k+1}},$$

where $\sum'$ extends over the values of k such that $\alpha_k = 1$, that is, $c_k + 1 = p_m$. Then by (2.2) and (1.4) we get

$$p_{k+1} \sim p_k \sim p_{p_m} \sim m\log^2 m. \tag{3.3}$$

Since the series $\sum_{m=2}^{\infty} \frac{1}{m\log^2 m}$ is convergent, it then follows that the series $\sum_{k=1}^{\infty} \frac{\alpha_k}{p_{k+1}}$ is also convergent. Now (3.2) implies that

$$\sum_{k=1}^{n} \left| \frac{c_{k+1}}{p_{k+1}} - \frac{c_k}{p_k} \right| = \sum_{k=1}^{n} \left| \frac{k+1}{p_{k+1}} - \frac{k}{p_k} \right| + O(1)$$

and the desired conclusion follows. □

Analogously to the Erdös-Prachar theorem, we shall prove the following fact.

Theorem 3.2. *We have*

$$\sum_{p_k \le x} \left| \frac{p_{k+1}}{c_{k+1}} - \frac{p_k}{c_k} \right| \asymp \log^2 x$$

for every $x > 1$.

Proof. We have

$$\frac{p_{k+1}}{c_{k+1}} - \frac{p_k}{c_k} = \left(\frac{p_{k+1}}{k+1} - \frac{p_k}{k} \right) + p_k \left(\frac{1}{k(k+1)} - \frac{c_{k+1}-c_k}{c_{k+1}c_k} \right) + \frac{(p_{k+1}-p_k)(k+1-c_{k+1})}{(k+1)c_{k+1}}. \tag{3.4}$$

By (1.9) we get

$$\begin{aligned}
&\sum_{p_k \le x} \left| \frac{(p_{k+1}-p_k)(k+1-c_{k+1})}{(k+1)c_{k+1}} \right| \\
&= -\sum_{p_k \le x} \frac{(p_{k+1}-p_k)u_{k+1}}{c_{k+1}\log(k+1)} \\
&= O\left(\sum_{k=2}^{\pi(x)} \frac{p_{k+1}-p_k}{k\log k} \right) \\
&= O\left(\frac{x}{\pi(x)\log \pi(x)} + \sum_{k=3}^{\pi(x)} p_k \left(\frac{1}{(k-1)\log(k-1)} - \frac{1}{k\log k} \right) \right).
\end{aligned}$$

By (1.3) we have $\pi(x)\log \pi(x) \sim x$,

$$p_k \left(\frac{1}{(k-1)\log(k-1)} - \frac{1}{k\log k} \right) \sim \frac{p_k \log k}{k^2\log^2 k} \sim \frac{1}{k}$$

and (1.7) implies

$$\sum_{p_k\le x}\frac{(p_{k+1}-p_k)(k+1-c_{k+1})}{(k+1)c_{k+1}}=O(\log x). \tag{3.5}$$

We have also

$$\sum_{p_k\le x}\left|p_k\left(\frac{1}{k(k+1)}-\frac{c_{k+1}-c_k}{c_{k+1}c_k}\right)\right|$$
$$\le {\sum_{p_k\le x}}''\left|p_k\left(\frac{1}{k(k+1)}-\frac{1}{c_k(c_k+1)}\right)\right|+2{\sum_{p_k\le x}}'\left|\frac{p_k}{c_k(c_k+2)}\right|,$$

where $\sum'$ extends over the values of k such that c_k+1 is a prime number, while $\sum''$ extends over the values of k such that $c_k+1=p_m$ is composed. By (1.9) we deduce

$${\sum_{p_k\le x}}''\left|\frac{p_k(c_k-k)(c_k+k+1)}{k(k+1)c_k(c_k+1)}\right|=O\left(\sum_{p_k\le x}\frac{k\log k}{k^4}\cdot k^2\log k\right)$$
$$=O\left(\sum_{p_k\le x}\frac{1}{k}\right)=O(\log x).$$

By (1.4) and (1.8) it follows that

$${\sum_{p_k\le x}}'\left|\frac{p_k}{c_k(c_k+2)}\right|\sim{\sum_{k\le\pi(x)}}'\frac{c_k\log c_k}{c_k(c_k+2)}$$
$$\sim{\sum_{c_k\le x}}'\frac{\log(c_k+1)}{c_k+1}$$
$$=\sum_{p_m\le x}\frac{\log p_m}{p_m}=O(\log x).$$

Thus

$$\sum_{p_k\le x}p_k\left|\frac{1}{k(k+1)}-\frac{c_{k+1}-c_k}{c_{k+1}c_k}\right|=O(\log x). \tag{3.6}$$

Now by (1.1), (3.4), (3.5) and (3.6) it follows that $\sum_{p_k\le x}\left|\frac{p_{k+1}}{c_{k+1}}-\frac{p_k}{c_k}\right|\asymp\log^2 x$ and the proof is completed. □

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