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ON THE UTILITY OF THE TELYAKOVSKIĬ'S CLASS S

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ABSTRACT. An illustration is given showing the advantage of the definition given by Telyakovskiĭ for the class introduced by Sidon. It is also verified that if a sequence $\{a_n\}$ belongs to the recently defined subclass S_γ of S, $\gamma > 0$, then the sequence $\{n^\gamma a_n\}$ belongs to the class S, but the converse statement does not hold.

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1. INTRODUCTION

A great number of mathematicians have studied the question 'What conditions for a sequence $\{a_n\}$ guarantee that the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

to be Fourier series, or to converge in L^1-metric?'. We refer only to W.H. Young [13], A.H. Kolmogorov [2], S. Sidon [6], S. A. Telyakovskiĭ [9] and the plentiful references given in [9] and in the excellent monograph by R.P. Boas, Jr. [1]. It is also known that conditions were established with monotone, quasi-monotone, convex and quasi-convex sequences, with null-sequences of bounded variation, and also sequences given by Sidon via a nice special construction.

In 1973 S. A. Telyakovskiĭ [10] introduced a very effective idea, defined a "new" class of coefficient sequences. He denoted this class by S; the letter S refers to an esteemed result of S. Sidon [6], and to the class defined by him in the same paper. Namely, Telyakovskiĭ also showed

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that his class and that of Sidon are identical, but to apply his definition is more convenient. This is the reason, in my view, that later most of the authors ([7], [8], [14]), dealing with similar problems, wanted to extend the definition of Telyakovskiĭ.

In [3] and [4] we showed that some of these “extensions” are equivalent to the class S, and some others are real extensions of S, but they are identical among themselves.

All of these facts show that the class S defined by Telyakovskiĭ plays a very important role in the studies of the problems mentioned above.

The definition of the class S is the following: A null-sequence $\mathbf{a} := \{a_n\}$ belongs to the class S, or briefly $\mathbf{a} \in S$, if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \le A_n$ hold for all n.

The aim of the present note is to give one further illustration which underlies the central position of the class S and the following theorems proved in the same paper where the definition of S was given.

In [10] Telyakovskiĭ, among others, proved the next two theorems.

Theorem 1.1. *Let the coefficients of the series (1.1) belong to the class S. Then the series (1.1) is a Fourier series and*

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \right| dx \le C \sum_{n=0}^{\infty} A_n,$$

where C is an absolute constant.

Theorem 1.2. *Let the coefficients of the series (1.2) belong to the class S. Then for any $p = 1, 2, \dots$*

$$\int_{\frac{\pi}{p+1}}^{\pi} \left| \sum_{n=1}^{\infty} a_n \sin nx \right| dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty} A_n \right)$$

holds uniformly.

In particular, the series (1.2) is a Fourier series if and only if

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty.$$

Recently Z. Tomovski [12] defined certain subclasses of S, and denoted them by S_r, $r = 1, 2, \dots$ (see also [11] and in [5] the definition of the class $S(\alpha)$). A null-sequence $\{a_n\}$ belongs to the class S_r, if there exists a monotonically decreasing sequence $\{A_n^{(r)}\}$ such that $\sum_{n=1}^{\infty} n^r A_n^{(r)} < \infty$ and $|\Delta a_n| \le A_n^{(r)}$ for all n. (For $r = 0$ clearly $S_0 = S$ and $A_n^{(0)} = A_n$.)

In [11] Tomovski established, among others, two theorems in connection with the classes S_r as follows:

Theorem 1.3. *Let the coefficients of the series (1.1) belong to the class S_r, $r = 0, 1, \dots$. Then the r-th derivative of the series (1.1) is a Fourier series and if $f^{(r)}(x)$ denotes its sum function we have that*

$$\int_0^{\pi} \left| f^{(r)}(x) \right| dx \le M \sum_{n=0}^{\infty} n^r A_n^{(r)}, \quad M = M(r) > 0.$$

Theorem 1.4. *Let the coefficients of the series (1.2) belong to the class S_r, $r = 0, 1, \dots$, furthermore let $g(x)$ denote the sum function of the series (1.2). Then for any $p = 1, 2, \dots$*

$$\int_{\frac{\pi}{p+1}}^{\pi} |g^{(r)}(x)| dx = \sum_{n=1}^{p} |a_n| n^{r-1} + O\left(\sum_{n=1}^{\infty} n^r A_n^{(r)} \right).$$

In particular, the r-th derivative of the series (1.2) is a Fourier series if and only if

$$\sum_{n=1}^{\infty} |a_n| n^{r-1} < \infty.$$

It is obvious that if $r = 0$ then the Theorems 1.3 and 1.4 reduce to the Theorems 1.1 and 1.2, respectively.

The proof of Theorem 1.3 has not yet appeared, the proof of Theorem 1.4 given in [11] is a constrictive one, follows similar lines as that of Telyakovskiĭ.

Now, we shall verify that if a sequence $\{a_n\}$ belongs to S_r, then the sequence $\{n^r a_n\}$ belongs to S, with such a sequence $\{A_n\}$ which satisfies the inequality

$$\sum_{n=1}^{\infty} A_n \le (r+1) \sum_{n=1}^{\infty} n^r A_n^{(r)}, \quad (A_n \equiv A_n^{(0)}). \tag{1.3}$$

Thus, this result and the Theorems 1.1 and 1.2 immediately imply the Theorems 1.3 and 1.4, respectively.

2. RESULTS

We shall deduce our assertion from a somewhat more general result. In the Introduction we have already referred to that in [5], we also defined a certain subclass of S as follows:

Let $\alpha := \{\alpha_n\}$ be a positive monotone sequence tending to infinity. A null-sequence $\{a_n\}$ belongs to the class $S(\alpha)$, if there exists a monotonically decreasing sequence $\{A_n^{(\alpha)}\}$ such that

$$\sum_{n=1}^{\infty} \alpha_n A_n^{(\alpha)} < \infty, \quad \text{and} \quad |\Delta a_n| \le A_n^{(\alpha)} \quad \text{for all} \quad n.$$

If we denote the class $S(\alpha)$, where $\alpha_n := n^\alpha$, $\alpha > 0$, by S_α, that is, if we introduce the definition $S_\alpha := S(n^\alpha)$, we immediately get the generalization of the classes S_r, $r = 1, 2, \ldots,$ for any positive α.

We shall prove our result for the classes S_α, $\alpha > 0$.

Theorem 2.1. *Let $\gamma \ge \beta > 0$. If $\{a_n\}$ belongs to the class S_γ, then the sequence $\{n^\beta a_n\}$ belongs to the class $S_{\gamma-\beta}$ and*

$$\sum_{n=1}^{\infty} n^{\gamma-\beta} A_n^{(\gamma-\beta)} \le (\beta+1) \sum_{n=1}^{\infty} n^\gamma A_n^{(\gamma)} \tag{2.1}$$

holds.

It is clear that if $\gamma = \beta = r$ then (2.1) gives (1.3). Thus the inequality (1.3), utilizing the assumptions of Theorem 1.3 and 1.4, and the statements of Theorems 1.1 and 1.2, implies the assortions of Theorems 1.3 and 1.4, respectively.

This is a new and short proof for the Theorems 1.3 and 1.4.

Remark 2.2. The statement of the theorem is not reversible in general.

3. PROOFS

Proof of Theorem 2.1. In order to prove our theorem we have to verify that there exists a monotonically decreasing sequence $\left\{A_n^{(\gamma-\beta)}\right\}$ such that (2.1) and

$$|\Delta(n^\beta a_n)| \le A_n^{(\gamma-\beta)} \tag{3.1}$$

hold. Since $\{a_n\} \in S_\gamma$ thus if $\beta \geq 1$ then

$$|\Delta(n^\beta a_n)| = |n^\beta(a_n - a_{n+1}) - a_{n+1}((n+1)^\beta - n^\beta)| \tag{3.2}$$
$$\leq n^\beta|\Delta a_n| + \beta(n+1)^{\beta-1}|a_{n+1}|$$
$$\leq n^\beta A_n^{(\gamma)} + \beta(n+1)^{(\beta-1)} \sum_{k=n+1}^{\infty} A_k^{(\gamma)}.$$

Now define

$$A_n^{(\gamma-\beta)} := n^\beta A_n^{(\gamma)} + \beta \sum_{k=n+1}^{\infty} k^{\beta-1} A_k^{(\gamma)}.$$

By this definition and (3.2) it is clear that (3.1) holds. Next we show that the sequence $\{A_n^{(\gamma-\beta)}\}$ is monotonic, that is

$$A_{n+1}^{(\gamma-\beta)} \leq A_n^{(\gamma-\beta)}.$$

Since $(n+1)^\beta \leq n^\beta + \beta(n+1)^{\beta-1}$ and $A_{n+1}^{(\gamma)} \leq A_n^{(\gamma)}$, thus

$$A_{n+1}^{(\gamma-\beta)} = (n+1)^\beta A_{n+1}^{(\gamma)} + \beta \sum_{k=n+2}^{\infty} k^{\beta-1} A_k^{(\gamma)}$$
$$\leq n^\beta A_n^{(\gamma)} + \beta(n+1)^{\beta-1} A_{n+1}^{(\gamma)} + \beta \sum_{k=n+2}^{\infty} k^{\beta-1} A_k^{(\gamma)} = A_n^{(\gamma-\beta)}.$$

Finally we verify (2.1). Since

$$\sum_{n=1}^{\infty} n^{\gamma-\beta} A_n^{(\gamma-\beta)} = \sum_{n=1}^{\infty} n^\gamma A_n^{(\gamma)} + \beta \sum_{n=1}^{\infty} n^{\gamma-\beta} \sum_{k=n+1}^{\infty} k^{\beta-1} A_k^{(\gamma)}$$
$$\leq \sum_{n=1}^{\infty} n^\gamma A_n^{(\gamma)} + \beta \sum_{k=2}^{\infty} k^{\beta-1} A_k^{(\gamma)} \sum_{n=1}^{k} n^{\gamma-\beta}$$
$$\leq (\beta+1) \sum_{n=1}^{\infty} n^\gamma A_n^{(\gamma)}.$$

If $0 < \beta < 1$ then, using the first equality of (3.2), we get that

$$|\Delta(n^\beta a_n)| \leq n^\beta A_n^{(\gamma)} + \beta n^{\beta-1} \sum_{k=n+1}^{\infty} A_k^{(\gamma)}.$$

Henceforth the proof follows the lines given for $\beta \geq 1$ if we define

$$A_n^{(\gamma-\beta)} := n^\beta A_n^{(\gamma)} + \beta n^{\beta-1} \sum_{k=n+1}^{\infty} A_k^{(\gamma)}.$$

Herewith the proof is complete. $\square$

Proof of Remark 2.2. It suffices to prove the remark for the case $\gamma = \beta = 1$. We know that if $\{a_n\} \in S_1$ then $\{na_n\} \in S$. Our next example will show that there exists a sequence $\{c_n\}$ such that $\{nc_n\} \in S$ but $\{c_n\} \notin S_1$. This verifies that the implication

$$\{a_n\} \in S_1 \Rightarrow \{na_n\} \in S$$

is not reversible.

Put

$$c_n := \frac{1}{n\log(n+1)}, \quad n \geq 1.$$

Then the sequence $\{nc_n\}$ is monotonically decreasing, tends to zero, and thus clearly belongs to the class S.

On the other hand

$$|\Delta c_n| \geq \frac{1}{n(n+1)\log(n+1)},$$

whence

$$\sum_{n=1}^{\infty} nA_n^{(1)} = \infty$$

obviously follows if $A_n^{(1)} \geq |\Delta c_n|$ holds, consequently $\{c_n\}$ does not belong to S_1.

This proves Remark 2.2. □

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