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SOME DISTORTION INEQUALITIES ASSOCIATED WITH THE FRACTIONAL DERIVATIVES OF ANALYTIC AND UNIVALENT FUNCTIONS

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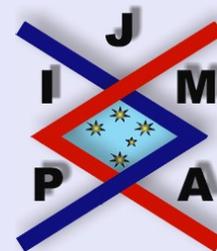
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Abstract

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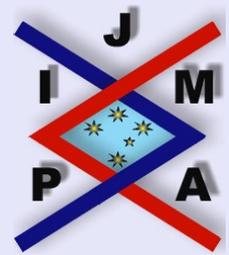


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Abstract

For the classes \mathcal{S} and \mathcal{K} of (normalized) univalent and convex analytic functions, respectively, a number of authors conjectured interesting extensions of certain known distortion inequalities in terms of a fractional derivative operator. While examining and investigating the validity of these conjectures, many subsequent works considered various generalizations of the distortion inequalities relevant to each of these conjectures. The main object of this paper is to give a direct proof of one of the known facts that these conjectures are false. Several further distortion inequalities involving fractional derivatives are also presented.

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1. Introduction and Definitions

Let \mathcal{A} denote the class of functions $f(z)$ normalized by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also, let \mathcal{S} and \mathcal{K} denote the subclasses of \mathcal{A} consisting of functions which are, respectively, univalent and convex in \mathcal{U} (see, for details, [4], [5], and [12]).

Geometric Function Theory is the study of the relationship between the *analytic* properties of $f(z)$ and the *geometric* properties of the *image* domain

$$\mathcal{D} = f(\mathcal{U}).$$

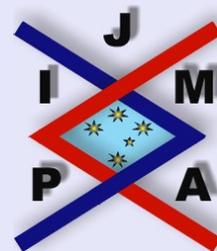
An excellent example of this interplay is provided by the following important result which validates a 1916 conjecture of Ludwig Bieberbach (1896-1982):

Theorem 1. de Branges [3]. *If the function $f(z)$ given by (1.1) is in the class \mathcal{S} , then*

$$(1.2) \quad |a_n| \leq n \quad (n \in \mathbb{N} \setminus \{1\}; \mathbb{N} := \{1, 2, 3, \dots\}),$$

where the equality holds true for all $n \in \mathbb{N} \setminus \{1\}$ only if $f(z)$ is any rotation of the Koebe function:

$$(1.3) \quad K(z) := \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \quad (z \in \mathcal{U}).$$



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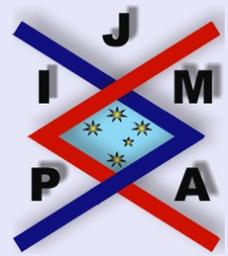


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The assertion (1.2) and its *well-known* (rather classical) analogue for the class \mathcal{K} (cf., e.g., [5, p. 117, Theorem 7]) lead us immediately to known distortion inequalities for the n th derivative of functions in the classes \mathcal{S} and \mathcal{K} , respectively. Each of the following conjectures, which were made in an attempt to extend these known distortion inequalities for the classes \mathcal{S} and \mathcal{K} , involves the *fractional derivative operator* D_z^λ of order λ , defined by (cf., e.g., [7] and [9])

$$(1.4) \quad D_z^\lambda f(z) := \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta & (0 \leq \lambda < 1) \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) & (n \leq \lambda < n+1; n \in \mathbb{N}), \end{cases}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z -plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Conjecture 1. [8, p. 88]. *If the function $f(z)$ is in the class \mathcal{S} , then*

$$(1.5) \quad |D_z^{n+\lambda} f(z)| \leq \frac{(n+\lambda+|z|) \Gamma(n+\lambda+1)}{(1-|z|)^{n+\lambda+2}} \\ (z \in \mathcal{U}; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; 0 \leq \lambda < 1),$$

where the equality holds true for the Koebe function $K(z)$ defined by (1.3).

Conjecture 2. [10, p. 225]. *If the function $f(z)$ is in the class \mathcal{K} , then*

$$(1.6) \quad \left| D_z^{n+\lambda} f(z) \right| \leq \frac{\Gamma(n+\lambda+1)}{(1-|z|)^{n+\lambda+1}}$$

$$(z \in \mathcal{U}; n \in \mathbb{N}_0; 0 \leq \lambda < 1),$$

where the equality holds true for the function $L(z)$ defined by

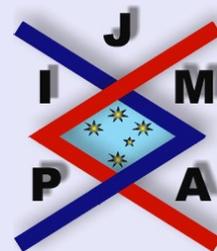
$$(1.7) \quad L(z) := \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \quad (z \in \mathcal{U}).$$

For $\lambda = 0$ and $n \in \mathbb{N}_0$, Conjectures 1 and 2 can easily be validated by means of the aforementioned known distortion inequalities. Each of these conjectures has indeed been proven to be false for $0 < \lambda < 1$ and $n \in \mathbb{N}_0$ (see, for details, [1], [2], and [6]; see also a recent work of Srivastava [11], which presents various *further* developments and generalizations relevant to the aforementioned conjectures). Our main objective in this paper is to give a *direct* proof of the fact that Conjecture 1 is not true for $0 < \lambda < 1$ and $n \in \mathbb{N}_0$. We also derive several further distortion inequalities involving fractional derivatives.

In our present investigation, we shall also make use of the *hypergeometric function* defined by

$$(1.8) \quad F(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

$$(a, b, c \in \mathbb{C}; c \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\}),$$



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where $(\lambda)_k$ denotes the Pochhammer symbol given, in terms of Gamma functions, by

$$(1.9) \quad (\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0) \\ \lambda(\lambda + 1) \dots (\lambda + k - 1) & (k \in \mathbb{N}). \end{cases}$$

The hypergeometric function is analytic in \mathcal{U} and

$$(1.10) \quad F(a, b; c; z) = F(b, a; c; z).$$

Furthermore, it possesses the following integral representation:

$$(1.11) \quad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt$$

$$(\Re(c) > \Re(b) > 0; |\arg(1-z)| \leq \pi - \varepsilon; 0 < \varepsilon < \pi).$$

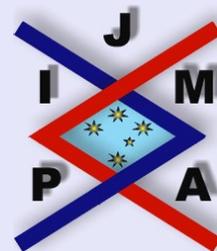
It is easily seen from the definition (1.4) that

$$(1.12) \quad D_z^\lambda \{z^{\mu-1}\} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)} z^{\mu-\lambda-1} \quad (0 \leq \lambda < 1; \mu > 0),$$

so that

$$(1.13) \quad D_z^\lambda \{z^{\mu-1} (1-z)^{-\nu}\} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)} z^{\mu-\lambda-1} F(\mu, \nu; \mu-\lambda; z)$$

$$(0 \leq \lambda < 1; \mu > 0; \nu \in \mathbb{R}; z \in \mathcal{U}).$$



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Thus, for the extremal functions $K(z)$ and $L(z)$ defined by (1.3) and (1.7), respectively, by suitably further specializing the fractional derivative formula (1.13) with $\mu = 2$, we obtain

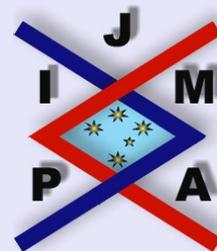
$$(1.14) \quad D_z^\lambda K(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} F(2, 2; 2-\lambda; z)$$

$$(0 \leq \lambda < 1; z \in \mathcal{U})$$

and (cf. [6])

$$(1.15) \quad D_z^\lambda L(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} F(2, 1; 2-\lambda; z)$$

$$(0 \leq \lambda < 1; z \in \mathcal{U}).$$



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2. Main Results Relevant to Conjecture 1

We begin by proving

Theorem 2. *Let $0 < \lambda < 1$. Then Conjecture 1 is not true for $n \in \mathbb{N}$.*

Proof. For $L(z) \in \mathcal{S}$, it follows from (1.15) and the definition (1.8) that

$$(2.1) \quad D_z^\lambda L(z) = z^{-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^k$$

$$(0 < \lambda < 1; z \in \mathcal{U} \setminus \{0\}),$$

where $z^{-\lambda}$ is analytic in $\mathcal{U} \setminus \{0\}$ and the multiplicity of $z^{-\lambda}$ is removed by requiring $\log z$ to be real when $z > 0$. Thus, by the definition (1.4), we have

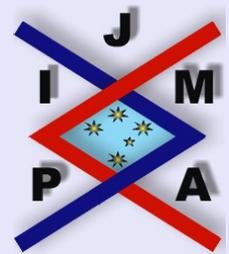
$$(2.2) \quad D_z^{1+\lambda} L(z) = (z^{-\lambda})' \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^k + z^{-\lambda} \left(\sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^k \right)'$$

$$= z^{-1-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda)} z^k \quad (0 < \lambda < 1; z \in \mathcal{U} \setminus \{0\}).$$

By the principle of mathematical induction, it can be shown by using (2.2) that

$$(2.3) \quad D_z^{n+\lambda} L(z) = z^{-n-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda-n+1)} z^k$$

$$= \frac{z^{1-n-\lambda}}{\Gamma(2-n-\lambda)} F(2, 1; 2-n-\lambda; z)$$



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$$(0 < \lambda < 1; n \in \mathbb{N}; z \in \mathcal{U} \setminus \{0\}).$$

Upon setting $z = r$ ($0 < r < 1$) in (2.3), if we let $r \rightarrow 0$, it is easily seen that

$$(2.4) \quad D_z^{n+\lambda} L(z) \Big|_{z=r} \rightarrow \infty \quad (r \rightarrow 0; 0 < \lambda < 1; n \in \mathbb{N}).$$

On the other hand, if Conjecture 1 is true, the claimed assertion (1.5) readily yields

$$(2.5) \quad |D_z^{n+\lambda} L(z)| \leq M(n; \lambda) \quad (|z| \rightarrow 0; 0 < \lambda < 1; n \in \mathbb{N}),$$

where $M(n; \lambda)$ is a (finite) constant depending only on n and λ . This contradiction with (2.4) evidently completes the proof of Theorem 2. \square

Next we prove

Theorem 3. *Let the function $f(z)$ be in the class \mathcal{S} . Then*

$$(2.6) \quad |D_z^\lambda f(z)| \leq \frac{r^{1-\lambda}}{\Gamma(1-\lambda)} \int_0^1 \frac{1+rt}{(1-t)^\lambda (1-rt)^3} dt$$

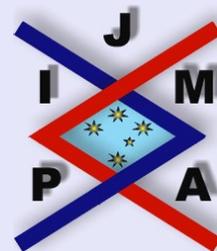
$$(r = |z|; z \in \mathcal{U}; 0 < \lambda < 1),$$

where the equality holds true for the Koebe function $K(z)$ given by (1.3).

Proof. Suppose that the function $f(z) \in \mathcal{S}$ is given by (1.1). Then, by using (1.12) in conjunction with (1.1), we obtain

$$(2.7) \quad D_z^\lambda f(z) = z^{-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} a_k z^k$$

$$(a_1 := 1; 0 < \lambda < 1; z \in \mathcal{U}),$$



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where the multiplicity of $z^{-\lambda}$ is removed as in Theorem 2.

By applying the assertion (1.2) of Theorem 1 on the right-hand side of (2.7), we have

$$\begin{aligned}
 |D_z^\lambda f(z)| &\leq r^{-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} k r^k \\
 (2.8) \qquad &= \frac{r^{1-\lambda}}{\Gamma(2-\lambda)} \sum_{k=0}^{\infty} \frac{(2)_k (1)_k}{(2-\lambda)_k} \frac{(k+1) r^k}{k!} \\
 &= \frac{r^{1-\lambda}}{\Gamma(2-\lambda)} (rF(2, 1; 2-\lambda; r))' \\
 &\quad (r = |z|; z \in \mathcal{U}; 0 < \lambda < 1).
 \end{aligned}$$

Since $0 < 1 < 2 - \lambda$ ($0 < \lambda < 1$), we can make use of the integral representation (1.11), and we thus find that

$$(2.9) \qquad (rF(2, 1; 2-\lambda; r))' = (1-\lambda) \int_0^1 \frac{1+rt}{(1-t)^\lambda (1-rt)^3} dt,$$

which, when substituted for in (2.8), immediately yields the assertion (2.6) of Theorem 3.

Finally, by taking the Koebe function $K(z)$ for $f(z)$ in (2.6), we can see that the result is sharp. \square

Remark 1. Theorem 3 can also be deduced by applying the case $n = 0$ of a known result due to Cho *et al.* [2, p. 120, Theorem 3].

Remark 2. By comparing the assertions (2.6) and (1.5) with $n = 0$, it readily follows that Conjecture 1 is not true also when $n = 0$ and $0 < \lambda < 1$.



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3. A Distortion Inequality Involving the Hypergeometric Function

In this section, we prove a distortion inequality involving the hypergeometric function, which is given by

Theorem 4. *Let the function $f(z)$ be in the class \mathcal{S} . Then*

$$(3.1) \quad \left| D_z^{1+\lambda} f(z) \right| \leq \frac{r^{-\lambda}}{\Gamma(1-\lambda)} (rF(2, 1; 1-\lambda; r))' \\ (r = |z|; z \in \mathcal{U} \setminus \{0\}; 0 < \lambda < 1),$$

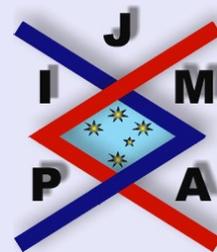
where the equality holds true for the Koebe function $K(z)$ given by (1.3).

Proof. For the function $f(z) \in \mathcal{S}$ given by (1.1), it follows from (2.7) and the definition (1.4) that

$$(3.2) \quad D_z^{1+\lambda} f(z) = z^{-1-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda)} a_k z^k \\ (a_1 := 1; 0 < \lambda < 1; z \in \mathcal{U} \setminus \{0\}),$$

since $z^{-\lambda}$ is analytic in $\mathcal{U} \setminus \{0\}$.

Applying the assertion (1.2) of Theorem 1 once again, we find from (3.2)



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that

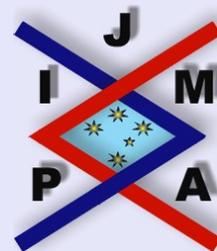
$$\begin{aligned}
 (3.3) \quad |D_z^{1+\lambda} f(z)| &\leq r^{-1-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda)} k r^k \\
 &= \frac{r^{-\lambda}}{\Gamma(1-\lambda)} \sum_{k=0}^{\infty} \frac{(2)_k (1)_k}{(1-\lambda)_k} \frac{(k+1) r^k}{k!} \\
 &= \frac{r^{-\lambda}}{\Gamma(1-\lambda)} (rF(2, 1; 1-\lambda; r))'
 \end{aligned}$$

$$(r = |z|; z \in \mathcal{U} \setminus \{0\}; 0 < \lambda < 1),$$

which proves the inequality (3.1).

By taking the Koebe function $K(z)$ for $f(z)$ in (3.1), we thus complete our direct proof of Theorem 4. \square

Remark 3. The assertion (3.1) of Theorem 4 can also be proven by appealing to the case $n = 1$ of the aforementioned known result due to Cho *et al.* [2, p. 120, Theorem 3].



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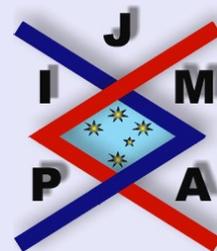
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References

- [1] M.-P. CHEN, H.M. SRIVASTAVA AND C.-S. YU, A note on a conjecture involving fractional derivatives of convex functions, *J. Fract. Calc.*, **5** (1994), 81–85.
- [2] N.E. CHO, S. OWA AND H.M. SRIVASTAVA, Some remarks on a conjectured upper bound for the fractional derivative of univalent functions, *Internat. J. Math. Statist. Sci.*, **2** (1993), 117–125.
- [3] L. DE BRANGES, A proof of the Bieberbach conjecture, *Acta Math.*, **154** (1985), 137–152.
- [4] P.L. DUREN, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.
- [5] A.W. GOODMAN, *Univalent Functions*, Vol. **I**, Polygonal Publishing House, Washington, New Jersey, 1983.
- [6] Y. LING, Y. GAI AND G. BAO, On distortion theorem for the n th order derivatives of starlike and convex functions of order α , *J. Harbin Inst. Tech.*, **3**(4) (1996), 5–8.
- [7] S. OWA, On the distortion theorems. I, *Kyungpook Math. J.*, **18** (1978), 53–59.
- [8] S. OWA, K. NISHIMOTO, S.K. LEE AND N.E. CHO, A note on certain fractional operator, *Bull. Calcutta Math. Soc.*, **83** (1991), 87–90.



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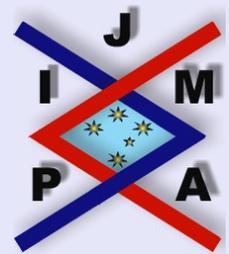
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- [9] S. OWA AND H.M. SRIVASTAVA, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, **39** (1987), 1057–1077.
- [10] S. OWA AND H.M. SRIVASTAVA, A distortion theorem and a related conjecture involving fractional derivatives of convex functions, in *Univalent Functions, Fractional Calculus, and Their Applications* (H.M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1989, 219–228.
- [11] H.M. SRIVASTAVA, Certain conjectures and theorems involving the fractional derivatives of analytic and univalent functions, in *Analysis and Topology* (C.A. Cazacu, O.E. Lehto, and Th. M. Rassias, Editors), World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1998, 653–676.
- [12] H.M. SRIVASTAVA AND S. OWA (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.



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