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MONOTONIC REFINEMENTS OF A KY FAN INEQUALITY

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ABSTRACT. It is well-known that inequalities between means play a very important role in many branches of mathematics. Please refer to [1, 3, 7], etc. The main aims of the present article are:

(i) to show that there are monotonic and continuous functions $H(t),\ K(t),\ P(t)$ and $Q(t)$ on $[0,1]$ such that for all $t \in [0,1]$,

$$H_n \leq H(t) \leq G_n \leq K(t) \leq A_n \quad \text{and}$$

$$H_n/(1-H_n) \leq P(t) \leq G_n/G'_n \leq Q(t) \leq A_n/A'_n,$$

where A_n, G_n and H_n are respectively the weighted arithmetic, geometric and harmonic means of the positive numbers $x_1, x_2, ..., x_n$ in $(0, 1/2]$, with positive weights $\alpha_1, \alpha_2, ..., \alpha_n$; while A'_n and G'_n are respectively the weighted arithmetic and geometric means of the numbers $1-x_1,\ 1-x_2, ..., 1-x_n$ with the same positive weights $\alpha_1, \alpha_2, ..., \alpha_n$;

(ii) to present more general monotonic refinements for the Ky Fan inequality as well as some inequalities involving means; and

(iii) to present some generalized and new inequalities in this connection.

Key words and phrases: Ky Fan inequality, monotonic refinements of inequalities, arithmetic, geometric and harmonic means.

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1. INTRODUCTION

Let n be a positive integer. To two given sequences of positive numbers $x_1,\ x_2, \ldots, x_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, we denote by $A_n,\ G_n$ and H_n respectively the weighted arithmetic, geometric and harmonic means, that is,

$$A_n = \sum_{i=1}^{n} \alpha_i x_i,$$

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$$G_n = \prod_{i=1}^{n} x_i^{\alpha_i},$$

$$H_n = \left(\sum_{i=1}^{n} \alpha_i / x_i \right)^{-1}.$$

We use the symbols $\mathbf{a}_n$, $\mathbf{g}_n$ and $\mathbf{h}_n$ to denote the corresponding unweighted arithmetic, geometric and harmonic means of the n positive numbers $x_1, x_2, \ldots, x_n$. The following well-known inequality has been proved, using many different methods: (Please refer to [3].)

$$H_n \leq G_n \leq A_n \tag{1.1}$$

Let the real numbers x_i be such that $0 < x_i \leq 1/2$, for all $i = 1, 2, \ldots, n$. We denote by A'_n, G'_n and H'_n the weighted arithmetic, geometric and harmonic means of the numbers $1 - x_1, 1 - x_2, \ldots, 1 - x_n$, namely,

$$A'_n = \sum_{i=1}^{n} \alpha_i \left(1 - x_i\right),$$

$$G'_n = \prod_{i=1}^{n} \left(1 - x_i\right)^{\alpha_i},$$

$$H'_n = \left(\sum_{i=1}^{n} \alpha_i / \left(1 - x_i\right) \right)^{-1}.$$

Also, let $\mathbf{a}'_n$, $\mathbf{g}'_n$ and $\mathbf{h}'_n$ denote the corresponding unweighted arithmetic, geometric and harmonic means of the numbers $1 - x_1, 1 - x_2, \ldots, 1 - x_n$ respectively. In recent years many interesting inequalities involving these mean values have been published, in particular, the following well-known Ky Fan and Wang-Wang inequalities :

$$\frac{H_n}{H'_n} \leq \frac{G_n}{G'_n} \leq \frac{A_n}{A'_n} \tag{1.2}$$

with equality holding if and only if $x_1 = \cdots = x_n$. Please refer to the following papers by H. Alzer [1] – [2] and Wang-Wang [10] or [7], etc. The right-hand inequality of (1.2) is the famous Ky Fan inequality; the left-hand inequality for the unweighted case was first discovered by Wang-Wang in 1984 [10]. The main purpose of this paper is to present some new monotonic continuous functions $H(\lambda), K(\lambda), P(\lambda)$ and $Q(\lambda)$ on $[0, 1]$ such that

$$H_n \leq H(\lambda) \leq G_n \leq K(\lambda) \leq A_n$$

and

$$H_n/(1 - H_n) \leq P(\lambda) \leq G_n/G'_n \leq Q(\lambda) \leq A_n/A'_n.$$

In fact, our theorems here generalize results of Wang and Yang in [9] and a theorem of K.M. Chong [6]. In Section 2, we shall generalize refinements of inequalities between means. In Section 3, we shall present generalizations to refinements of the Ky Fan inequality, with some new inequalities deduced. Finally, in Section 4, we shall show that Theorem 3.1 can be used to deduce many other established refinements of the Ky Fan inequality.

In a recent paper of Wang and Yang [9], the following two interesting theorems were put forward. In fact, they are refinements of inequalities (1.1) and (1.2) in Section 1, for the discrete unweighted case. They are restated here without proof. For the details of the proof, please refer to [9].

Theorem 1.1. *Given a sequence* $\{ x_1, x_2, \ldots, x_n \}$ *of positive numbers, which are not all equal:*

(a) *For any* t *in* $[0,1/n]$, *let*

$$h(t)=\prod_{i=1}^{n}\left[\frac{1}{x_i}+t\sum_{j=1}^{n}\left(\frac{1}{x_j}-\frac{1}{x_i}\right)\right]^{-1/n} \tag{1.3}$$

Then, $h(t)$ *is continuous, strictly decreasing and* $\mathbf{h}_n=h(1/n)\le h(t)\le h(0)=\mathbf{g}_n$ *on* $[0,1/n]$.

(b) *For any* t *in* $[0,1/n]$, *let*

$$k(t)=\prod_{i=1}^{n}\left[x_i+t\sum_{j=1}^{n}(x_j-x_i)\right]^{1/n} \tag{1.4}$$

Then, $k(t)$ *is continuous, strictly increasing and* $\mathbf{g}_n=k(0)\le k(t)\le k(1/n)=\mathbf{a}_n$ *on* $[0,1/n]$.

Theorem 1.2. *Given a sequence* $\{x_1,x_2,\ldots,x_n\}$ *with* x_i *in* $(0,1/2]$, $i=1,2,\ldots,n$, *which are not all equal:*

(a) *For any* t *in* $[0,1/n]$, *let*

$$p(t)=\prod_{i=1}^{n}\left[\frac{1}{x_i}+t\sum_{j=1}^{n}\left(\frac{1}{x_j}-\frac{1}{x_i}\right)-1\right]^{-1/n} \tag{1.5}$$

Then, $p(t)$ *is continuous, strictly decreasing, and* $\mathbf{h}_n/(1-\mathbf{h}_n)=p(1/n)\le p(t)\le p(0)=\mathbf{g}_n/\mathbf{g}_n'$ *on* $[0,1/n]$.

(b) *For any* t *in* $[0,1/n]$, *let*

$$q(t)=\frac{\prod\limits_{i=1}^{n}\left[x_i+t\sum\limits_{j=1}^{n}(x_j-x_i)\right]^{1/n}}{\prod\limits_{i=1}^{n}\left[1-x_i-t\sum\limits_{j=1}^{n}(x_j-x_i)\right]^{1/n}} \tag{1.6}$$

Then, $q(t)$ *is continuous, strictly increasing and* $\mathbf{g}_n/\mathbf{g}_n'=q(0)\le q(t)\le q(1/n)=\mathbf{a}_n/\mathbf{a}_n'$ *on* $[0,1/n]$.

2. SOME GENERALIZATIONS

In this section, we are going to present and prove a generalization of Theorem 1.1 and Theorem 1.2(a), in particular, to the case for weighted means. Its statement runs as follows:

Theorem 2.1. *Let* $a_1,a_2,\ldots,a_n$, *and* $\alpha_1,\alpha_2,\ldots,\alpha_n$ *be two sequences of positive numbers, with* a_i *not all equal and* $\sum_{i=1}^{n}\alpha_i=1$. *Let* a *be any positive number such that* $A_n\le a$, *where* $A_n=\sum_{i=1}^{n}\alpha_i a_i$, *and* k *is a constant such that* $k<a_i$, *for all* $i=1,2,\ldots,n$. *Let*

$$F(\lambda)=\prod_{i=1}^{n}\left[\lambda a+(1-\lambda)a_i-k\right]^{\alpha_i} \tag{2.1}$$

$$G(\lambda)=\prod_{i=1}^{n}\left[\lambda a+(1-\lambda)a_i-k\right]^{-\alpha_i} \tag{2.2}$$

Then,

(i) $F(\lambda)$ *is continuous and strictly increasing on* $[0,1]$;

(ii) $G(\lambda)$ *is continuous and strictly decreasing on* $[0,1]$.

Proof. (i) Taking the logarithm of $F(\lambda)$, we have,

$$\ln F(\lambda) = \sum_{i=1}^{n} \alpha_i \ln [\lambda a + (1-\lambda) a_i - k]$$

Differentiating the last expression with respect to λ, we have:

$$\frac{F'(\lambda)}{F(\lambda)} = \sum_{i=1}^{n} \frac{\alpha_i (a - a_i)}{[\lambda a + (1-\lambda) a_i - k]}$$

Differentiating again, we obtain:

$$\left[\frac{F'(\lambda)}{F(\lambda)}\right]' = -\sum_{i=1}^{n} \frac{\alpha_i (a - a_i)^2}{[\lambda a + (1-\lambda) a_i - k]^2} < 0 \tag{2.3}$$

for all λ in $[0,1]$, as the a_i are not all equal. Hence, $F'(\lambda)/F(\lambda)$ is strictly decreasing on $[0,1]$. Also, as $A_n \leq a$ and $k < a$, we have :

$$\frac{F'(1)}{F(1)} = \sum_{i=1}^{n} \frac{\alpha_i (a - a_i)}{a - k} = \frac{a - A_n}{a - k} \geq 0 \tag{2.4}$$

Therefore, $F'(\lambda)/F(\lambda) > 0,$ for all λ in $[0,1).$ As $F(\lambda)$ is positive for all λ in $[0,1]$, $F'(\lambda) > 0$ for λ in $[0,1)$. Hence, $F(\lambda)$ is strictly increasing on $[0,1]$. The continuity of $F(\lambda)$ on $[0,1]$ is obvious.

(ii) As $F(\lambda)$ is positive for all λ in $[0,1]$ and $G(\lambda) = 1/F(\lambda),$ $G(\lambda)$ is continuous and strictly decreasing on $[0,1]$. Hence, the proof of Theorem 2.1 is complete.

□

Now, we use Theorem 2.1 to deduce some established theorems.

Remark 2.2. (i) From Theorem 2.1, we have, for all $\lambda \in (0,1)$,

$$F(0) < F(\lambda) < F(1),$$

which yields for not all equal a_i,

$$\prod_{i=1}^{n} (a_i - k)^{\alpha_i} < a - k. \tag{2.5}$$

In particular, if $a = A_n$, for not all equal a_i we have,

$$\prod_{i=1}^{n} (a_i - k)^{\alpha_i} < A_n - k, \tag{2.6}$$

which is a generalization of the weighted arithmetic-geometric means inequality.

(ii) Again, in Theorem 2.1, we let $k = 0, a = \sum_{i=1}^{n} \alpha_i a_i = A_n$. Then, $F(\lambda)$ will reduce to, say

$$K(\lambda) = \prod_{i=1}^{n} [\lambda A_n + (1-\lambda) a_i]^{\alpha_i} \tag{2.7}$$

It is clear that $K(\lambda)$ is continuous and strictly increasing on $[0,1]$, and for all $\lambda \in (0,1)$,

$$K(0) = G_n < K(\lambda) < K(1) = A_n. \tag{2.8}$$

This is a refinement of the weighted arithmetic-geometric means inequality.

(iii) Furthermore, if we put $\lambda = nt, \alpha_i = \frac{1}{n}, i = 1, 2, \ldots, n$, into $K(\lambda)$, we obtain for all $t \in [0, 1/n]$,

$$K(nt) = \prod_{i=1}^{n} [ntA_n + (1-nt)a_i]^{1/n} = \prod_{i=1}^{n} \left[a_i + t \sum_{j=1}^{n} (a_j - a_i) \right]^{1/n}.$$

The last expression is in fact the function $k(t)$ of Theorem 1.1(b). Hence, we have shown that Theorem 1.1(b) is a particular case of Theorem 2.1.

Remark 2.3. If, in Theorem 2.1, we let $k = 0,\ a_i = \frac{1}{x_i},\ i = 1, 2, \ldots, n,\ a = \frac{1}{H_n} = \frac{\alpha_1}{x_1} + \cdots + \frac{\alpha_n}{x_n}$, then $G(\lambda)$ will reduce to,

$$H(\lambda) = \prod_{i=1}^{n} \left[\frac{\lambda}{H_n} + (1-\lambda)\frac{1}{x_i} \right]^{-\alpha_i}. \tag{2.9}$$

Then, $H(\lambda)$ is continuous and strictly decreasing on $[0, 1]$, and for all $\lambda \in (0, 1)$,

$$H(1) = H_n < H(\lambda) < H(0) = G_n. \tag{2.10}$$

(2.10) is a refinement of the weighted means inequality. Furthermore, if we put $\lambda = nt, \alpha_i = \frac{1}{n}$, $a_i = 1/x_i,\ i = 1, 2, \ldots, n,$ into $H(\lambda)$, we obtain for all t in $[0, 1/n]$,

$$H(nt) = \prod_{i=1}^{n} \left[nta + (1-nt)\frac{1}{x_i} \right]^{-1/n} = \prod_{i=1}^{n} \left[\frac{1}{x_i} + t \sum_{j=1}^{n} \left(\frac{1}{x_j} - \frac{1}{x_i} \right) \right]^{-1/n}$$

This is the function $h(t)$ in Theorem 1.1(a). Hence, we have deduced Theorem 1.1(a) as a particular case of Theorem 2.1.

Theorem 2.4. *Let $x_1, x_2, \ldots, x_n$ be n positive numbers, not all equal, with $x_i \in (0, 1/2]$ for all $i = 1, 2, .., n$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the corresponding weights, i.e., $\alpha_i > 0,\ i = 1, 2, \ldots, n$ and $\alpha_1 + \cdots + \alpha_n = 1$. Let γ be a constant such that $\gamma < \frac{1}{x_i}$, for all $i = 1, 2, \ldots, n$. We define $P(\lambda)$ as :*

$$P(\lambda) = \prod_{i=1}^{n} \left[\frac{\lambda}{H_n} + (1-\lambda)\frac{1}{x_i} - \gamma \right]^{-\alpha_i} \tag{2.11}$$

Then,

(i) *$P(\lambda)$ is continuous and strictly decreasing on $[0, 1]$;*

(ii) *for all $\lambda \in (0, 1)$, we have,*

$$P(1) = \frac{H_n}{1 - \gamma H_n} < P(\lambda) < P(0) = \prod_{i=1}^{n} \left(\frac{x_i}{1 - \gamma x_i} \right)^{\alpha_i}. \tag{2.12}$$

Proof. (i) $P(\lambda)$ is continuous and strictly decreasing on $[0, 1]$, as we get $P(\lambda)$ from the continuous and strictly decreasing function $G(\lambda)$, by putting $k = \gamma$, $a_i = 1/x_i$ with $x_i \in (0, 1/2],\ i = 1, 2, \ldots, n, a = 1/H_n = \alpha_1/x_1 + \cdots + \alpha_n/x_n$ into $G(\lambda)$ of Theorem 2.1.

(ii) We have : $P(0) = G(0) = \prod_{i=1}^{n} \left(\frac{x_i}{1-\gamma x_i} \right)^{\alpha_i}$,

$$P(1) = G(1) = H_n/(1 - \gamma H_n).$$

Hence, for all $\lambda \in (0, 1)$,

$$H_n/(1 - \gamma H_n) < P(\lambda) < \prod_{i=1}^{n} \left(\frac{x_i}{1 - \gamma x_i} \right)^{\alpha_i}$$

This completes the proof of Theorem 2.4. □

Remark 2.5. If we put $\lambda = nt$, $\alpha_i = 1/n,\ i = 1, 2, \ldots, n$, $\gamma = 1$ into $P(\lambda)$ of Theorem 2.4, we obtain for any t in $[0, 1/n]$,

$$P(nt) = \prod_{i=1}^{n} \left[\frac{nt}{\mathbf{h}_n} + (1 - nt)\, a_i - 1\right]^{-1/n} = \prod_{i=1}^{n} \left[\frac{1}{x_i} + t \sum_{j=1}^{n} \left(\frac{1}{x_j} - \frac{1}{x_i}\right) - 1\right]^{-1/n}$$

This is the function $p(t)$ of Theorem 1.2(a), and we have deduced Theorem 1.2(a) as a particular case of Theorem 2.1.

The only part in Section 1, which is not yet dealt with, is Theorem 1.2(b). Its proof is postponed to the next section, with some additional theorems. We end this section by considering another similar theorem. In [6], K.M. Chong presented the following theorem:

Theorem 2.6. *Let $a_1, a_2, \ldots, a_n$ be positive numbers and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be their corresponding weights, i.e. $\alpha_i > 0,\ i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let $f(\lambda)$ be defined as:*

$$f(\lambda) = \prod_{i=1}^{n} \left[\lambda \sum_{j=1}^{n} \alpha_j a_j + (1 - \lambda)\, a_i\right]^{\alpha_i} \tag{2.13}$$

Then $f(\lambda)$ is a strictly increasing function of λ for $\lambda \in [0, 1]$, unless $a_1 = a_2 = \cdots = a_n$; in which case $f(0) = G_n = A_n = f(1)$.

Proof. It is obvious that when $k = 0$ and $a = A_n$ in Theorem 2.1 we obtain K.M. Chong's theorem at once. □

3. MONOTONIC REFINEMENTS OF THE KY FAN INEQUALITY

In the previous section, we have seen that Theorem 2.1 is a generalization of various theorems. In this section, we shall present a refinement of the well-known Ky Fan inequality, which is a generalization of Theorem 1.2(b).

Theorem 3.1. *Let $x_1, x_2, \ldots, x_n$ be n positive numbers, not all equal, such that $x_i \in (0, 1/2]$ for all $i = 1, 2, \ldots, n$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be their corresponding weights i.e. $\alpha_i > 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let β and δ be two constants such that $\delta \le \beta$ and $\beta < x_i$, for all $i = 1, 2, \ldots, n$. Let $r(\lambda)$ be defined as :*

$$r(\lambda) = \frac{\prod_{i=1}^{n} [\lambda z + (1 - \lambda) x_i - \beta]^{\alpha_i}}{\prod_{i=1}^{n} [\lambda(1 - z) + (1 - \lambda)(1 - x_i) - \delta]^{\alpha_i}} \tag{3.1}$$

for any $\lambda \in [0, 1]$, and any real number z such that $\sum_{i=1}^{n} \alpha_i x_i \le z \le 1/2$.
Then,

(i) *$r(\lambda)$ is continuous and strictly increasing on $[0, 1]$; and*
(ii) *when $\beta = \delta = 0$, we have, for all $\lambda \in (0, 1)$,*

$$G_n / G_n' = r(0) < r(\lambda) < r(1) = \frac{z}{1 - z}. \tag{3.2}$$

Proof. (i) Taking the logarithm, we have,

$$\ln\{r(\lambda)\} = \sum_{i=1}^{n} \alpha_i \ln[\lambda z + (1 - \lambda) x_i - \beta] - \sum_{i=1}^{n} \alpha_i \ln[\lambda(1 - z) + (1 - \lambda)(1 - x_i) - \delta]$$

Differentiating with respect to λ, we have :

$$\begin{aligned}\frac{r'(\lambda)}{r(\lambda)} &= \sum_{i=1}^{n} \alpha_i \frac{z - x_i}{\lambda z + (1-\lambda)x_i - \beta} - \sum_{i=1}^{n} \alpha_i \frac{(1-z) - (1-x_i)}{\lambda(1-z) + (1-\lambda)(1-x_i) - \delta} \\ &= \sum_{i=1}^{n} \alpha_i \frac{z - x_i}{\lambda z + (1-\lambda)x_i - \beta} + \sum_{i=1}^{n} \alpha_i \frac{z - x_i}{\lambda(1-z) + (1-\lambda)(1-x_i) - \delta}.\end{aligned}$$

Let $u(\lambda) = \dfrac{r'(\lambda)}{r(\lambda)}$.

We are going to show that $\ln\{r(\lambda)\}$ and hence $r(\lambda)$ are both strictly increasing by showing that $u(\lambda) > 0$ for all $\lambda \in [0, 1)$.

Differentiating $u(\lambda)$ with respect to λ, we have :

$$u'(\lambda) = -\sum_{i=1}^{n} \frac{\alpha_i (z - x_i)^2}{[\lambda z + (1-\lambda)x_i - \beta]^2} + \sum_{i=1}^{n} \frac{\alpha_i (z - x_i)^2}{[\lambda(1-z) + (1-\lambda)(1-x_i) - \delta]^2} < 0$$

as

$$\frac{1}{[\lambda z + (1-\lambda)x_i - \beta]^2} > \frac{1}{[\lambda(1-z) + (1-\lambda)(1-x_i) - \delta]^2}$$

$i = 1, 2, \ldots, n$, unless $z = x_1 = x_2 = \ldots = x_n = 1/2$, and $\beta = \delta$.

Hence $u(\lambda)$ is strictly decreasing on $[0, 1]$.

$$\begin{aligned}u(1) &= \sum_{i=1}^{n} \alpha_i \frac{z - x_i}{z - \beta} + \sum_{i=1}^{n} \alpha_i \frac{z - x_i}{1 - z - \delta} \\ &= \frac{1}{z - \beta} \sum_{i=1}^{n} \alpha_i (z - x_i) + \frac{1}{1 - z - \delta} \sum_{i=1}^{n} \alpha_i (z - x_i) \\ &= \left(z - \sum_{i=1}^{n} \alpha_i x_i\right) \frac{1 - \beta - \delta}{(z - \beta)(1 - z - \delta)} \geq 0, \qquad \text{for } \sum_{i=1}^{n} \alpha_i x_i \leq z \leq 1/2.\end{aligned}$$

Hence, $u(\lambda) = \frac{r'(\lambda)}{r(\lambda)} > 0$, for all $\lambda \in [0, 1)$.

As $r(\lambda)$ is always positive, we have $r'(\lambda) > 0$ for all $\lambda \in [0, 1)$ and $r(\lambda)$ is strictly increasing on $[0, 1]$.

(ii) It is easy to see that when $\beta = \delta = 0$, $r(0) = G_n/G'_n,\ r(1) = \frac{z}{1-z}$ and $r(0) < r(\lambda) < r(1)$ on $(0, 1)$.

□

It is remarked, that if $\beta = \delta = 0$, $z = \sum_{i=1}^{n} \alpha_i x_i$ in Theorem 3.1, then the chain of inequalities in (3.2), with $r(\lambda)$ replaced by $Q(\lambda)$, will become : for any $\lambda \in (0, 1)$,

$$\frac{G_n}{G'_n} = Q(0) < Q(\lambda) < Q(1) = \frac{z}{1-z} = \frac{A_n}{A'_n} \tag{3.3}$$

This is a refinement of the Ky Fan inequality.

Remark 3.2. (3.3) is a refinement of the weighted Ky Fan inequality and we have $r(0) < r(\lambda) < r(1)$, unless $x_1 = x_2 = \ldots = x_n$. In general, (3.2) yields a generalization of the Ky Fan inequality as follows :

For $A_n \le z \le 1/2$ and $\delta \le \beta < x_i \in (0, 1/2]$, for all $i = 1, 2, \ldots, n$, we have,

$$r(0) = \frac{\prod_{i=1}^{n} [x_i - \beta]^{\alpha_i}}{\prod_{i=1}^{n} [1 - x_i - \delta]^{\alpha_i}} \le \frac{z - \beta}{1 - z - \delta} = r(1), \tag{3.4}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

If in (3.4) we let $z = A_n$ the weighted arithmetic mean of $x_1, x_2, \ldots, x_n$, we obtain a generalization of the weighted Ky Fan inequality :

$$\frac{\prod_{i=1}^{n} [x_i - \beta]^{\alpha_i}}{\prod_{i=1}^{n} [1 - x_i - \delta]^{\alpha_i}} \le \frac{A_n - \beta}{A_n' - \delta}, \tag{3.5}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Remark 3.3. If we put $\alpha_i = 1/n$, $i = 1, 2, \ldots, n$, $z = \sum_{i=1}^{n} \alpha_i x_i$, $\beta = \delta = 0$ and $\lambda = nt$ into (3.1), we then obtain after simplification,

$$\begin{aligned} r(nt) &= \frac{\prod_{i=1}^{n} \left[nt \sum_{j=1}^{n} \frac{1}{n} x_j + (1 - nt) x_i \right]^{1/n}}{\prod_{i=1}^{n} \left[nt \left(1 - \sum_{j=1}^{n} \frac{1}{n} x_j \right) + (1 - nt)(1 - x_i) \right]^{1/n}} \\ &= \frac{\prod_{i=1}^{n} \left[x_i + t \sum_{j=1}^{n} (x_j - x_i) \right]^{1/n}}{\prod_{i=1}^{n} \left[1 - x_i - t \sum_{j=1}^{n} (x_j - x_i) \right]^{1/n}} = q(t), \qquad \text{for } t \in [0, 1/n]. \end{aligned}$$

This is the function $q(t)$ in Theorem 1.2(b), showing that Theorem 3.1 is a generalization of Theorem 1.2(b).

Remark 3.4. In [5], we have the following theorem, which can be easily seen to follow as a particular case of Theorem 3.1, when $\beta = 0$ and $\delta = 0$:

Theorem 3.5. *Let $x_1, x_2, \ldots, x_n$ be n positive numbers, such that $x_i \in (0, 1/2]$, for all $i = 1, 2, \ldots, n$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be their corresponding positive weights, with $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Let z be a constant such that $A_n \le z \le 1/2$, where $A_n = \sum_{i=1}^{n} \alpha_i x_i$. We define $w(\lambda)$, for any $\lambda \in [0, 1]$, to be the function :*

$$w(\lambda) = \frac{\prod_{i=1}^{n} [\lambda z + (1 - \lambda) x_i]^{\alpha_i}}{\prod_{i=1}^{n} [\lambda (1 - z) + (1 - \lambda)(1 - x_i)]^{\alpha_i}}. \tag{3.6}$$

Then,

(i) $w(\lambda)$ *is continuous and strictly increasing on* $[0, 1]$, *unless* $x_1 = x_2 = \cdots = x_n$;

(ii) $G_n / G_n' = w(0) \le w(\lambda) \le w(1) = \frac{z}{1 - z}$, *for* $\lambda \in [0, 1]$.

4. SECOND PROOF OF THEOREM 2.1

In this section, we shall show that Theorem 3.1 is not only a generalization of Theorem 1.2(b), but also it can be used to deduce some elementary theorems.

Second proof of Theorem 2.1. Suppose the n numbers $x_1, x_2, \dots, x_n$ are not all equal.

For $x_i \in (0, 1/2],\ i = 1, 2, \dots, n$, z lying between $\sum_{i=1}^{n} \alpha_i x_i$ and $1/2$, the function $r(\lambda)$ of Theorem 3.1 is strictly increasing on $[0,1]$, where for $\lambda \in [0,1]$, $r(\lambda)$ is defined as :

$$r(\lambda) = \frac{\prod_{i=1}^{n} [\lambda z + (1-\lambda)x_i - \beta]^{\alpha_i}}{\prod_{i=1}^{n} [\lambda(1-z) + (1-\lambda)(1-x_i) - \delta]^{\alpha_i}}. \tag{4.1}$$

Now, we put $x_i = \frac{a_i}{l},\ i = 1, 2, \dots, n$, where l is a large positive number, and let $z = \frac{a}{l},\ \beta = \frac{k}{l}$ with $\delta = \beta$. Then, the function $r(\lambda)$ becomes :

$$r(\lambda) = \frac{\frac{1}{l} \prod_{i=1}^{n} [\lambda a + (1-\lambda)a_i - k]^{\alpha_i}}{\prod_{i=1}^{n} \left[\lambda(1 - \frac{a}{l}) + (1-\lambda)(1 - \frac{a_i}{l}) - \beta\right]^{\alpha_i}}. \tag{4.2}$$

By Theorem 3.1,

$$v(\lambda) = \frac{\prod_{i=1}^{n} [\lambda a + (1-\lambda)a_i - k]^{\alpha_i}}{\prod_{i=1}^{n} \left[\lambda(1 - \frac{a}{l}) + (1-\lambda)(1 - \frac{a_i}{l}) - \frac{k}{l}\right]^{\alpha_i}} \tag{4.3}$$

is strictly increasing as λ increases from 0 to 1.

We let l tend to $+\infty$, the denominator tends to 1 and we have shown that the function in Theorem 2.1

$$F(\lambda) = \prod_{i=1}^{n} [\lambda a + (1-\lambda)a_i - k]^{\alpha_i} \tag{4.4}$$

is an increasing function on $[0,1]$.

Differentiation calculations as in Theorem 2.1 easily reveal that in fact $F(\lambda)$ is strictly increasing on $[0,1]$. This completes the proof of Theorem 2.1. □

Remark 4.1. From the discussions in Section 2, it can be seen that Theorem 2.1 generalizes Theorem 1.1, Theorem 1.2(a), Theorem 2.4 and Theorem 2.6. From the discussions of the last two sections, it can be seen that Theorem 3.1 generalizes Theorem 1.2(b), Theorem 3.5 and Theorem 2.1. As a whole, we have shown that Theorem 3.1 is a generalization of all other refinements of inequalities (1.1) and (1.2), appearing in this paper.

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