

# Journal of Inequalities in Pure and Applied Mathematics

## SUB-SUPER SOLUTIONS AND THE EXISTENCE OF EXTREMAL SOLUTIONS IN NONCOERCIVE VARIATIONAL INEQUALITIES

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©2000 [Victoria University](#)  
ISSN (electronic): 1443-5756  
033-00



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volume 2, issue 2, article 20,  
2001.

*Received 13 September, 2000;  
accepted 28 February, 2001.*

*Communicated by: [A.M. Rubinov](#)*

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

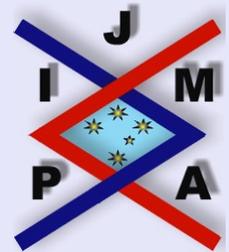
The paper is concerned with the solvability of variational inequalities that contain second-order quasilinear elliptic operators and convex functionals. Appropriate concepts of sub- and supersolutions (for inequalities) are introduced and existence of solutions and extremal solutions are discussed.

*2000 Mathematics Subject Classification:* 35J25, 35J60, 35J65, 35J85

*Key words:* Subsolution, Supersolution, Extremal solution, Variational inequality

## Contents

|   |                              |    |
|---|------------------------------|----|
| 1 | Introduction .....           | 3  |
| 2 | General Settings .....       | 8  |
| 3 | Main Existence Results ..... | 14 |
| 4 | Some Examples .....          | 25 |
|   | 4.1 .....                    | 25 |
|   | 4.2 .....                    | 29 |
|   | References                   |    |



---

### Sub- Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 37

# 1. Introduction

We are concerned in this paper with the existence of solutions and extremal solutions of noncoercive variational inequalities of the form:

$$(1.1) \quad \begin{cases} \langle L(u), v - u \rangle - \langle G(u), v - u \rangle + j(v) - j(u) \geq 0, \quad \forall v \in W_0^{1,p}(\Omega) \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

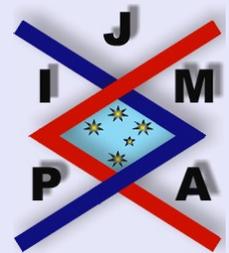
Here  $\Omega$  is a bounded region in  $\mathbb{R}^N$  with smooth boundary.  $L$  is (the weak form of) the second order quasi-linear elliptic operator

$$(1.2) \quad - \sum_{i=1}^N \frac{\partial}{\partial x_i} [A_i(x, u, \nabla u)] + A_0(x, u, \nabla u)$$

and  $G$  is the lower-order term (cf. (2.1) and (2.6)).  $j$  is a convex functional, representing obstacles or unilateral conditions imposed on the solutions. Depending on the choice of  $j$ , the variational inequality (1.1) is the weak form of an equation or a complementarity problem that contains the operator (1.2) with various types of free boundaries or constraints (cf. e.g. [14, 3, 12]).

Since  $G$  may have superlinear growth, the operator  $L - G$  is noncoercive in general. The solvability of problem (1.1) can be studied by several approaches, for example, bifurcation methods (cf. [16, 27, 29, 19], etc.), recession arguments ([4, 2, 28, 1, 21, 22, 20], etc.), variational approaches ([24, 32, 21], etc.), or topological/fixed point methods ([30, 31], etc.).

We are concerned here with another way to study the solvability of (1.1), based on sub- and supersolutions. Recession methods have been quite popular



---

**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

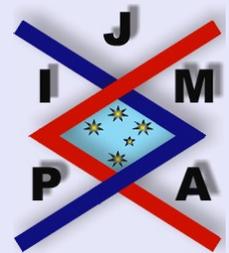
Quit

Page 3 of 37

recently in studying noncoercive problems. There are essential differences between these two approaches. Following recession approaches, the solvability of the problem is usually established by assuming conditions on asymptotic behaviors (i.e., behaviors when the involved variables are large) of the lower order terms (e.g.,  $G$  in problem (1.1)). Problems of type (1.1) have been investigated in detail by recession arguments in [21, 22]. Improvements on the existence results based on recession approaches in [2, 28, 1, 21, 22] were presented recently in [20].

Compared to other methods, the sub-supersolution approach when applicable (i.e, when sub- and supersolutions exist) usually permits more flexible requirements on the growth rate of the perturbing term  $G$  (normally, one only needs to know the behaviors of  $G$  on bounded intervals). Moreover, based on the lattice structure of the space  $W^{1,p}(\Omega)$ , the sub- and supersolution method could also give insight into the ordering properties of the solution set between the sub- and supersolutions, and especially, the existence of maximal and minimal solutions. We refer the reader to [26] or [18] for more discussions on the difficulties arising when the sub-supersolution method is extended from equations (with natural symmetric structure) to variational inequalities (without symmetric settings), together with advantages of the method. More remarks on our approach here for (1.1), compared with the recession approach, are given in Remark 4.1.

This paper is the next step of our study plan proposed in [18] on sub- supersolution methods applied to variational inequalities. In that paper, we consider inequalities on closed convex sets, that is the particular case where  $j$  is the indi-




---

**Sub- Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

Quit

Page 4 of 37

cator function of a closed convex set  $K$ :

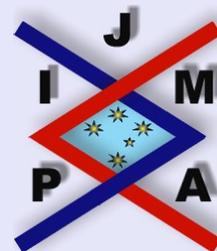
$$j(u) = \begin{cases} 0 & \text{if } u \in K \\ \infty & \text{if } u \notin K. \end{cases}$$

However, many interesting problems in mechanics and applied mathematics lead to other types of convex functionals, for example,

$$j(u) = \int_{\Omega} \psi(x, u(x)) dx \quad \text{or} \quad j(u) = \int_{\partial\Omega} \psi(x, u|_{\partial\Omega}(x)) dS,$$

(cf. [12, 11]). Because of the nonsymmetric nature of the problem, sub- supersolution methods for smooth equations (cf. e.g. [13], [10], [9], or [15]) and also the arguments in [18] for inequalities on convex sets are not directly applicable to (1.1). The goal of this paper is to study the variational inequality (1.1) with more generality on the convex functional  $j$  by a sub- supersolution approach. The main difficulty we face here is defining sub- and supersolutions for the inequality (1.1) in an appropriate way such that the truncation–penalization machinery used for smooth, symmetric equations and for inequalities on convex sets can be extended to our nonsmooth, nonsymmetric case. Basically, we need to define sub- and supersolutions of (1.1) such that: (i) under reasonable conditions, one can show the existence of solutions and extremal solutions between sub- and supersolutions, (ii) there is some way to find sub- supersolutions or to check whether a given function is a sub- or supersolution, and (iii) sub- and supersolutions in inequalities extend those in equations.

To meet these requirements, in the next section, we need to make non straight-forward extensions on the usual sub- and supersolution concepts for equations




---

**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

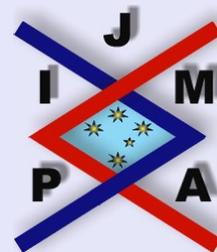
Close

Quit

Page 5 of 37

and also on those presented in [18] to the more general situation of inequality (1.1) (cf. Definition 2.1). In Section 3, we prove several existence results for the inequality (1.1) based on the sub- supersolution concepts in Section 2. It is shown in Theorems 3.1 and 3.3 that if there exist a subsolution and a supersolution or merely a subsolution (or a supersolution) and a one-sided growth condition, then problem (1.1) is solvable. We also consider the existence of maximal or minimal (extremal) solutions, which are the biggest and smallest solutions of (1.1) (in certain ordering) within the interval between a subsolution and a supersolution (Theorems 3.2, 3.4). In Section 4, we consider some examples where one can actually find sub- and supersolutions. Combining with the results in Section 3, we obtain the existence of nonnegative nontrivial solutions and extremal solutions in eigenvalue problems for variational inequalities. The first problem is about an inequality containing a quasilinear elliptic operator and the convex term is given by an integral. By using constants as sub- and supersolutions, we find conditions such that the inequality has bounded solutions. The second example is an eigenvalue problem for an inequality that contains the  $p$ -Laplacian. By using sub- and supersolutions constructed from the principal eigenfunctions of the  $p$ -Laplacian, we show the existence of positive solutions of the inequality.

Compared with sub- supersolution methods for equations or for inequalities on convex sets, the development of the method for inequalities with general convex functionals (not necessarily indicators of convex sets) requires some nontrivial adaptation and modifications and new arguments in several places. Note that our presentation here is somewhat related to the results in [6, 8, 7] about sub- supersolution methods for differential inclusions with convex terms given by certain integrals. The concepts of sub- and supersolutions there are




---

**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



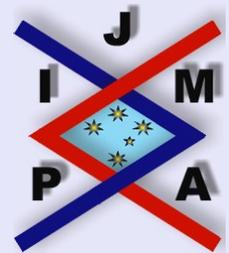
Go Back

Close

Quit

Page 6 of 37

for (pointwise) inclusion are defined mostly pointwise, while our concepts here are for inequalities and are based on the dual between  $W^{1,p}(\Omega)$  and  $[W^{1,p}(\Omega)]^*$ . An interesting question is to possibly compare the approach here with that in [6, 8, 7].



---

**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

Quit

Page 7 of 37

## 2. General Settings

In this section, we consider the assumptions imposed on the inequality (1.1) and next define sub- and supersolutions for it. We use the notation  $X := W^{1,p}(\Omega)$  and  $X_0 := W_0^{1,p}(\Omega)$  for the usual first-order Sobolev spaces. In (1.1),  $L$  is a mapping from  $X$  to  $X^*$ , defined by

(2.1)

$$\langle L(u), v \rangle = \int_{\Omega} \left[ \sum_{i=1}^N A_i(x, u, \nabla u) \partial_i v + A_0(x, u, \nabla u) v \right] dx, \quad \forall u, v \in X,$$

where, for each  $i \in \{0, 1, \dots, N\}$ ,  $A_i$  is a Carathéodory function from  $\Omega \times \mathbb{R}^{N+1}$  to  $\mathbb{R}$ . For  $i \in \{1, \dots, N\}$ ,

$$(2.2) \quad |A_i(x, u, \xi)| \leq a_0(x) + b_0(|u|^{p-1} + |\xi|^{p-1}),$$

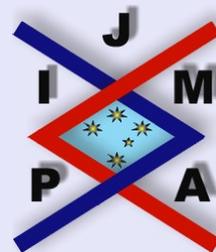
and

$$(2.3) \quad |A_0(x, u, \xi)| \leq a_1(x) + b_1(|u|^{q-1} + |\xi|^{\frac{p}{q}}),$$

for almost all  $x \in \Omega$ , all  $u \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^N$  with  $b_0, b_1 > 0$ ,  $a_0 \in L^{p'}(\Omega)$ ,  $a_1 \in L^{p'}(\Omega)$ ,  $1 \leq q < p^*$ . (As usual,  $p'$  is the Hölder conjugate of  $p$  and  $p^*$  is its Sobolev conjugate.) Moreover,

(2.4)

$$\sum_{i=1}^N [A_i(x, u, \xi) - A_i(x, u', \xi')] (\xi_i - \xi'_i) + [A_0(x, u, \xi) - A_0(x, u', \xi')] (u - u') > 0,$$



**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 8 of 37

if  $(u, \xi) \neq (u', \xi')$ , and

$$(2.5) \quad \sum_{i=1}^N A_i(x, u, \xi) \xi_i + A_0(x, u, \xi) u \geq \alpha(|\xi|^p + |u|^p) - \beta(x),$$

for a.e.  $x \in \Omega$ , all  $u \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^N$ , where  $\alpha > 0$  and  $\beta \in L^1(\Omega)$ . The lower-order operator  $G$  is defined by

$$(2.6) \quad \langle G(u), v \rangle = \int_{\Omega} F(x, u, \nabla u) v dx,$$

where  $F : \Omega \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}$  is a Carathéodory function with certain growth conditions to be specified later. We also assume that  $j$  is a mapping from  $X$  to  $\mathbb{R} \cup \{\infty\}$  such that the restriction  $j|_{W_0^{1,p}(\Omega)}$  is convex and lower semicontinuous on  $W_0^{1,p}(\Omega)$  with non empty effective domain. Before stating our theorem about existence of solutions, we need to define subsolutions and supersolutions for inequalities with convex functionals. These definitions extend those definitions presented in [18] for inequalities on closed convex sets. As seen in the following definitions, they are more complicated. As usual, we use the notation

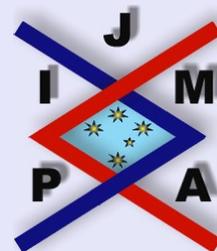
$$u \vee v = \max\{u, v\}, \quad u \wedge v = \min\{u, v\}.$$

and

$$A * B = \{a * b : a \in A, b \in B\},$$

where  $A, B \subset W^{1,p}(\Omega)$  and  $*$   $\in$   $\{\wedge, \vee\}$ . As is well known,  $W^{1,p}(\Omega)$  and  $W_0^{1,p}(\Omega)$  are closed under the operations  $\vee$  and  $\wedge$ , that is,

$$u, v \in W^{1,p}(\Omega) \text{ (resp. } W_0^{1,p}(\Omega)) \implies u \vee v, u \wedge v \in W^{1,p}(\Omega) \text{ (resp. } W_0^{1,p}(\Omega)).$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 9 of 37

**Definition 2.1.** A function  $\underline{u} \in W^{1,p}(\Omega)$  is called a  $W$ -subsolution of (1.1) if there exists a functional  $J$  (depending on  $u$ ):

$$J = J_{\underline{u}} : W^{1,p}(\Omega) \rightarrow \mathbb{R} \cup \{\infty\},$$

such that

- (2.7)
- (i)  $\underline{u} \leq 0$  on  $\partial\Omega$
  - (ii)  $F(\cdot, \underline{u}, \nabla \underline{u}) \in L^{q'}(\Omega)$
  - (iii)  $J(\underline{u}) < \infty$ , and

(2.8) 
$$j(v \vee \underline{u}) + J(v \wedge \underline{u}) \leq j(v) + J(\underline{u}), \quad \forall v \in W_0^{1,p}(\Omega) \cap D(j),$$

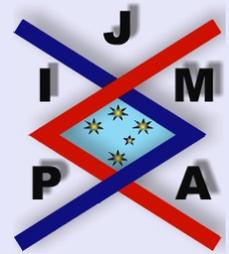
and

(2.9)

- (iv)  $\langle L(\underline{u}), v - \underline{u} \rangle - \langle G(\underline{u}), v - \underline{u} \rangle + J(v) - J(\underline{u}) \geq 0, \quad v \in \underline{u} \wedge [W_0^{1,p}(\Omega) \cap D(j)],$

$(D(j) = \{v \in X : j(v) < \infty\})$  is the effective domain of  $j$ . We have a similar definition for  $W$ -supersolution  $\bar{u}$ :  $\bar{u}$  is a  $W$ -supersolution of (1.1) if there exists  $J = J_{\bar{u}} : W^{1,p}(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$  such that:

- (2.10)
- (i)  $\bar{u} \geq 0$  on  $\partial\Omega$
  - (ii)  $F(\cdot, \bar{u}, \nabla \bar{u}) \in L^{q'}(\Omega)$
  - (iii)  $J(\bar{u}) < \infty$ , and



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 10 of 37

$$(2.11) \quad j(v \wedge \bar{u}) + J(v \vee \underline{u}) \leq j(v) + J(\bar{u}), \quad \forall v \in W_0^{1,p}(\Omega) \cap D(j),$$

and

$$(2.12) \quad \text{(iv) } \langle L(\bar{u}), v - \bar{u} \rangle - \langle G(\bar{u}), v - \bar{u} \rangle + J(v) - J(\bar{u}) \geq 0, \quad v \in \bar{u} \vee [W_0^{1,p}(\Omega) \cap D(j)].$$

A subsolution of (1.1) is a finite maximum of  $W$ -subsolutions and a supersolution is a finite minimum of  $W$ -supersolutions.

Suppose there exist a subsolution  $\underline{u} = \max\{u_i : 1 \leq i \leq k\}$  and a supersolution  $\bar{u} = \min\{\bar{u}_l : 1 \leq l \leq m\}$  of (1.1). We assume that  $F$  has the following growth condition:

$$(2.13) \quad |F(x, u, \xi)| \leq a_2(x) + b_2|\xi|^{p/q'}$$

for a.e.  $x \in \Omega$ , all  $\xi \in \mathbb{R}^N$ , all  $u$  such that  $\underline{u}_0(x) \leq u \leq \bar{u}_0(x)$ , where  $a_2 \in L^q(\Omega)$ ,  $b_2 \geq 0$ ,  $q < p^*$  ( $p^*$  is the Sobolev conjugate of  $p$ ), and

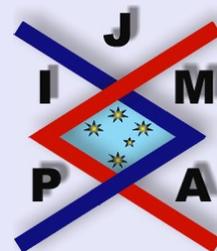
$$\underline{u}_0 = \min\{u_i : 1 \leq i \leq k\}, \quad \bar{u}_0 = \max\{\bar{u}_l : 1 \leq l \leq m\}.$$

We conclude this section with some remarks.

**Remark 2.1.** (i) If  $u$  is a solution of (1.1), then  $u$  is a subsolution of (1.1), provided  $j$  satisfies the following condition:

$$(2.14) \quad j(v \vee u) + j(v \wedge u) \leq j(v) + j(u),$$

for all  $u, v \in W^{1,p}(\Omega)$ . In fact, if  $u$  is a solution of (1.1) then it satisfies (i) – (ii). By choosing  $J = j$ , we see that (2.8) follows from (2.14). If  $v = u \wedge w$ ,  $w \in$




---

**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

Quit

Page 11 of 37

$W_0^{1,p}(\Omega)$ , then  $v = 0$  on  $\partial\Omega$ , i.e.,  $v \in W_0^{1,p}(\Omega)$ . Hence, (2.9) is a consequence of (1.1). Similarly, if (2.14) holds, then any solution is a supersolution.

(ii) (2.14) is satisfied for several usual convex functionals  $j$ . For example, if  $j$  is given by

$$(2.15) \quad j(u) = \int_E \psi(x, u) dx,$$

where  $E$  is a subset of  $\Omega$  or  $\partial\Omega$ ,  $\psi : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ , is a Carathéodory function such that

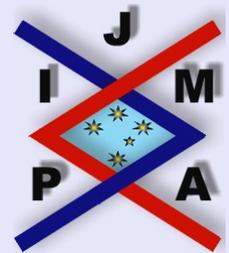
$$(2.16) \quad \psi(x, u) \geq a_3(x) + b_3|u|^s, \quad x \in \Omega, \quad u \in \mathbb{R},$$

where  $a_3 \in L^1(\Omega)$  and  $0 \leq s < p^*$ .  $j$  is well defined from  $W^{1,p}(\Omega)$  to  $\mathbb{R} \cup \{\infty\}$  and  $j$  is convex if  $\psi(x, \cdot)$  is convex for a.e.  $x \in \Omega$ . Also, by Fatou's lemma,  $j$  is weakly lower semicontinuous. Let  $u, v \in W^{1,p}(\Omega)$  and denote

$$\Omega_1 = \{x \in \Omega : v(x) < u(x)\}, \quad \Omega_2 = \{x \in \Omega : v(x) \geq u(x)\}.$$

Then,

$$(2.17) \quad \begin{aligned} j(v \wedge u) + j(v \vee u) &= \left( \int_{\Omega_1} + \int_{\Omega_2} \right) \psi(v \wedge u) + \left( \int_{\Omega_1} + \int_{\Omega_2} \right) \psi(v \vee u) \\ &= \int_{\Omega_1} \psi(x, u) + \int_{\Omega_2} \psi(x, v) + \int_{\Omega_1} \psi(x, v) + \int_{\Omega_2} \psi(x, u) \\ &= \int_{\Omega} \psi(x, u) + \int_{\Omega} \psi(x, v) \\ &= j(u) + j(v). \end{aligned}$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 12 of 37

Hence, (2.14) is satisfied. Note that from (2.16),  $\psi(x, u)$  is bounded from below by a function in  $L^1(\Omega)$ . Thus, the integrals in (2.17) are in  $\mathbb{R} \cup \{\infty\}$  and we can split and combine them as done.

(iii) If  $j = I_K$ ,  $K$  is a closed convex set in  $W_0^{1,p}(\Omega)$ , then we recover the cases considered in [18]. Moreover, (2.14) holds provided  $K$  satisfied the condition

$$(2.18) \quad u, v \in K \implies u \wedge v, u \vee v \in K.$$

As noted in [18], (2.18) is satisfied whenever  $K$  is defined by obstacles or by certain conditions on the gradients. We can also check that by using (2.15).

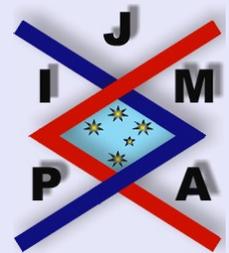
(iv) If  $j = 0$ , we have an equation in (1.1). By choosing  $J = 0$  also, we see that (2.8) – (2.11) obviously hold and (2.9) – (2.12) reduce to the usual definitions of sub- and supersolutions of equations. If  $j = I_K$  as in (iii), then by choosing  $J = 0$ , we see that the definition of subsolutions in [18] is equivalent to the definition in (i) – (iv) here. Thus, Definition 2.1 is an extension of that in [18].

(v) By choosing  $J = 0$  in (2.8) and (2.9), we see that if  $\underline{u}$  is a subsolution of the equation

$$\langle L(u), v \rangle - \langle G(u), v \rangle = 0, \quad \forall v \in W_0^{1,p}(\Omega)$$

and  $j(v \vee \underline{u}) \leq j(v)$ ,  $\forall v \in W_0^{1,p}(\Omega) \cap D(j)$ , then  $\underline{u}$  is a subsolution of (1.1). Similar observations hold for supersolutions.

(vi) Compared to the definitions in [13, 10, 9, 15, 18], the new ingredient here is the introduction of the functional  $J$  in Definition 2.1, which permits more flexibility in constructing sub- and supersolutions (by choosing different  $J$ ).



**Sub- Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 13 of 37

### 3. Main Existence Results

In this section, we state and prove our existence results for solutions and extremal solutions of (1.1), based on the concepts of sub- and supersolutions in Section 2.

**Theorem 3.1.** *Assume (1.1) has a subsolution  $\underline{u}$  and a supersolution  $\bar{u}$  such that  $\underline{u} \leq \bar{u}$  and that (2.13) holds. Then, (1.1) has a solution  $u$  such that  $\underline{u} \leq u \leq \bar{u}$ .*

*Proof.* We follow the usual truncation–penalization technique as in [13, 9, 15] or [18]. Therefore, we just outline the main arguments and present only the different points and modifications needed for our situation here. Let  $b$  be defined by (cf. [18])

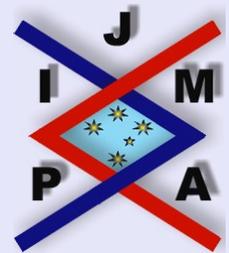
$$(3.1) \quad b(x, t) = \begin{cases} [t - \bar{u}(x)]^{q-1} & \text{if } t > \bar{u}(x) \\ 0 & \text{if } \underline{u}(x) \leq t \leq \bar{u}(x) \\ -[-t + \underline{u}(x)]^{q-1} & \text{if } t < \underline{u}(x). \end{cases}$$

We have the following estimates (cf. (49) and (50) in [18]):

$$|b(x, t)| \leq a_3(x) + c_3|t|^{q-1},$$

with  $a_3 \in L^{q'}(\Omega)$ , and

$$\int_{\Omega} b(\cdot, u)u \geq c_4\|u\|_{L^q(\Omega)}^q - c_5,$$



Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 14 of 37

for all  $u \in L^q(\Omega)$ , where the  $c_i$ 's ( $i = 3, 4, 5$ ) are positive constants independent of  $u$ . We define  $T_{il}$  ( $1 \leq i \leq k$ ,  $1 \leq l \leq m$ ) and  $T$  by:

$$T_{il}(u)(x) = \begin{cases} \underline{u}_i(x) & \text{if } u(x) < \underline{u}_i(x) \\ u(x) & \text{if } \underline{u}_i(x) \leq u(x) \leq \bar{u}_l(x) \\ \bar{u}_l(x) & \text{if } u(x) > \bar{u}_l(x), \end{cases}$$

and

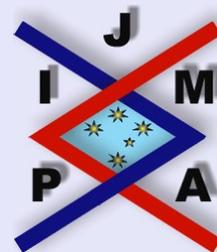
$$T(u)(x) = \begin{cases} \underline{u}(x) & \text{if } u(x) < \underline{u}(x) \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \bar{u}(x) \\ \bar{u}(x) & \text{if } u(x) > \bar{u}(x). \end{cases}$$

Let us consider the variational inequality

$$(3.2) \quad \begin{cases} \langle L(u) + \beta B(u) - C(u), v - u \rangle + j(u) - j(v) \geq 0, \quad \forall v \in W_0^{1,p}(\Omega) \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

with  $\beta > 0$  sufficiently large. Consider the (nonlinear) operators  $B$  and  $C$  given by

$$\langle B(u), \phi \rangle = \int_{\Omega} b(\cdot, u)\phi,$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 15 of 37

and

$$(3.3) \langle C(u), \phi \rangle = \int_{\Omega} [F(\cdot, T(u), \nabla T(u)) + \sum_{i,l} |F(\cdot, T_{il}(u), \nabla T_{il}(u)) - F(\cdot, T(u), \nabla T(u))|] \phi, \\ \forall u, \phi \in W_0^{1,p}(\Omega).$$

Let us prove that  $H = L + \beta B - C$  is pseudo-monotone on  $W^{1,p}(\Omega)$ . In fact, assume  $w_n \rightharpoonup w$  in  $W^{1,p}(\Omega)$  (" $\rightharpoonup$ " denotes the weak convergence) and

$$(3.4) \quad \limsup_{n \rightarrow \infty} \langle H(w_n), w_n - w \rangle \leq 0.$$

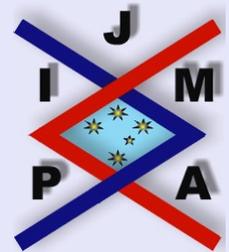
We show that

$$(3.5) \quad \lim_{n \rightarrow \infty} \langle H(w_n), w_n - v \rangle \geq \langle H(w), w - v \rangle, \quad \forall v \in W^{1,p}(\Omega).$$

Since the embedding  $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$  is compact, we have  $w_n \rightarrow w$  in  $L^p(\Omega)$ . By passing to a subsequence, if necessary, we can assume that there is a function  $h$  in  $L^p(\Omega)$  such that

$$(3.6) \quad \begin{cases} w_n \rightarrow w \text{ a.e. in } \Omega, \text{ and} \\ |w_n| \leq h \text{ a.e. in } \Omega, \forall n. \end{cases}$$

Since the sequence  $\{w_n\}$  is bounded in  $W^{1,p}(\Omega)$ , the sequences  $\{F(\cdot, T_{il}(w_n), \nabla T_{il}(w_n))\}$  and  $\{F(\cdot, T(w_n), \nabla T(w_n))\}$  are uniformly bounded



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 16 of 37

in  $L^{q'}(\Omega)$ . From (3.3), it follows that the sequence  $\{(\beta B - C)(w_n)\}$  is bounded in  $L^{q'}(\Omega)$ . (3.6) thus implies that

$$\langle (\beta B - C)(w_n), w_n - w \rangle \rightarrow 0.$$

Hence, from (3.4),

$$(3.7) \quad \limsup \langle L(w_n) - L(w), w_n - w \rangle = \limsup \langle L(w_n), w_n - w \rangle \leq 0.$$

Since  $\{A_i\}$  ( $i = 0, 1, \dots, n$ ) satisfy (2.2) – (2.4), it follows from (3.6) and (3.7) that  $w_n \rightarrow w$  in  $W^{1,p}(\Omega)$ . Consequently,  $H(w_n) \rightarrow H(w)$  in  $L^{q'}(\Omega)$  and (3.5) follows. This shows that  $L + \beta B - C$  is pseudo-monotone. Using arguments similar to those in [18], we can prove that  $L + \beta B - C$  is coercive on  $W_0^{1,p}(\Omega)$ . Moreover, this mapping is obviously continuous and bounded. Classical existence results for variational inequalities (cf. e.g. [23, 14]) give the existence of at least one solution  $u \in W_0^{1,p}(\Omega)$  of (3.2). Also, it is clear that  $u \in D(j)$ . We prove that  $\underline{u} \leq u$ . Let  $\underline{u}_q$  ( $1 \leq q \leq k$ ) be a  $W$ -subsolution. Since  $u \in W_0^{1,p}(\Omega) \cap D(j)$ , (2.9) with  $\underline{u} = \underline{u}_q$  and  $v = \underline{u} \wedge u$  gives

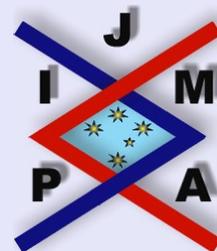
$$\langle (\underline{u}_q), \underline{u}_q \wedge u - \underline{u}_q \rangle - \langle G(\underline{u}_q), \underline{u}_q \wedge u - \underline{u}_q \rangle + J(\underline{u}_q \wedge u) - J(\underline{u}_q) \geq 0.$$

Since  $\underline{u}_q \wedge u = \underline{u}_q - (\underline{u}_q - u)^+$ , the above inequality becomes

$$(3.8) \quad -\langle (\underline{u}_q), (\underline{u}_q - u)^+ \rangle - \langle G(\underline{u}_q), (\underline{u}_q - u)^+ \rangle + J(\underline{u}_q \wedge u) - J(\underline{u}_q) \geq 0.$$

On the other hand, since  $\underline{u}_q \vee u = 0$  on  $\partial\Omega$ ,  $v = \underline{u}_q \vee u \in W_0^{1,p}(\Omega)$ . Letting  $v$  into (3.2) and noting that  $\underline{u}_q \vee u = u + (\underline{u}_q - u)^+$ , we get

$$(3.9) \quad \langle L(u) + \beta B(u) - C(u), (\underline{u}_q - u)^+ \rangle + j(\underline{u}_q \vee u) - j(u) \geq 0.$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 17 of 37

Adding (3.8) and (3.9), one gets

$$\begin{aligned} & \langle L(u) - L(\underline{u}_q), (\underline{u}_q - u)^+ \rangle + \langle G(\underline{u}_q), (\underline{u}_q - u)^+ \rangle \\ & + \langle \beta B(u) - C(u), (\underline{u}_q - u)^+ \rangle j(\underline{u}_q \vee u) - j(u) + J(\underline{u}_q \wedge u) - J(\underline{u}_q) \geq 0. \end{aligned}$$

From (2.8),

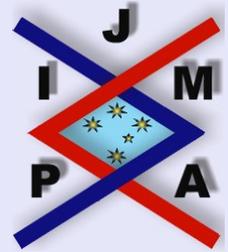
$$j(\underline{u}_q \vee u) - j(u) + J(\underline{u}_q \wedge u) - J(\underline{u}_q) \leq 0.$$

Using the integral formulation of  $B, C$  and  $G$ , we get

$$\begin{aligned} (3.10) \quad & \langle L(u) - L(\underline{u}_q), (\underline{u}_q - u)^+ \rangle + \int_{\Omega} F(x, \underline{u}_q, \nabla \underline{u}_q) (\underline{u}_q - u)^+ \\ & + \beta \int_{\Omega} b(x, u) (\underline{u}_q - u)^+ - \int_{\Omega} \left[ F(x, T(u), \nabla T(u)) + \sum_{i,l} |F(x, T_{il}(u), \nabla T_{il}(u)) \right. \\ & \quad \left. - F(x, T(u), \nabla T(u)) \right] (\underline{u}_q - u)^+ \geq 0. \end{aligned}$$

We have

$$\begin{aligned} & \langle L(u) - L(\underline{u}_q), (\underline{u}_q - u)^+ \rangle + \beta \int_{\Omega} b(\cdot, u) (\underline{u}_q - u)^+ \\ & + \int_{\Omega} \left[ F(\cdot, \underline{u}_q) - F(\cdot, T(u)) - \sum_{i,l} |F(\cdot, T_{il}(u)) - F(\cdot, T(u))| \right] (\underline{u}_q - u)^+ \geq 0. \end{aligned}$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 18 of 37

Using

$$\begin{aligned}
 & F(x, \underline{u}_q(x)) - F(x, T(u)(x)) - \sum_{i,l} |F(x, T_{il}(u)(x)) - F(x, T(u)(x))| \\
 & \leq F(x, \underline{u}_q(x)) - F(x, T(u)(x)) - |F(x, T_{q_1}(u)(x)) - F(x, T(u)(x))| \\
 & = F(x, \underline{u}_q(x)) - F(x, T(\underline{u})(x)) - |F(x, \underline{u}_q(x)) - F(x, \underline{u}(x))| \\
 & \leq 0,
 \end{aligned}$$

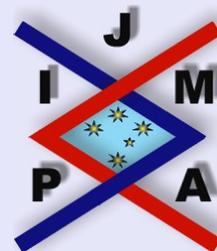
we obtain

(3.11)

$$\begin{aligned}
 & \int_{\Omega} \left[ F(\cdot, \underline{u}_q) - F(\cdot, T(u)) - \sum_{i,l} |F(\cdot, T_{il}(u)) - F(\cdot, T(u))| \right] (\underline{u}_q - u)^+ \\
 & = \int_{\{\underline{u}_q > u\}} \left[ F(\cdot, \underline{u}_q) - F(\cdot, T(u)) - \sum_{i,l} |F(\cdot, T_{il}(u)) - F(\cdot, T(u))| \right] (\underline{u}_q - u)^+ \\
 & \leq 0.
 \end{aligned}$$

Using the fact that

$$\begin{aligned}
 & \langle L(u) - L(\underline{u}_q), (\underline{u}_q - u)^+ \rangle \\
 & = - \int_{\{\underline{u}_q - u > 0\}} \left\{ \sum_{i=1}^N [A_i(x, \underline{u}_q, \nabla \underline{u}_q) - A_i(x, u, \nabla u)] (\partial_i \underline{u}_q - \partial_i u) \right. \\
 & \quad \left. + [A_0(x, \underline{u}_q, \nabla \underline{u}_q) - A_0(x, u, \nabla u)] (\underline{u}_q - u) \right\} \\
 & \leq 0,
 \end{aligned}$$



**Sub-Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 19 of 37

(by (2.4)), we have from (3.10) and (3.11) the following estimate

$$\begin{aligned}
 0 &\leq \int_{\Omega} b(\cdot, u)(\underline{u}_q - u)^+ \\
 &= \int_{\{\underline{u}_q > u\}} b(\cdot, u)(\underline{u}_q - u) \\
 &= - \int_{\{\underline{u}_q > u\}} (\underline{u} - u)^{q-1}(\underline{u}_q - u) \quad (\text{since } a < \underline{u}_q \leq \underline{u}) \\
 &\leq 0.
 \end{aligned}$$

Thus,

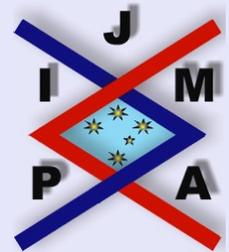
$$0 = \int_{\Omega} [(\underline{u}_q - u)^+]^q dx,$$

and  $(\underline{u}_q - u)^+ = 0$  a.e. in  $\Omega$ , i.e.,  $u \geq \underline{u}_q$  a.e. in  $\Omega$ . Using these arguments for all  $q \in \{1, \dots, k\}$ , we see that  $u \geq \underline{u}$ . We can show in the same way that  $u \leq \bar{u}$ . Now, from (3.1), we have  $b(x, u(x)) = 0$  for almost all  $x \in \Omega$ , i.e.,  $B = 0$ . Also,  $T_{il}(u) = T(u) = u$ , for all  $i, l$  and thus

$$\langle C(u), \phi \rangle = \int_{\Omega} F(\cdot, u, \nabla u) \phi = \langle G(u), \phi \rangle.$$

Hence, since  $u$  satisfies (3.2), it also satisfies (1.1), i.e.,  $u$  is a solution of (1.1) and  $\underline{u} \leq u \leq \bar{u}$ .  $\square$

We now prove that (1.1) has a maximal and a minimal solution within the interval between  $\underline{u}$  and  $\bar{u}$ .



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 20 of 37

**Theorem 3.2.** Assume (1.1) has a subsolution  $\underline{u}$  and a supersolution  $\bar{u}$  such that  $\underline{u} \leq \bar{u}$ . Moreover, (2.13) and (2.14) hold. Then, (1.1) has a maximal solution  $u^*$  and a minimal solution  $u_*$  such that

$$(3.12) \quad \underline{u} \leq u_* \leq u^* \leq \bar{u},$$

that is,  $u_*$  and  $u^*$  are solutions of (1.1) that satisfy (3.12) and if  $u$  is a solution of (1.1) such that  $\underline{u} \leq u \leq \bar{u}$  then  $u_* \leq u \leq u^*$  on  $\Omega$ .

The proof is similar to that of the particular case  $j = I_K$ , which was already presented in [18]. Therefore, it is omitted.

As in the case of variational inequalities on convex sets, we still have existence of solutions and extremal solutions provided only subsolutions (or supersolutions) exist together with certain one-sided growth conditions. We have in fact the following result.

**Theorem 3.3.** Assume (1.1) has a subsolution  $\underline{u}$  and  $F$  has the growth condition

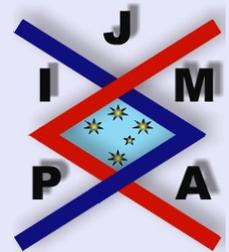
$$(3.13) \quad |F(x, u, \xi)| \leq a_3(x) + b_3(|u|^\sigma + |\xi|^\sigma)$$

for a.e.  $x \in \Omega$ , all  $u$  such that  $\underline{u}_0(x) \leq u$ , all  $\xi \in \mathbb{R}^N$ , where  $0 \leq \sigma < p-1$ ,  $a \in L^{p'}(\Omega)$ , and

$$\underline{u}_0 = \min\{\underline{u}_i : 1 \leq i \leq k\}.$$

Hence, (1.1) has a solution  $u$  such that  $u \geq \underline{u}$ .

The idea of the proof of this result is a combination of Theorem 3.1 stated above and an extension of Theorem 1 in [18]. We omit the proof and refer the reader to [18] for more details.



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 21 of 37

By looking closely at the set of solutions of (1.1), one can improve Theorem 3.3 and get the following stronger result.

**Theorem 3.4.** *Under the assumptions of Theorem 3.3, (1.1) has a maximal solution  $u^*$  and a minimal solution  $u_*$  such that*

$$(3.14) \quad \underline{u} \leq u_* \leq u^* \leq \bar{u},$$

that is,  $u_*$  and  $u^*$  are solutions of (1.1) that satisfy (3.14) and if  $u$  is a solution of (1.1) such that  $\underline{u} \leq u \leq \bar{u}$  then  $u_* \leq u \leq u^*$  on  $\Omega$ .

*Proof.* The proof follows the same line as that in Theorem 2, [18]. A main ingredient of the proof is the boundedness of the set

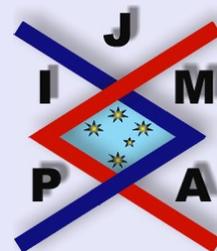
$$S = \{u \in W_0^{1,p}(\Omega) : u \geq \underline{u}, u \text{ is a solution of (1.1)}\}$$

in  $W_0^{1,p}(\Omega)$ . Proving that  $S$  is bounded requires some different arguments from those in [18]. From (1.1) with  $v = \phi$  being a fixed element in  $D(j)$ , we have

$$\langle L(u), \phi - u \rangle - \langle G(u), \phi - u \rangle + j(\phi) - j(u) \geq 0.$$

Therefore,

$$\begin{aligned} \langle L(u), \phi \rangle &= \int_{\Omega} \left[ \sum_i A_i(x, u, \nabla u) \partial_i u + A_0(x, u, \nabla u) u \right] \\ &\geq \alpha \int_{\Omega} (|\nabla u|^p + |u|^p) - \int_{\Omega} \beta dx \quad (\text{by (2.5)}) \\ &\geq \alpha \|u\|_{W_0^{1,p}(\Omega)}^p - c. \end{aligned}$$




---

**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

Quit

Page 22 of 37

Hence,

$$\begin{aligned}
 |\langle L(u), \phi \rangle| &\leq c \left[ \sum_i \|A_i(\cdot, u, \nabla u)\|_{L^{p'}(\Omega)} \|\partial_i \phi\|_{L^p(\Omega)} \right. \\
 &\quad \left. + \|A_0(\cdot, u, \nabla u)\|_{L^{p'}(\Omega)} \|\phi\|_{L^p(\Omega)} \right] \\
 &\leq c \left[ \sum_i \|a_0 + b_0|u|^{p-1} + b_0|\nabla u|^{p-1}\|_{L^{p'}(\Omega)} \|\partial_i \phi\|_{L^p(\Omega)} \right. \\
 &\quad \left. + \|a_1 + b_1|u|^{p-1} + |\nabla u|^{p-1}\|_{L^{p'}(\Omega)} \|\phi\|_{L^p(\Omega)} \right] \\
 &\leq c(1 + \|u\|_{L^p(\Omega)}^{p-1} + \|\nabla u\|_{L^p(\Omega)}^{p-1}) \\
 &\leq c(1 + \|u\|_{W^{1,p}(\Omega)}^{p-1}),
 \end{aligned}$$

( $c$  denotes a generic constant). From (3.13), we have

$$|\langle G(u), \phi \rangle| \leq c(1 + \|u\|_{W^{1,p}(\Omega)}^\sigma)$$

and

$$|\langle G(u), u \rangle| \leq c(1 + \|u\|_{W^{1,p}(\Omega)}^\sigma) \|u\|_{W^{1,p}(\Omega)} \leq c(1 + \|u\|_{W^{1,p}(\Omega)}^{\sigma+1}).$$

Since  $j$  is convex and lower semi-continuous, there exist  $a_4, b_4 \in \mathbb{R}$  such that

$$j(u) \geq a_4 + b_4 \|u\|_{W_0^{1,p}(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

Hence,

$$\alpha \|u\|_{W^{1,p}(\Omega)}^p - c \leq c(1 + \|u\|_{W^{1,p}(\Omega)}^{p-1} + \|u\|_{W^{1,p}(\Omega)}^\sigma + \|u\|_{W^{1,p}(\Omega)}).$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

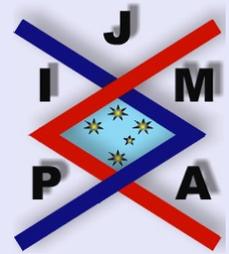
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Page 23 of 37

Since  $\sigma < p$ , this shows that  $\|u\|_{W^{1,p}(\Omega)} \leq c$  for all solutions  $u$  of (1.1) such that  $u \geq \underline{u}$ . Hence,  $S$  is bounded in  $W^{1,p}(\Omega)$ .

The remainder of the proof is similar to that of Theorem 3 in [18]. □

**Remark 3.1.** Note that if  $A_i = A_i(x, \xi)$  ( $i = 1, \dots, N$ ) do not depend on  $u$ , then we can choose  $A_0 = 0$  and all the results stated above still hold.



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**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

Quit

Page 24 of 37

## 4. Some Examples

We now apply these general results to establish the existence of solutions and extremal solutions in some particular variational inequalities.

### 4.1.

In this example, we study a quasi-linear elliptic variational inequality that contains a "unilateral" term given by an integral. Assume that for  $i = 0, 1, \dots, N$ ,  $A_i$  satisfies

$$(4.1) \quad A_i(x, u, 0) = 0$$

for a.e.  $x \in \Omega$ , all  $u \in \mathbb{R}$  and consider the variational inequality

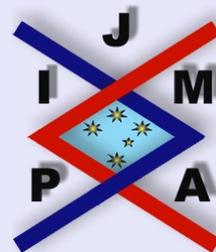
$$(4.2) \quad \begin{cases} \langle L(u), v - u \rangle - \lambda \int_{\Omega} F(x, u, \nabla u)(v - u) + j(v) - j(u) \geq 0, \quad \forall v \in W_0^{1,p}(\Omega) \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

Here,  $L$  and  $A$  are defined as in (2.1), (2.2), and (2.3) of Section 2.  $\lambda$  is a real parameter and

$$(4.3) \quad j(u) = \int_{\Omega} \psi(x, u(x)) dx,$$

where  $\psi : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$  is a Carathéodory function such that

$$(4.4) \quad \psi(x, u) \geq -a(x) - b|u|^p,$$



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**Sub-Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

Quit

Page 25 of 37

where  $a \in L^1(\Omega)$ ,  $b \geq 0$ . It follows from this inequality that for  $u \in W^{1,p}(\Omega)$ ,  $\psi(x, u(x))$  is measurable and since  $-a(x) - b|u(x)|^p \in L^1(\Omega)$ ,  $j$  is well defined and  $j(u) \in \mathbb{R} \cup \{\infty\}$ . Assume also that for almost all  $x \in \Omega$ ,  $\psi(x, \cdot)$  is convex. Hence,  $j$  is convex on  $W^{1,p}(\Omega)$ . It follows from Fatou's lemma that  $j$  is lower semicontinuous on that space. The following lemma shows the existence of constant sub- and supersolutions of (4.2).

**Lemma 4.1.** (a) Assume  $B \in \mathbb{R}$ ,  $B \leq 0$  is such that

$$(4.5) \quad \begin{aligned} & \text{(i)} \quad F(x, B, 0) \geq 0 \text{ for a.e. } x \in \Omega \\ & \text{(ii)} \quad F(\cdot, B, 0) \in L^{q'}(\Omega) \end{aligned}$$

and

$$(4.6) \quad \text{(iii)} \quad \psi(x, B) \leq \psi(x, v), \quad \forall v \leq B,$$

then  $B$  is a subsolution of (4.2).

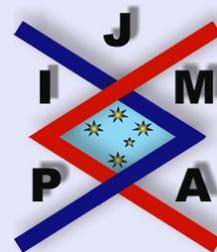
(b) Similarly, if  $A \in \mathbb{R}$ ,  $A \geq 0$  and

$$(4.7) \quad \begin{aligned} & \text{(i)} \quad F(x, A, 0) \leq 0 \text{ for a.e. } x \in \Omega \\ & \text{(ii)} \quad F(\cdot, A, 0) \in L^{q'}(\Omega) \end{aligned}$$

and

$$(4.8) \quad \text{(iii)} \quad \psi(x, A) \geq \psi(x, v), \quad \forall v \geq A,$$

then  $A$  is a supersolution of (4.2).



**Sub- Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 26 of 37

*Proof.* (a) Choosing  $J = 0$ , we see that  $\underline{u} = B$  satisfies conditions (i) – (iii) of Definition 2.1. Moreover, (2.8) becomes, in this case,

$$(4.9) \quad j(v \vee B) \leq j(v), \quad v \in W_0^{1,p}(\Omega) \cap D(j),$$

i.e.,

$$\int_{\Omega} \psi(x, v(x) \vee B) dx \leq \int_{\Omega} \psi(x, v(x)) dx.$$

In view of (4.3) and (4.4), this is equivalent to

$$(4.10) \quad \int_{\{x \in \Omega : v(x) > B\}} \psi(x, v) dx + \int_{\{x \in \Omega : v(x) \leq B\}} \psi(x, B) dx \\ \leq \int_{\{x \in \Omega : v(x) > B\}} \psi(x, v) dx + \int_{\{x \in \Omega : v(x) \leq B\}} \psi(x, v) dx.$$

Now, from (4.6), we have

$$\psi(x, B) \leq \psi(x, v(x)) \quad \text{on } \{x \in \Omega : v(x) \leq B\}$$

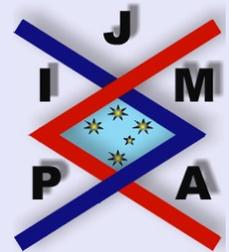
and thus

$$\int_{\{x \in \Omega : v(x) \leq B\}} \psi(x, B) \leq \int_{\{x \in \Omega : v(x) \leq B\}} \psi(x, v),$$

which implies (4.10) and thus (4.9).

To check (2.9), we assume that  $v = B \wedge w$  with some  $w \in W_0^{1,p}(\Omega) \cap D(j)$ . From (4.1) and the definition of  $L$ ,  $L(B) = 0$ . Since  $v - B \leq 0$ , we have from (4.5)(i) that

$$\langle G(B), v - B \rangle = \int_{\Omega} F(x, B, 0)(v - B) dx \leq 0.$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 27 of 37

This implies (2.9), completing the proof of (a). The proof of (b) is similar.  $\square$

By using Theorems 3.2, 3.4, and Lemma 4.1, we have the following existence result for (4.2).

**Theorem 4.2.** (a) Assume  $B \in \mathbb{R}$  satisfies (4.5),  $\psi$  satisfies (4.6), and that

$$|F(x, u, \xi)| \leq a(x) + b(|u|^\sigma + |\xi|^\sigma)$$

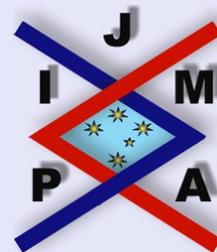
for a.e.  $x \in \Omega$ ,  $u \geq B$ ,  $\xi \in \mathbb{R}^N$ , with  $0 \leq \sigma < p - 1$ ,  $a \in L^{p'}(\Omega)$ . Then, (4.2) has a minimal solution  $u_*$  and a maximal solution  $u^*$  such that  $B \leq u_* \leq u^*$ .

(b) Assume  $A, B \in \mathbb{R}$  ( $A \geq B$ ) satisfy (4.5) – (4.8) and that  $F$  has the growth condition

$$|F(x, u, \xi)| \leq a(x) + b(|\xi|^{p/q'})$$

for a.e.  $x \in \Omega$ ,  $\xi \in \mathbb{R}^N$ ,  $u \in [A, B]$  with  $q < p^*$ ,  $a \in L^{p'}(\Omega)$ . Then, (4.2) has a minimal solution  $u_*$  and a maximal solution  $u^*$  such that  $B \leq u_* \leq u^* \leq A$ .

**Remark 4.1.** As shown in Theorem 4.2 (see also Theorem 4.4 below), comparing sub-supersolution with recession method, we note that more flexible conditions are usually required in the first method. In fact, in Theorem 4.2, if there are  $A, B \in \mathbb{R}$  satisfying (4.5) – (4.8), then the growth condition for  $F$  is limited to only  $u \in [A, B]$ . On the other hand, when recession arguments are used (cf. e.g. Theorems 3.4, 3.16 in [2], Theorems 2.3, 4.3 in [5], Theorems 2.5, 4.4, Corollary 6.10 in [21], or Theorems 1, 2, 3 in [20], etc.) conditions on behaviors of the functional  $G$  containing  $F$  at infinity are assumed, which is completely different from our approach here.



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 28 of 37

Another advantage of the method here is that we obtain, in addition to the solvability of (1.1), ordering properties of the solution sets, especially the existence of maximal and minimal solutions. This cannot be obtained by recession arguments. However, sub-supersolution method works only in function spaces with some lattice structure (such as  $W^{1,p}(\Omega)$ ). That is the reason why the method is normally restricted to problems with second-order operators, such as (1.1). Recession methods, on the other hand, are applicable to higher order problems.

## 4.2.

We consider in this example a variational inequality that contains the  $p$ -Laplacian, that is, the inequality (1.1) with

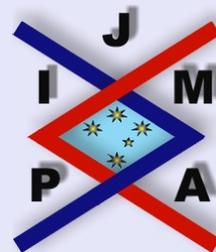
$$\langle L(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx,$$

In this case,  $A_i = |\nabla u|^{p-2} \partial_i u$ , ( $1 \leq i \leq N$ ) and  $A_0 = 0$ . The coefficients  $A_i$  ( $i = 0, 1, \dots, N$ ) clearly satisfy (2.2) and (2.3). For each  $K > 0$ , suppose that the function

$$(4.11) \quad x \mapsto \sup\{|F(x, u, \xi)| : 0 \leq u \leq K, |\xi| \leq K\}$$

belongs to  $L^q(\Omega)$ . We also assume the following behavior of  $F(x, u, \xi)$  when  $u$  is very small or very large:

$$(4.12) \quad \liminf_{u \rightarrow 0^+, |\xi| \rightarrow 0} \frac{F(x, u, \xi)}{u^{p-1}} > \frac{\lambda_0}{\lambda} > \limsup_{u \rightarrow \infty, \xi \in \mathbb{R}^N} \frac{F(x, u, \xi)}{u^{p-1}},$$



### Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 29 of 37

where  $\lambda_0$  is the principal eigenvalue of the  $p$ -Laplacian,

$$\lambda_0 = \inf \left\{ \left( \int_{\Omega} |u|^p dx \right)^{-1} \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

Let  $\phi_0$  be the (unique) eigenfunction corresponding to  $\lambda_0$  such that  $\phi_0(x) > 0$  for all  $x \in \Omega$ . (It is known, see e.g. [25], that  $\phi_0 \in C^{1,\alpha}(\bar{\Omega})$  for some  $\alpha \in (0, 1)$ .) By choosing  $J = 0$  and using the arguments in [17] (Lemma 1), we can show that the function  $\underline{u} = \epsilon\phi_0$  satisfies (2.9) for all  $\epsilon > 0$  sufficiently small. On the other hand, let  $\tilde{\Omega}$  be a bounded open region that contains  $\bar{\Omega}$  and let  $\tilde{\lambda}$  be the principal eigenvalue of the  $p$ -Laplacian on  $\tilde{\Omega}$  and  $\tilde{\phi}$  the corresponding eigenfunction on  $\tilde{\Omega}$  such that  $\tilde{\phi} > 0$  on  $\tilde{\Omega}$ . Then, we can prove that  $\bar{u} = R\tilde{\phi}|_{\tilde{\Omega}}$  satisfies (2.12) (with  $J = 0$ ) for  $R > 0$ , sufficiently large. The proofs of these statements are somewhat lengthy; we refer the reader to [17] for more details. The following lemma is about the construction of sub- and supersolutions of (1.1) based on the eigenfunctions  $\phi_0$  and  $\tilde{\phi}$  of the  $p$ -Laplacian.

**Lemma 4.3.** (a) *If there exists  $C_1 > 0$  such that  $\psi$  is nonincreasing on  $(-\infty, C_1)$ , i.e.,*

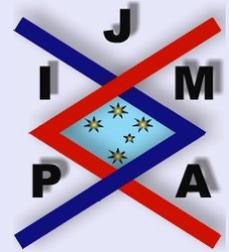
$$(4.13) \quad \psi(x, u) \leq \psi(x, v), \text{ for a.e. } x \in \Omega, \text{ for all } u, v \text{ such that } v \leq u < C_1,$$

*then, for  $\epsilon > 0$  sufficiently small,  $\underline{u} = \epsilon\phi_0$  is a subsolution of (4.2).*

(b) *Similarly, if there exists  $C_2 > 0$  such that  $\psi$  is nondecreasing on  $(C_2, \infty)$ , i.e.,*

$$(4.14) \quad \psi(x, u) \geq \psi(x, v), \text{ for a.e. } x \in \Omega, \text{ for all } u, v \text{ such that } u \geq v > C_2,$$

*then for  $R > 0$  sufficiently large,  $\bar{u} = R\tilde{\phi}|_{\tilde{\Omega}}$  is a supersolution of (4.2).*



**Sub- Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 30 of 37

*Proof.* (a) We need only to check (2.8), i.e.,

$$j(v \vee \epsilon\phi_0) \leq j(v), \quad \forall v \in W_0^{1,p}(\Omega) \cap D(j).$$

This is equivalent to

$$\begin{aligned} \int_{\{x \in \Omega: v < \epsilon\phi_0\}} \psi(x, \epsilon\phi_0) dx + \int_{\{x \in \Omega: v \geq \epsilon\phi_0\}} \psi(x, v) dx \\ \leq \left( \int_{\{x \in \Omega: v < \epsilon\phi_0\}} + \int_{\{x \in \Omega: v \geq \epsilon\phi_0\}} \right) \psi(x, v) dx, \end{aligned}$$

that is,

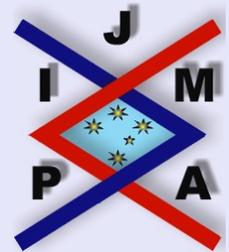
$$(4.15) \quad \int_{\{x \in \Omega: v < \epsilon\phi_0\}} \psi(x, \epsilon\phi_0) dx \leq \int_{\{x \in \Omega: v < \epsilon\phi_0\}} \psi(x, v) dx.$$

Now, since  $\phi_0 \in L^\infty(\Omega)$ ,  $\epsilon\phi_0(x) < C_1$ , for a.e.  $x \in \Omega$  for  $\epsilon > 0$  small. Hence, for  $v < \epsilon\phi_0 < C_1$ , (4.13) implies  $\psi(x, v(x)) \geq \psi(x, \epsilon\phi_0(x))$  for a.e.  $x \in \Omega$ . This implies (4.15). Hence,  $\epsilon\phi_0$  is a subsolution of (4.2). The proof of (b) is similar.  $\square$

As a consequence of Lemma 4.3 and Theorem 3.2, we have the following result.

**Theorem 4.4.** *Under the conditions (4.12) and (4.11), there exist a subsolution  $u_*$  and a supersolution  $u^*$  of (4.2) such that*

$$(0 <) \epsilon\phi_0 \leq u_* \leq u^* \leq R\tilde{\phi}|_\Omega,$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 31 of 37

where  $\epsilon > 0$  sufficiently small and  $R > 0$  sufficiently large. In particular, if  $F$  has the growth condition (4.11) and  $\lambda$  satisfies (4.12), then, (4.2) has a positive solution.

**Remark 4.2.** (a) (4.2) can be seen as an eigenvalue problem for a variational inequality. We have proved that for  $\lambda$  in certain appropriate interval (given by (4.12)), then (4.2) has positive eigenfunction.

(b) One can replace (2.4) by a somewhat different condition, concentrating only on the higher-order coefficients  $A_i$  ( $1 \leq i \leq N$ ). Namely, we assume that  $A_0 = 0$  and instead of (2.4),

$$\sum_{i=1}^N [A_i(x, u, \xi) - A_i(x, u', \xi)](\xi_i - \xi'_i) > 0,$$

for a.e.  $x \in \Omega$ , all  $u \in \mathbb{R}$ , all  $\xi, \xi' \in \mathbb{R}^N$ ,  $\xi \neq \xi'$ , and in (2.5),

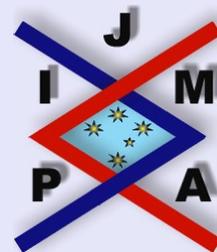
$$\sum_{i=1}^N A_i(x, u, \xi)\xi_i \geq \alpha|\xi|^p - \beta, \quad \text{a.e. } x \in \Omega, \quad \forall u \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,$$

( $\alpha > 0$ ). Also, we need a Hölder-continuity type of assumption with respect to  $u$ :

$$|A_i(x, u, \xi) - A_i(x, u', \xi)| \leq [k(x) + |u|^{p-1} + |u'|^{p-1} + |\xi|^{p-1}]\omega(|u - u'|),$$

for a.e.  $x \in \Omega$ , all  $u, u' \in \mathbb{R}$ , all  $\xi \in \mathbb{R}^N$ , where  $\omega : [0, \infty) \rightarrow [0, \infty)$  satisfies

$$\int_{0+} \frac{dr}{\omega^{p'}(r)} = \infty$$



**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

Title Page

Contents



Go Back

Close

Quit

Page 32 of 37

(cf. [7]). Assume also that  $j$  is given by an integral:

$$j(u) = \int_{\Omega} \psi(x, u(x)) dx,$$

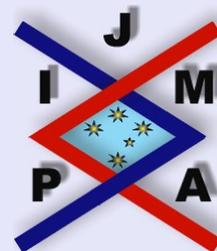
where  $\psi$  satisfies the following growth condition (instead of (2.16)):

$$|\psi(x, u)| \leq a_3(x) + b_3|u|^s, \quad \text{a.e. } x \in \Omega, \forall u \in \mathbb{R},$$

with  $a_3 \in L^1(\Omega)$ ,  $0 \leq s < p^*$ . It can be checked that  $j$  is continuous. By using the arguments in [7] (see also [18]), we can prove the following result:

**Theorem 4.5.** *If  $u_1$  and  $u_2$  satisfy (2.9) with  $J = j$ , then  $u = \max\{u_1, u_2\}$  also satisfies (2.9) with  $J = j$ .*

It follows that if  $u_1, u_2$  are solutions of (1.1), then  $\max\{u_1, u_2\}$  satisfies (2.9).




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**Sub-Supersolutions and the Existence of Extremal Solutions in Noncoercive Variational Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

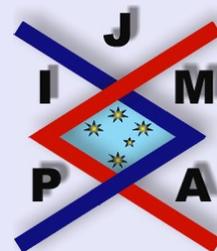
Close

Quit

Page 33 of 37

## References

- [1] D.D. ANG, K. SCHMITT AND V.K. LE, Noncoercive variational inequalities: some applications, *Nonlinear Analysis, TMA*, **15** (1990), 497–512.
- [2] C. BAIOCCHI, G. BUTTAZZO, F. GASTALDI AND F. TOMARELLI, General existence theorems for unilateral problems in continuum mechanics, *Arch. Rational Mech. Anal.*, **100** (1988), 149–189.
- [3] C. BAIOCCHI AND A. CAPELO, *Variational and Quasivariational Inequalities: Applications to Free Boundary Problems*, Wiley, New York, 1984.
- [4] C. BAIOCCHI, F. GASTALDI AND F. TOMARELLI, Some existence results on noncoercive variational inequalities, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **13** (1986), 617–659.
- [5] G. BUTTAZZO AND F. TOMARELLI, Compatibility conditions for nonlinear neumann problems, *Adv. Math.*, **89** (1991), 127–143.
- [6] S. CARL, On the existence of extremal weak solutions for a class of quasilinear parabolic problems, *Diff. and Integral Equations*, **6** (1993), 1493–1505.
- [7] —, Leray–Lions operators perturbed by state-dependent subdifferentials, *Nonlinear World*, **3** (1996), 505–518.
- [8] S. CARL AND H. DIEDRICH, The weak upper and lower solution method for quasilinear elliptic equations with generalized subdifferentiable perturbations, *Appl. Anal.*, **56** (1995), 263–278.



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Sub-Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities

Vy Khoi Le

---

Title Page

Contents



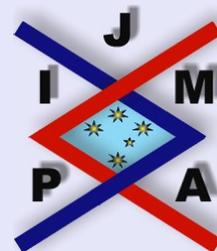
Go Back

Close

Quit

Page 34 of 37

- [9] S. CARL AND S. HEIKKILÄ, On extremal solutions of an elliptic boundary value problem involving discontinuous nonlinearities, *Diff. and Integral Equations*, **5** (1992), 581–589.
- [10] E.N. DANCER AND G. SWEERS, On the existence of a maximal weak solution for a semilinear elliptic equation, *Diff. and Integral Equations*, **2** (1989), 533–540.
- [11] G. DUVAUT AND J.L. LIONS, *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, 1972.
- [12] A. FRIEDMAN, *Variational Principles and Free Boundary Value Problems*, Wiley-Interscience, New York, 1983.
- [13] P. HESS, On the solvability of nonlinear elliptic boundary value problems, *Indiana Univ. Math. J.*, **25** (1976), 461–466.
- [14] D. KINDERLEHRER AND G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
- [15] T. KURA, The weak supersolution-subsolution method for second order quasilinear elliptic equations, *Hiroshima Math. J.*, **19** (1989), 1–36.
- [16] M. KUČERA, A global continuation theorem for obtaining eigenvalues and bifurcation points, *Czech. Math. J.*, **38** (1988), 120–137.
- [17] V.K. LE, Existence of positive solutions of variational inequalities by a subsolution–supersolution approach, *J. Math. Anal. Appl.*, **252** (2000), 65–90.




---

**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



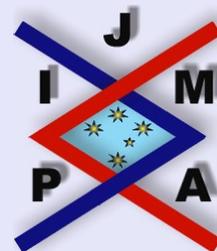
Go Back

Close

Quit

Page 35 of 37

- [18] V.K. LE, Subsolution–supersolution method in variational inequalities, *Nonlinear Analysis*, **45** (2001), 775–800.
- [19] V.K. LE, On global bifurcation of variational inequalities and applications, *J. Diff. Eq.*, **141** (1997), 254–294.
- [20] V.K. LE, Some existence results for noncoercive nonconvex minimization problems with fast or slow perturbing terms, *Num. Func. Anal. and Optim.*, **20** (1999), 37–58.
- [21] V.K. LE and K. SCHMITT, Minimization problems for noncoercive functionals subject to constraints, *Trans. Amer. Math. Soc.*, **347** (1995), 4485–4513.
- [22] ———, Minimization problems for noncoercive functionals subject to constraints, Part II, *Adv. Differential Equations*, **1** (1996), 453–498.
- [23] J.L. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, 1969.
- [24] E. MIERSEMANN, Eigenvalue problems for variational inequalities, *Contemporary Mathematics*, **4** (1984), 25–43.
- [25] M. ÔTANI AND T. TESHIMA, On the first eigenvalue of some quasilinear elliptic equations, *Proc. Japan Acad.*, **64** (1988), 8–10.
- [26] N. PAPAGEORGIOU, F. PAPALINI AND S. VERCILLO, Minimal solutions of nonlinear parabolic problems with unilateral constraints, *Houston J. Math.*, **23** (1997), 189–201.




---

**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



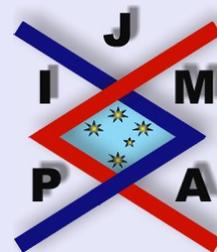
Go Back

Close

Quit

Page 36 of 37

- [27] P. QUITTNER, Solvability and multiplicity results for variational inequalities., *Comment. Math. Univ. Carolin.*, **30** (1989), 281–302.
- [28] B. REDDY AND F. TOMARELLI, The obstacle problem for an elastoplastic body, *Appl. Math. Optim.*, **21** (1990), 89–110.
- [29] F. SCHURICHT, Minimax principle for eigenvalue problems of variational inequalities in convex sets, *Math. Nachr.*, **163** (1993), 117–132.
- [30] A. SZULKIN, On a class of variational inequalities involving gradient operators, *J. Math. Anal. Appl.*, **100** (1982), 486–499.
- [31] —, Positive solutions of variational inequalities: A degree theoretic approach, *J. Diff. Equa.*, **57** (1985), 90–111.
- [32] —, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **3** (1986), 77–109.




---

**Sub- Supersolutions and the  
Existence of Extremal Solutions  
in Noncoercive Variational  
Inequalities**

Vy Khoi Le

---

Title Page

Contents



Go Back

Close

Quit

Page 37 of 37