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IMPROVED INCLUSION-EXCLUSION INEQUALITIES FOR SIMPLEX AND ORTHANT ARRANGEMENTS

[(1)]DANIEL Q. NAIMAN AND [(2)]HENRY P. WYNN

[(1)]DEPARTMENT OF MATHEMATICAL SCIENCES, JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218
daniel.naiman@jhu.edu
URL: http://mts.jhu.edu/~dan

[(2)]DEPARTMENT OF STATISTICS, WARWICK UNIVERSITY
COVENTRY CV4 7AL, UK
hpw@stats.warwick.ac.uk

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ABSTRACT. Improved inclusion-exclusion inequalities for unions of sets are available wherein terms usually included in the alternating sum formula can be left out. This is the case when a key *abstract tube* condition, can be shown to hold. Since the abstract tube concept was introduced and refined by the authors, several examples have been identified, and key properties of abstract tubes have been described. In particular, associated with an abstract tube is an inclusion-exclusion identity which can be truncated to give an inequality that is guaranteed to be at least as sharp as the inequality obtained by truncating the classical inclusion-exclusion identity.

We present an abstract tube corresponding to an orthant arrangement where the inclusion-exclusion formula terms are obtained from the incidence structure of the boundary of the union of orthants. Thus, the construction of the abstract tube is similar to a construction for Euclidean balls using a Voronoi diagram. However, the proof of the abstract tube property is a bit more subtle and involves consideration of abstract tubes for arrangements of simplicies, and intricate geometric arguments based on their Voronoi diagrams.

Key words and phrases: Orthant arrangments, Inclusion-exclusion.

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1. INTRODUCTION

This paper continues work by the authors on a special class of indicator function and probability bounds of the inclusion-exclusion type [8, 9]. These are are based on the *abstract tube* concept and give improvements over bounds produced by truncating the classical inclusion-exclusion identity.

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Definition 1.1. An *abstract tube* is a finite collection of sets $\{A_1, \ldots, A_n\}$ and a finite simplicial complex $\mathcal{S}$ with the following properties:

(i) every vertex of $\mathcal{S}$ corresponds to an index $i \in \{1, \ldots, n\}$, so that $\mathcal{S}$ can be viewed as a collection of subsets of $\{1, \ldots, n\}$, and
(ii) whenever $x \in \bigcup_{i=1}^n A_i$ the subsimplicial complex $\mathcal{S}(x) = \{J \in \mathcal{S} \ : \ x \in \bigcap_{i \in J} A_i\}$ is contractible.

Definition 1.1 is slightly more general than the one in [9] in that we do not require a one-to-one correspondence between vertices in the simplicial complex $\mathcal{S}$ and the index set $\{1, \ldots, n\}$. That is, the index set can be a superset of the set of vertices. All of the properties of abstract tubes given in [9] remain valid for this more general notion of abstract tube. In particular, associated with an abstract tube is an inclusion-exclusion identity for $I_{\bigcup_{i=1}^n A_i}$ based on the terms in $\mathcal{S}$, which can be truncated to give an upper or lower bound. Furthermore, abstract tubes with smaller simplicial complexes leade to sharper truncation inequalities.

Since abstract tubes were introduced, there has been much interest by the authors and others in uncovering new examples of them, while at the same time, there has been reason to suspect that the interesting abstract tubes from a *geometric* point of view always arise from convex polyhedra. Certainly, for the key examples appearing in [9] (see also [7]) where the sets A_i involved are Euclidean balls or unions of half-spaces, a convex polyhedron is present, or lurking, and plays a fundamental role in that it's face incidence structure defines the simplicial complex. Furthermore, the construction of these abstract tubes always involves the nerve of a Voronoi diagram associated with the arrangement of sets.

Dohmen [2, 3, 4, 5] has discovered some new classes of abstract tubes and has demonstrated the utility of the abstract tube concept to network reliability. While these classes of tubes provide many elegant examples with far-reaching applications, the constructions tend to be graph-theoretic and the tubes are defined in *combinatorial* rather than geometric terms. Thus, they do not appear to shed light on the question as to the generality of the Voronoi construction since they apparently correspond to a different class of abstract tubes than the ones considered in [9]. In fact, the authors have not been able to show that the abstract tube formed using balls and the associated Delauney simplicial complex can be realized as one Dohmen's class of abstract tubes.

In this paper, we address the above-mentioned question by describing a pair of new and related examples of abstract tubes, associated with simplex arrangements and orthant arrangements, based on the Voronoi-type construction. The abstract tube property for simplex arrangements is used to derive the abstract tube property for orthant arrangments. While these examples are geometric, the connection with polyhedra is considerably more complex, and the proof of the abstract tube property uses a somewhat more intricate geometric argument than in [9]. There remains the open question as to whether this more general proof technique can be used to verify the abstract tube property for other examples. In Section 2, we develop the tools needed to give the abstract tube associated with arrangements of simplices. The results of this section are key ingredients in Section 3 where we treat abstract tubes based on orthant arrangements.

Aside from being of intrinsic geometric interest the abstract tube for orthants can be used to derive improved reliability bounds for coherent systems. This idea is developed in [10] and used there, in particular, to give a new inclusion-exclusion identity for a k out of n system.

2. VORONOI DECOMPOSITION AND ABSTRACT TUBE BASED ON SIMPLEX ARRANGEMENTS

The results of this section concern arrangements consisting of copies of a regular simplex in $\mathbb{R}^d$, that is, translates of dilations of a simplex, and a certain related Voronoi-type diagram.

Simplex arrangements are closely related to arrangements consisting of translates of a single orthant in $\mathbb{R}^{d+1}$. In fact, the former is obtained by slicing the latter, and this point of view is very important for what follows. It is also the case that, analogous to a certain construction for balls (see [6]) properties of the Voronoi diagram are obtained by projecting the boundary of the orthant arrangement onto the slicing subspace.

For convenience, because of the connection with orthant arrangements, we identify $\mathbb{R}^d$ with the hyperplane

$$H = \left\{ x \in \mathbb{R}^{d+1} \ : \ \sum_{i=1}^{n} x_i = 0 \right\},$$

and we let $\pi_H : \mathbb{R}^{d+1} \to H$ denote the linear projection onto this hyperplane, so that $\pi_H(y) = y - \overline{y}\mathbf{1}$, where $\overline{y} = \frac{1}{d+1}\sum_{i=1}^{d+1} y_i$, and $\mathbf{1}$ denotes the vector whose coordinates are all equal to 1. Let $e^{(1)}, \ldots, e^{(d+1)}$ denote the usual orthonormal basis for $\mathbb{R}^{d+1}$. In order to simplify the notation below, we let $\overline{e} = \frac{1}{d+1}\sum_{i=1}^{d+1} e^{(i)} = \frac{1}{d+1}\mathbf{1}$, and let $\omega_d = \| \overline{e} - e^{(i)} \| = \sqrt{\frac{d}{d+1}}$. Let

$$u^{(i)} = \frac{-\pi_H(e^{(i)})}{\| \pi_H(e^{(i)}) \|} = \omega_d^{-1}(\overline{e} - e^{(i)}), \text{ for } i = 1, \ldots, d+1,$$

so that

$$\langle u^{(i)}, u^{(j)} \rangle = \begin{cases} -1/d & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Having established a coordinate system for $\mathbb{R}^{d+1}$ we can introduce the notation $x \preceq x^*$ for points $x, x^* \in \mathbb{R}^{d+1}$ to mean that $x_i \leq x_i^*$ for all $i = 1, \ldots, d+1$, and we use $x \prec x^*$ to mean that all of the inequalities are strict. We also use the notation $x \succeq x^*$ and $x \succ x^*$ with the obvious reverse interpretation.

Each point $y \in \mathbb{R}^{d+1}$ defines a closed orthant

$$O_y = \left\{ x \in \mathbb{R}^{d+1} \ : \ x \succeq y \right\}$$

which is a translation $y + O_0$, of the usual nonnegative orthant. For $b \in H$ and $r \geq 0$ define the *regular d-simplex in H,*

$$A_{b,r} = \bigcap_{i=1}^{d+1} \left\{ x \in H \ : \ \langle x, u^{(i)} \rangle \leq \langle b, u^{(i)} \rangle + r \right\}.$$

It is easy to see that $A_{b,r}$ is the convex hull of the points $b - rdu^{(i)}, \ i = 1, \ldots, d+1$. This simplex has barycenter b, the Euclidean distance from b to any of the bounding hyperplanes of $A_{b,r}$ is r, and the Euclidean distance from b to any vertex is rd.

More generally, we allow $r < 0$ and still refer to the *simplex* $A_{b,r}$ corresponding to the ordered pair (b, r). This level of generality, where we allow for *virtual simplices*, is very important for the main result of the next section. Thus, the notation $A_{b,r}$ has a dual meaning as it can represent a set (possibly empty) or an ordered pair. It will be clear from the context below which interpretation is appropriate. Generally speaking, when we use $A_{b,r}$ to define a distance, we use the pair (b, r). On the other hand, when we consider Boolean operations involving simplices, then we use the notion of $A_{b,r}$ as a set.

We will use the term *arrangement of orthants* in $\mathbb{R}^{d+1}$ to mean a finite collection $\{O_{y^{(i)}}, \ i = 1, \ldots, n\}$, where $y^{(i)}$ are distinct elements of $\mathbb{R}^{d+1}$ (Figure 2.1) and the term *arrangement of simplices* in $\mathbb{R}^d$ to mean a finite collection $\{A_{b^{(i)},r^{(i)}}, i = 1, \ldots, n\}$, where $b^{(i)} \in H$ and $r^{(i)} \in \mathbb{R}$ and the pairs $(b^{(i)}, r^{(i)})$ are distinct. Note that simplices in an arrangement are allowed to be empty when viewed as sets. Figure 2.1 shows an orthant arrangement.

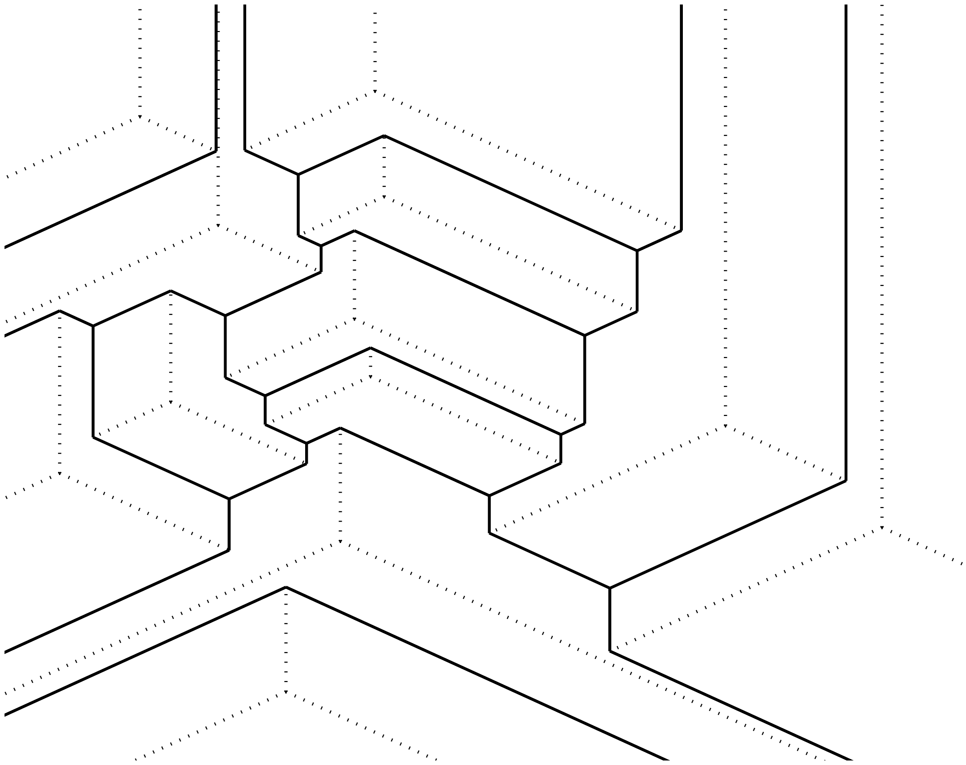

Figure 2.1: An orthant arrangement. The vertices of the orthants are the points where dotted line segments meet, and the solid line segments show where the orthants share common boundaries.

We introduce the *distance* to a simplex in $\mathbb{R}^d$ (H) by defining

$$d_{A_{b,r}}(x) = \max_{i=1,\ldots,d+1} \langle x - b, u^{(i)} \rangle - r, \text{ for } x \in H.$$

Observe that the simplex distance $d_{A_{b,r}}(x)$ is negative, zero, or positive depending on whether x lies in the interior, the boundary or the complement of the simplex $A_{b,r}$. If $r < 0$ then the distance is always negative, which is consistent with the fact that as a set $A_{b,r}$ is empty.

We use this simplex distance to associate a *Voronoi-type* diagram in H with any arrangement of simplices in H. Given an arrangement $\{A_i = A_{b^{(i)},r^{(i)}}, i = 1, \ldots, n\}$ of simplices in H, (we allow for $r^{(i)} \leq 0$) we define

$$S(i|j) = \left\{ x \in H \ : \ d_{A_i}(x) \leq d_{A_j}(x) \right\},$$

and

$$V_i = \bigcap_{j=1}^{n} S(i|j) = \left\{ x \in H \ : \ d_{A_i}(x) = \min_{j=1,\ldots,d} d_{A_j}(x) \right\}.$$

An important tool for constructing a simplicial complex from a collection of sets is the *nerve* construction.

Definition 2.1. The nerve corresponding to a collection of sets $\{V_i, i = 1, \ldots, n\}$ is the simplicial complex consisting of all index sets $J \subseteq \{1, \ldots, n\}$ for which $\bigcap_{i \in J} V_i \neq \emptyset$.

The following theorem, due to Borsuk [1], gives a topological connection between $\bigcup_{i=1}^{n} V_i$ and the nerve of the collection $\{V_i, i = 1, \ldots, n\}$.

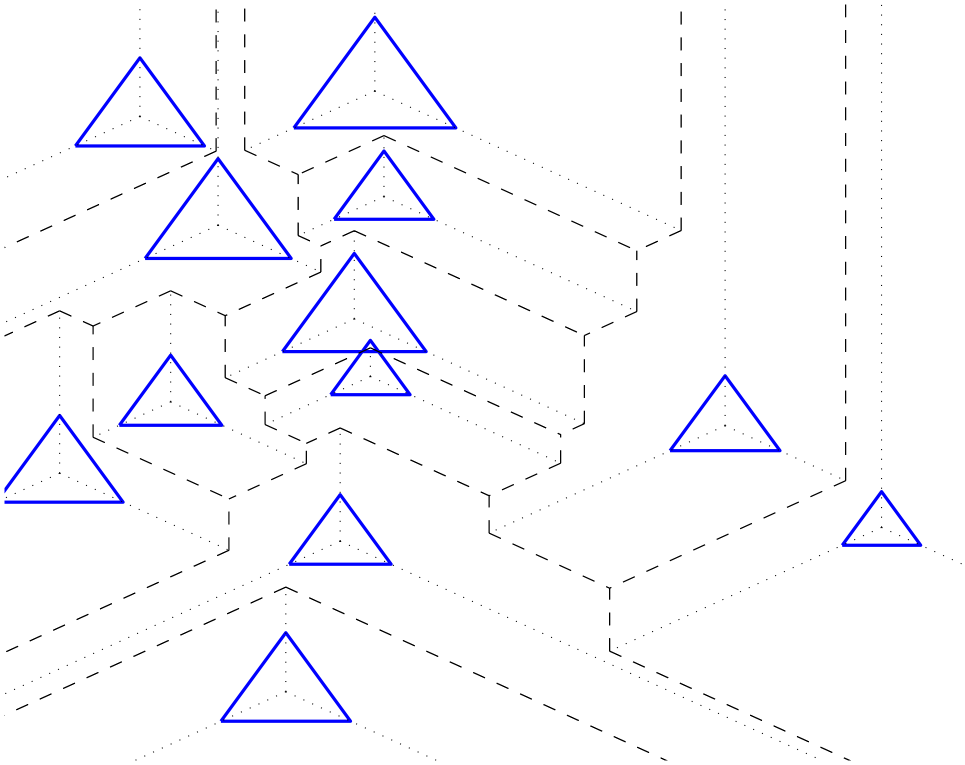

Figure 2.2: Simplex arrangement obtained by slicing the orthant arrangement in Figure 2.1 with the hyperplane H.

Theorem 2.1. *Given a collection of polyhedra $\{V_i, i = 1, \dots, n\}$ in R^d with the property that the intersection $\bigcap_{i \in J} V_i$ is either empty or contractible for all $J \subseteq \{1, \dots, n\}$, the set $\bigcup_{i=1}^n V_i$ and a geometric realization of the nerve of $\{V_i, i = 1, \dots, n\}$ have the same homotopy type.*

Now we can state the main result of this section.

Theorem 2.2. *Given a simplex arrangement $\{A_{b^{(i)},r^{(i)}}, i = 1, \dots, n\}$ let $\mathcal{S}$ be the nerve of the corresponding Voronoi sets. Then the pair $\left(\{A_{b^{(i)},r^{(i)}}, i = 1, \dots, n\}, \mathcal{S}\right)$ forms an abstract tube.*

The proof of this theorem requires several preliminary geometric propositions and lemmas, which we present first. The proofs of these may be found in Section 4. For the remainder of this section we fix a simplex arrangement $\{A_{b^{(i)},r^{(i)}}, i = 1, \dots, n\}$ with Voronoi sets $V_1, \dots, V_n$ as described above.

Proposition 2.3. *Given a point $y \in \mathbb{R}^{d+1}$ with $\overline{y} \leq 0$, we have $O_y \cap H = A_{b,r}$ where $b = y - \overline{y}\mathbf{1}$ and $r = -\overline{y}/\omega_d$.*

We refer to the simplex in Proposition 2.3 as the *simplex corresponding to* the orthant O_y. More generally, we allow for $\overline{y} > 0$ and we can still refer to the simplex $A_{b,r}$, as the simplex corresponding to the orthant O_y, if $b = y - \overline{y}\mathbf{1}$ and $r = -\overline{y}/\omega_d$. Also, we can invert this operation and find a unique orthant O_y *corresponding to* any given simplex $A_{b,r}$ by taking $y = b - r\omega_d\mathbf{1}$. This orthant has the property that $A_{b,r} = O_y \cap H$, if $r \geq 0$. This construction also allows us to associate an orthant arrangement in $\mathbb{R}^{d+1}$ with any arrangement of simplices in $\mathbb{R}^d$, and vice versa. Figure 2 gives the simplex arrangement obtained by slicing the orthant arrangement in Figure 2.1 with the hyperplane H.

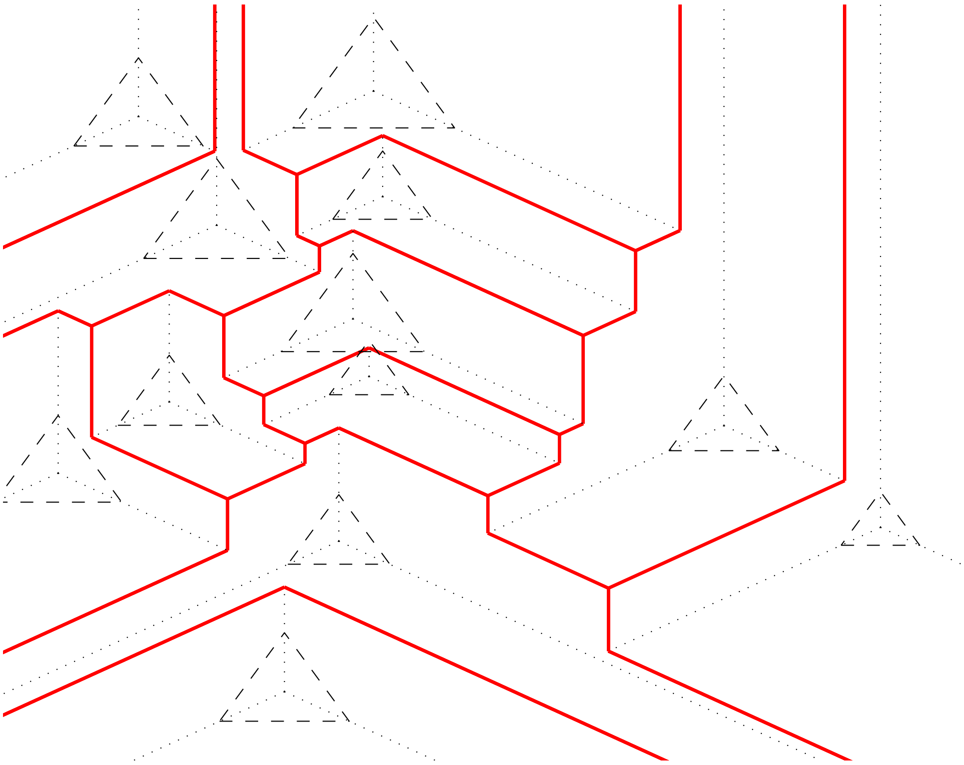

Figure 2.3: The Voronoi diagram associated with the simplex arrangement in Figure 2. Observe that the boundaries of the Voronoi sets correspond to the dashed line segments in Figure 2 and the solid lines in Figure 2.1.

In addition, a ball (with respect to this distance) about a simplex is a simplex. In fact, it is easy to see that

$$\left\{x \in H \;:\; d_{A_{b,r}}(x) \le s\right\} = A_{b,r+s}$$

as subsets of H.

Proposition 2.4. *If $\bigcap_{i=1}^{k} A_{b^{(i)},r^{(i)}} \neq \emptyset$ then $\bigcap_{i=1}^{k} A_{b^{(i)},r^{(i)}} = A_{b,r}$ where $b = -\omega_d(c - \bar{c}\mathbf{1})$, $r = -\bar{c}$, and where $c \in \mathbb{R}^{d+1}$ has coordinates*

$$c_p = \min_{i=1,\ldots,k} \langle b^{(i)}, u^{(p)} \rangle + r^{(i)}, \quad \textit{for } p = 1, \ldots, d+1.$$

In addition, $\max_{i=1,\ldots,k} d_{A_{b^{(i)},r^{(i)}}} = d_{A_{b,r}}$.

Observe that for a given point $b \in H$, the polyhedral cones

$$C_b^{(k)} = \left\{ b - \sum_{q \neq k} \lambda_q u^{(q)} \;:\; \lambda_q \ge 0 \right\} \subseteq H,$$

(with vertex b) cover H and meet only on their (relative) boundaries, which we denote by $\partial C_b^{(k)}$, so that a point $x \in H \backslash \bigcup_{k=1}^{d+1} \partial C_b^{(k)}$ lies in the interior of $C_b^{(k)}$ for a unique choice of index k. We express the simplex distance for a point in one of these cones in the following.

Proposition 2.5. *Given $b \in H$ and $r \ge 0$ and a point $x = b - \sum_{q \neq k} \lambda_q u^{(q)} \in C_b^{(k)}$, we have $d_{A_{b,r}}(x) = \frac{1}{d} \sum_{q \neq k} \lambda_q - r$.*

Proposition 2.6. *The set $\left\{ s \in \mathbb{R} \;:\; x + s\mathbf{1} \in O_{b - r\omega_d \mathbf{1}} \right\}$ forms an interval $[\omega_d d_{A_{b,r}}(x), +\infty)$, for all $r \in \mathbb{R}$ and $b, x \in H$.*

Let $y^{(i)} = b^{(i)} - r^{(i)}\omega_d\mathbf{1}$ so that the orthant $O_i = O_{y^{(i)}}$ corresponds to A_i. As an immediate consequence of Proposition 2.6, we see that

$$\left\{ s \in \mathbb{R} \ : \ x + s\mathbf{1} \in \bigcup_{i=1}^{n} O_i \right\} = \bigcup_{i=1}^{n} [\omega_d d_{A_i}(x), +\infty) = [\omega_d \min_{i=1,\ldots,n} d_{A_i}(x), +\infty).$$

for any point $x \in H$. Thus, the map $\Psi : H \to \partial \left\{ \bigcup_{i=1}^{n} O_i \right\}$ taking x to $x + \omega_d \min_{i=1,\ldots,n} d_{A_i}(x)\mathbf{1}$ gives a homeomorphism between H and $\partial \left\{ \bigcup_{i=1}^{n} O_i \right\}$ whose inverse is the restriction of the projection map π_H to $\partial \left\{ \bigcup_{i=1}^{n} O_i \right\}$. Using Proposition 2.6, it follows that

$$\Psi(V_i) = O_i \backslash \left(\bigcup_{j=1}^{n} O_j \right)^{int}$$

The following two Lemmas form a crucial step in establishing the abstract tube property below. It ensures that Borsuk's Theorem 2.1 can be applied to equate the homotopy type of the union of a collection of Voronoi sets with the nerve of the collection of Voronoi sets. These same results were essential in proving the abstact tube property for balls appearing in [8].

Lemma 2.7. *For every $J \subseteq \{1, \ldots, n\}$ the intersection $\bigcap_{i \in J} V_i$ is either empty or contractible.*

Lemma 2.8. *If $J \subseteq \{1, \ldots, n\}$ then*

$$\bigcup_{i \in J} V_i = \bigcup_{i \in J} \bigcap_{j \notin J} S(i|j).$$

The following result, which is specific to simplex arrangements and their Voronoi diagrams, gives a crucial geometric observation leading to the proof of Theorem 2.2.

Lemma 2.9. *If $x^* \in \bigcup_{i=1}^{n} A_i$ and $J = \{i \ : \ x^* \in A_i\}$ then $\bigcap_{i \in I} S(i|j)$ is nonempty and star-shaped with respect to the barycenter b of $\bigcap_{i \in I} A_i$, for all $I \subseteq J$ and $j \notin J$.*

Figure 2 illustrates the star-shaped property in Lemma 2.9.

Proof of Theorem 2.2. Fix $x^* \in \bigcup_{i=1}^{n} A_i$. We must show the subsimplicial complex $\mathcal{S}(x^*) = \{I \in \mathcal{S} \ : \ x^* \in \bigcap_{i \in I} A_i\}$ is contractible. Let $J = \{i \ : \ x^* \in A_i\}$ so that $\mathcal{S}(x^*)$ is the nerve of the collection $\{V_i, i \in J\}$. By Lemma 2.7 and Borsuk's Theorem 2.1, $\mathcal{S}(x^*)$ has the same homotopy type as $\bigcup_{i \in J} V_i$. By Lemma 2.8, we can write

$$\bigcup_{i \in J} V_i = \bigcup_{i \in J} T_i,$$

where

$$T_i = \bigcap_{j \notin J} S(i|j), \ \text{for } i \in J.$$

If $I \subseteq J$ and we write $\bigcap_{i \in I} A_i = A_{b,r}$ as in Proposition 2.4, then Lemma 2.9 guarantees that $\bigcap_{i \in I} S(i|j)$ is star-shaped with respect to the barycenter b for all $j \notin J$. It follows that

$$\bigcap_{i \in I} T_i = \bigcap_{i \in I} \bigcap_{j \notin J} S(i|j) = \bigcap_{j \notin J} \bigcap_{i \in I} S(i|j)$$

is also star-shaped with respect to b. Since every such intersection is star-shaped, and hence contractible, Borsuk's Theorem 2.1 allows us to conclude that $\bigcup_{i \in J} T_j$ has the same homotopy type as the nerve of the collection $\{T_j, j \in J\}$. But every intersection $\bigcap_{i \in I} T_i$ is nonempty, so the nerve forms a simplex, which is contractible. □

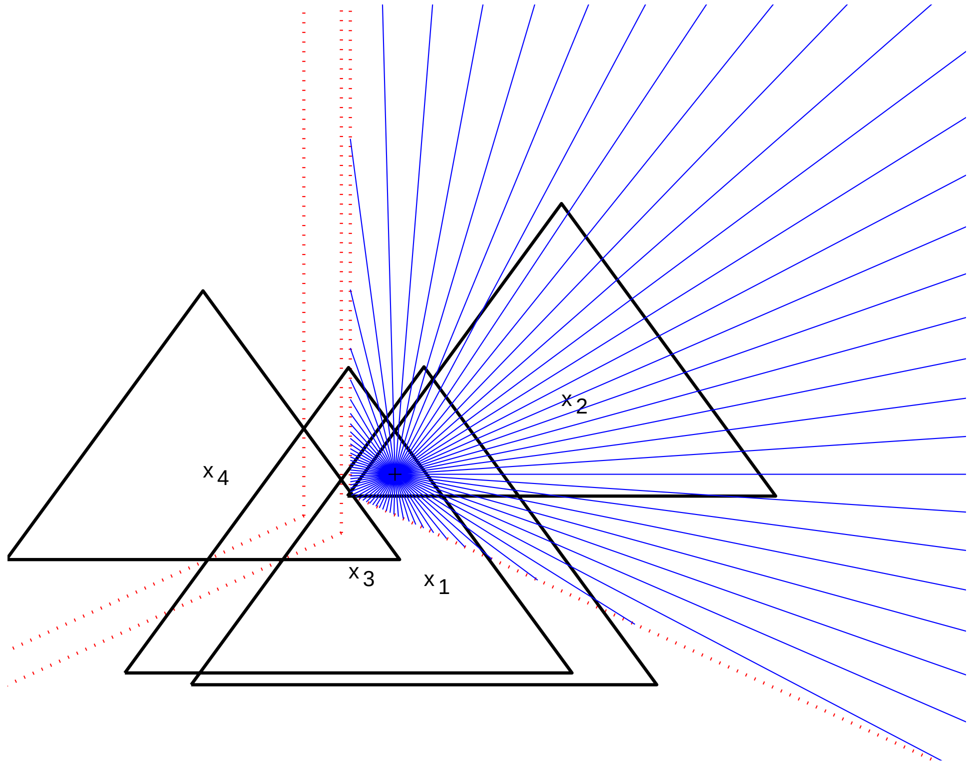

Figure 2.4: Illustration of the star-shaped property in Lemma 2.9. There are 4 triangles with centers labeled $x_1, \ldots, x_4$. The triangles centered at x_1, x_2 and x_3 intersect to form another triangle T, and the set $\bigcap_{i=1,2,3} S(i|4)$, is star-shaped with respect to the barycenter of T. The boundaries of the regions $S(i|4)$, $i = 1, 2, 3$ are drawn using dotted lines.

Remark 2.10. It is of interest to compare the proof of the abstract tube property with the proof appearing in [8] for the case of balls of equal radius, when the nerve of the usual Voronoi diagram is used to form the simplicial complex. There, contractibility of the subsimplicial complex $\mathcal{S}(x^*)$ follows from the fact that the union of Voronoi sets $\bigcup_{i \in J} V_i$ is star-shaped with respect to x^*. In the present case, we do not in general have star-shapedness of this set, but we are able to prove contractibility by representing this union as a union of pieces which always intersect in nonempty star-shaped pieces.

Remark 2.11. Since the distance to a simplex $A_{b,r}$ satisfies $d_{A_{b,r+c}}(x) = d_{A_{b,r}}(x) + c$, it follows that the Voronoi decomposition of H (and hence the associated simplicial complex $\mathcal{S}$) corresponding to a simplex arrangement $\{A_{b^{(i)},r^{(i)}}, i = 1, \ldots, n\}$ is unaffected if we add the same constant to each $r^{(i)}$.

3. ORTHANT ARRANGEMENTS

3.1. A Voronoi decomposition and abstract tube based on orthant arrangements.

Now we apply the results of the previous section to give an analogous result for arrangements consisting of translates of the orthants. To keep the notation consistent with that of the last section, we consider translates of the negative orthant in $\mathbb{R}^{d+1}$. We first introduce an *orthant distance,* which measures the distance to an orthant O_y. Let

$$\tilde{d}_{O_y}(x) = \max_{j=1,\ldots,d+1} \{y_j - x_j\}.$$

Observe that $\tilde{d}_{O_y}(x)$ is less than, equal to, or greater than 0 respectively, depending on whether x lies in the interior, boundary or exterior of O_y.

A collection of orthants $\{O_{y^{(i)}}, i = 1, \ldots, n,\}$ where the $y^{(i)}$ are distinct, will be referred to as an *orthant arrangement* in $\mathbb{R}^{d+1}$. Given such an arrangement, the orthant distance is used to define a Voronoi decomposition of $\mathbb{R}^{d+1}$ by letting

$$\tilde{V}_i = \bigcap_{j=1}^{n} \tilde{S}(i|j)$$

where

$$\tilde{S}(i|j) = \left\{ x \in \mathbb{R}^{d+1} \ : \ \tilde{d}_{O_{y^{(i)}}}(x) \leq \tilde{d}_{O_{y^{(j)}}}(x) \right\}.$$

The main result of this section is the following.

Theorem 3.1. *If $\{O_{y^{(i)}}, i = 1, \ldots, n\}$ is an orthant arrangement in $\mathbb{R}^{d+1}$, then the pair $(\{O_{y^{(i)}}, i = 1, \ldots, n\}, \tilde{\mathcal{S}})$ forms an abstract tube, where $\tilde{\mathcal{S}}$ denotes the nerve of the corresponding Voronoi decomposition $\{\tilde{V}_i, i = 1, \ldots, n\}$ of $\mathbb{R}^{d+1}$.*

Some preliminary Propositions will play a key role in the proof of Theorem 3.1. Proofs of the results presented in this section, except for the proof of the main result, Theorem 3.1, appear in Section 5.

Proposition 3.2. *Given an orthant arrangement $\{O_{y^{(i)}}, i = 1, \ldots, n\}$, the nerve of the corresponding Voronoi decomposition $\{\tilde{V}_i, i = 1, \ldots, n\}$ coincides with the nerve of $\{\tilde{V}_i \cap H, i = 1, \ldots, n\}$.*

The Voronoi decomposition for orthants is closely related to the one in the last section, and exploiting this connection is the key to proving the main result of this section. The basic idea is to introduce a *simplex arrangement associated with a given orthant arrangement* (as in the remark following Propositon 2.3)

$$\{O_{y^{(i)}}, i = 1, \ldots, n\},$$

by taking

$$\{A_{b^{(i)}, r^{(i)}}, i = 1, \ldots, n\},$$

where $b^{(i)} = y^{(i)} - \overline{y}^{(i)}\mathbf{1}$ and $r^{(i)} = -\overline{y}^{(i)}/\omega_d$.

Proposition 3.3. *Given any orthant arrangement $\{O_{y^{(i)}}, i = 1, \ldots, n\}$, in $\mathbb{R}^{d+1}$, let $\{V_i, i = 1, \ldots, n\}$ be the Voronoi decomposition for the associated simplex arrangement. Then the Voronoi decomposition for the orthant and simplex arrangements are related in that*

$$\tilde{V}_i \cap H = V_i,$$

and consequently the nerves of the decompositions coincide.

Finally, we will need the following.

Proposition 3.4. *Given $x, y \in \mathbb{R}^{d+1}$ we have $x \in O_y$ if and only if $x - \overline{x}\mathbf{1} \in A_{y - \overline{y}\mathbf{1}, (\overline{x} - \overline{y})/\omega_d}$.*

Proof of Theorem 3.1. Fix $x \in \bigcup_{i=1}^{n} O_{y^{(i)}}$ and let $\tilde{J} = \{i \ : \ x \in O_{y^{(i)}}\}$. We need to show that the subsimplicial complex defined by

$$\tilde{\mathcal{S}}(x) = \{I \in \tilde{\mathcal{S}} \ : \ x \in \bigcap_{i \in I} O_{y^{(i)}}\} = \{I \in \tilde{\mathcal{S}} \ : \ I \subseteq \tilde{J}\}$$

is contractible.

Consider the simplex arrangement obtained by applying the same construction in Proposition 3.4 to each of the orthants $O_{y^{(i)}}$, that is, take $\{A_{b^{(i)}, r^{(i)}}, i = 1, \ldots, n\}$, where $b^{(i)} = y^{(i)} - \overline{y}^{(i)}\mathbf{1}$, and $r^{(i)} = (\overline{x} - \overline{y}^{(i)})/\omega_d$. Let $\{V_i, \ i = 1, \ldots, n\}$ be the Voronoi decomposition for this simplex arrangment, and let $\mathcal{S}$ denote the corresponding nerve. This Voronoi decomposition is unchanged if we subtract the same constant ($\overline{x}/\omega_d$) from all of the $r^{(i)}$, but this modification leads to the simplex arrangement associated with the original orthant arrangement. We conclude that $\{V_i, i = 1, \ldots, n\}$ is also the Voronoi diagram for this simplex arrangement. By Proposition 3.3 we conclude that $\mathcal{S} = \tilde{\mathcal{S}}$.

By Theorem 2.2 ($\{A_{b^{(i)},r^{(i)}}, i = 1, \ldots, n\}, \mathcal{S}$) forms an abstract tube so if we let $J = \{i : x - \overline{x}\mathbf{1} \in A_{b^{(i)},r^{(i)}}\}$ then the subsimplicial complex defined by

$$\mathcal{S}(x - \overline{x}\mathbf{1}) = \{I \in \mathcal{S} : x - \overline{x}\mathbf{1} \in \bigcap_{i \in I} A_{b^{(i)},r^{(i)}}\} = \{I \in \mathcal{S} : I \subseteq J\}$$

is contractible. But Proposition 3.4 guarantees that $J = \tilde{J}$ so using the fact that $\tilde{\mathcal{S}} = \mathcal{S}$ we can conclude that

$$\tilde{\mathcal{S}}(x) = \mathcal{S}(x - \overline{x}\mathbf{1})$$

so $\tilde{\mathcal{S}}(x)$ is contractible. □

3.2. **Properties of the Orthant Voronoi decomposition.** The Voronoi decomposition $\{\tilde{V}_i, i = 1, \ldots, n\}$ corresponding to a given orthant arrangement $\{O_{y^{(i)}}, i = 1, \ldots, n\}$ has a simple description in terms of the decomposition of the boundary of the union of the orthants. This description helps us in calculating the simplicial complex $\tilde{\mathcal{S}}$.

Let

$$B_i = \partial O_{y^{(i)}} \backslash \left(\bigcup_{i=1}^{n} O_{y^{(i)}} \right)^{int}$$

so that the B_i define a decomposition of the boundary

$$B = \partial \left(\bigcup_{i=1}^{n} O_{y^{(i)}} \right) = \bigcup_{i=1}^{n} B_i.$$

Proposition 3.5. *For a nonempty index set J, we have $J \in \tilde{\mathcal{S}}$ if and only if $\bigcap_{i \in J} B_i \neq \emptyset$. In other words, the nerve of the Voronoi decomposition $\{\tilde{V}_i, i = 1, \ldots, n\}$ coincides with the nerve of the decomposition $\{B_i, i = 1, \ldots, n\}$ of B.*

Definition 3.1. An orthant $O_{y^{(i)}}$ in an orthant arrangement $\{O_{y^{(i)}}, i = 1, \ldots, n\}$ is *exposed* if $O_{y^{(i)}} \not\subseteq \bigcup_{j=1}^{n} O_{y^{(j)}}^{int}$.

Observe that $O_{y^{(i)}} \subseteq \bigcup_{j=1}^{n} O_{y^{(j)}}^{int}$ if and only if $y^{(i)} \in \bigcup_{j=1}^{n} O_{y^{(j)}}^{int}$, and this is in turn is equivalent to $y^{(i)} \succ y^{(j)}$ for some $j \neq i$. Thus, the exposed orthants correspond to those indices i for which $y^{(i)} \not\succ y^{(j)}$ for all j.

We use the notation $\max_{j \in J} y^{(j)}$ to mean the coordinatewise maximum of the $y^{(j)}$ for $j \in J$. As consequence of Proposition 3.5 we have the following description of the faces of the nerve of the Voronoi decomposition.

Corollary 3.6. *The faces of $\tilde{\mathcal{S}}$ correspond to the (nonempty) index sets J for which $\max_{i \in J} y^{(i)} \not\succ y^{(j)}$ for all j. In particular, the vertices of $\tilde{\mathcal{S}}$ (the single element faces) correspond to the exposed orthants.*

Following Corollary 3.6 we can say equivalently that the index set J, or the point $y = \max_{i \in J} y^{(i)}$, or the orthant O_y is *covered*.

3.3. **General position and dimension.** For a generic orthant arrangement $\{O_{y^{(i)}}, i = 1, \ldots, n\}$ in $\mathbb{R}^{d+1}$ the simplicial complex defining the tube above, that is, the nerve $\tilde{\mathcal{S}}$ of the Voronoi decomposition, has dimension $d+1$. As a consequence, the inclusion-exclusion identity has depth $d+2$ instead of n, which can lead to a dramatic improvement. We make this rigorous as follows.

Definition 3.2. An orthant arrangement $\{O_{y^{(i)}}, i = 1, \ldots, n\}$ in $\mathbb{R}^{d+1}$ is in *general position* if for every coordinate index j the values $y_j^{(i)}, i = 1, \ldots, n$ are distinct. In other words, $y_j^{(i)} = y_j^{(k)}$ for some i, j, k implies $i = k$.

The orthant arrangements that fail to be in general position define a set of Lebesgue measure zero in the set of orthant arrangements. Under the general position assumption the dimension of the simplicial complex defining the tube has the *right* dimension.

Proposition 3.7. *If an orthant arrangement is in general position then the simplicial complex $\tilde{\mathcal{S}}$ defining the abstract tube in Theorem 3.1 has dimension at most $d+1$.*

When an orthant arrangement fails to be in general position, it is still possible to perturb it slightly to attain general position, and use the modified arrangement to obtain improved inclusion-exclusion identities and inequalities that are valid almost everywhere. This idea is explored in [9] for abstract tubes related to polyhedra, and an analogous result can be used in the present context. In [10], abstract tubes based on orthant arrangements are used to derive new reliability bounds for coherent systems, and in that context, perturbation is used to give even further improved inclusion-exclusion indicator identities and inequalities.

3.4. **Inclusion-Exclusion Inequalities and Identities for Orthant Unions.** Using Theorem 4 in [9] the abstract tube property leads immediately to the following.

Theorem 3.8. *Given a finite collection of distinct points $y^{(i)}, i = 1, \ldots, n$ in $\mathbb{R}^d$, define*

$$\mathcal{S} = \{J \subseteq \{1, \ldots, n\} \ : \ \max_{i \in J} y^{(i)} \not\succ y^{(j)}, \text{ for all } j = 1, \ldots, n\}.$$

Then the following indicator function inequalities hold

$$(-1)^{m+1} I_{\bigcup_{i=1}^n O_i} \le (-1)^{m+1} \left\{ \sum_{k=1}^{m} (-1)^{k+1} \sum_{J \in \mathcal{S} \,:\, |J| = m} I_{\bigcap_{j \in J} O_j} \right\}, \text{ for } m = 1, 2, \ldots, D,$$

where O_i denotes $O_{y^{(i)}}$, and $D = \max\{|J| \ : \ J \in \mathcal{S}\}$. In addition, equality holds for $m = D$. Each inequality is at least as sharp as the corresponding classical inclusion-exclusion inequality

$$(-1)^{m+1} I_{\bigcup_{i=1}^n O_i} \le (-1)^{m+1} \left\{ \sum_{k=1}^{m} (-1)^{k+1} \sum_{J \subseteq \{1,\ldots,n\} \,:\, |J| = m} I_{\bigcap_{j \in J} O_j} \right\},$$

corresponding to the abstract tube using a simplicial complex composed of all nonempty index sets.

The theorem also holds if we use negative orthants instead of positive ones, that is, if we use as the definition of $O_{y^{(i)}}$

$$\{x \in \mathbb{R}^d \ : \ x \preceq y^{(i)}\},$$

and if we redefine $\mathcal{S}$ to be

$$\{J \subseteq \{1, \ldots, n\} \ : \ \min_{i \in J} y^{(i)} \not\prec y^{(j)}, \text{ for all } j = 1, \ldots, n\}.$$

4. PROOFS OF PROPOSITIONS AND LEMMAS IN SECTION 2

Proof of Proposition 2.3 Here, we are viewing a simplex as a *set.* For any $x \in H$ we have $x \succeq y$ if and only if

$$\langle x, -e^{(i)}\rangle \leq \langle y, -e^{(i)}\rangle, \text{ for } i = 1, \dots, d+1.$$

Since $\langle z, \overline{e}\rangle = 0$ for $z \in H$, this is equivalent to

$$\langle x, \overline{e} - e^{(i)}\rangle \leq \langle y - \overline{y}\mathbf{1}, \overline{e} - e^{(i)}\rangle - \langle \overline{y}\mathbf{1}, e^{(i)}\rangle \text{ for } i = 1, \dots, d+1,$$

which, upon dividing by $\omega_d = \| e^{(i)} - \overline{e} \|$ leads to the equivalent condition

$$\langle x, u^{(i)}\rangle \leq \langle y - \overline{y}\mathbf{1}, u^{(i)}\rangle - \overline{y}\langle \mathbf{1}, e^{(i)}\rangle / \omega_d \text{ for } i = 1, \dots, d+1.$$

□

Proof of Proposition 2.4. For the first claim, we can use the comment following Proposition 2.3 to write $A_{b^{(i)},r^{(i)}} = O_{y^{(i)}} \cap H$, where $y^{(i)} = b^{(i)} - r^{(i)}\omega_d\mathbf{1}$. Then we have

$$\bigcap_{i=1}^{k} A_{b^{(i)},r^{(i)}} = \bigcap_{i=1}^{k} O_{y^{(i)}} \cap H = O_z \cap H$$

where $z = \max_{i=1,\dots,k} y^{(i)}$, the maximum being coordinatewise. A straightforward calculation gives

$$z_p = -\omega_d \min_{i=1,\dots,k} \left\{ \langle y^{(i)}, u^{(p)}\rangle + r^{(i)} \right\},$$

so the result follows from the application of Propsition 2.3 For the second claim, we have

$$\max_{i=1,\dots,k} d_{A_{b^{(i)},r^{(i)}}}(x) = \max_{i=1,\dots,k} \max_{p=1,\dots,d+1} \langle x, u^{(p)}\rangle - \langle b^{(i)}, u^{(p)}\rangle - r^{(i)}$$

$$= \max_{p=1,\dots,d+1} \langle x, u^{(p)}\rangle - \min_{i=1,\dots,k} \left\{ \langle b^{(i)}, u^{(p)}\rangle + r^{(i)} \right\} = \max_{p=1,\dots,d+1} \langle x, u^{(p)}\rangle - \left\{ \langle b, u^{(p)}\rangle + r \right\}$$

$$= d_{A_{b,r}}(x).$$

□

Proof of Proposition 2.5. We have

$$d_{A_{b,r}}(x) = \max_p \left\{ \langle - \sum_{q \neq k} \lambda_q u^{(q)}, u^{(p)}\rangle \right\} - r = -\min_p \left\{ \sum_{q \neq k} \lambda_q \langle u^{(q)}, u^{(p)}\rangle \right\} - r.$$

The result then follows from the fact that

$$\sum_{q \neq k} \lambda_q \langle u^{(q)}, u^{(p)}\rangle = \begin{cases} -\frac{1}{d} \sum_{q \neq k} \lambda_q & \text{if } p = k \\ -\frac{1}{d} \sum_{q \neq p,k} \lambda_q + \lambda_p & \text{if } p \neq k. \end{cases}$$

□

Proof of Proposition 2.6. Since $x + s\mathbf{1} \in O_{b - r\omega_d \mathbf{1}}$ if and only if $x + s\mathbf{1} \succeq b - r\omega_d\mathbf{1}$, we see that the set $\left\{ s \in \mathbb{R} \;:\; x + s\mathbf{1} \in O_{b - r\omega_d \mathbf{1}} \right\}$, forms an interval that is closed on the left and extends to infinity on the right. The minimum value of s in this interval is given by

$$\max_{i=1,\dots,d+1} -(x_i - b_i) - r\omega_d = \max_{i=1,\dots,d+1} -\langle x - b, e^{(i)}\rangle - r\omega_d = \max_{i=1,\dots,d+1} -\langle x - b, e^{(i)} - \overline{e}\rangle - r\omega_d$$

$$= \max_{i=1,\dots,d+1} \langle x - b, \omega_d u^{(i)}\rangle - r\omega_d = \omega_d d_{A_{b,r}}(x).$$

□

Proof of Lemma 2.7. Since Ψ is a homeomorphism, and

$$\Psi(\bigcap_{i\in J} V_i) = \bigcap_{i\in J}\Psi(V_i) = \left(\bigcap_{j\in J} O_j\right)\setminus\left(\bigcup_{i=1}^{n} O_i\right)^{int}$$

it suffices to show that if the set

$$W = \left(\bigcap_{j\in J} O_j\right)\setminus\left(\bigcup_{i=1}^{n} O_i\right)^{int}$$

is nonempty, then it is contractible.

Suppose $z \in W$ so that $z \succeq y^{(j)}$ for all $j \in J$ and $z \not\succ y^{(i)}$, for $i = 1, \dots, n$. If we define $v = \max_{j\in J} y^{(j)}$ then observe that $v \in \bigcap_{j\in J} O_j$, and $z \succeq v$. Furthermore, if it were the case that $v \in O_i^{int}$ for some index i, so that $v \succ y^{(i)}$, then we would have $z \succ y^{(i)}$ and this is a contradiction. We conclude that $v \in W$.

We proceed to show W is star-shaped with respect to v. Suppose $w \in W$ and $\lambda \in [0, 1]$ then $w_\lambda = (1 - \lambda)w + \lambda v \in O_j$ for all $j \in J$ by convexity. We proceed to show $w_\lambda \notin O_i^{int}$ for $i = 1, \dots, n$. Since $w \in \bigcap_{j\in J} O_j$ we have $w \succeq v$, and it follows that $w \succeq w_\lambda \succeq v$. Consequently, if $w_\lambda \in O_i^{int}$ we obtain $w \in O_i^{int}$, which is a contradiction. □

Proof of Lemma 2.8. On the one hand

$$\bigcup_{i\in J} V_i = \bigcup_{i\in J}\bigcap_{j=1}^{n} S(i|j) \subseteq \bigcup_{i\in J}\bigcap_{j\notin J} S(i|j).$$

On the other hand, suppose $x \in \bigcup_{i\in J}\bigcap_{j\notin J} S(i|j)$ so that for some $i^* \in J$ we have $d_{A_{i^*}}(x) \le d_{A_j}$, for all $j \notin J$. Let $i^{**} \in J$ minimize $d_{A_{i^{**}}}(x)$. It follows that $d_{A_{i^{**}}}(x) \le d_{A_j}(x)$ for all $j = 1, \dots, n$, that is $x \in \bigcup_{i\in J}\bigcap_{j=1}^{n} S(i|j)$. □

Proof of Lemma 2.9. We can use Proposition 2.4 to write $\bigcap_{i\in I} A_i = A_{b,r}$ since $\bigcap_{i\in I} A_i \ne \emptyset$, b being the barycenter of the simplex $A_{b,r}$. Using the second part of Proposition 2.4, we see that

$$\bigcap_{i\in I} S(i|j) = \bigcap_{i\in I}\{x \ : \ d_{A_i}(x) \le d_{A_j}(x)\}$$
$$= \{x \ : \ \max_{i\in I} d_{A_i}(x) \le d_{A_j}(x)\} = \{x \ : \ d_{A_{b,r}}(x) \le d_{A_j}(x)\}.$$

To prove the claim of star-shapedness of $\bigcap_{i\in I} S(i|j)$ it suffices to show that the intersection of any ray emanating from the barycenter b with the set $\bigcap_{i\in I} S(i|j)$ forms a line segment containing b. So fix a ray, say $\{b - \eta\sum_{q\ne k}\lambda_q u^{(q)} \ : \ \eta \ge 0\}$, for some index k and nonnegative constants λ_q for $q \ne k$, and define

$$f(\eta) = d_{A_{b,r}}(b - \eta\sum_{q\ne k}\lambda_q u^{(q)}),$$

and

$$g(\eta) = d_{A_j}(b - \eta\sum_{q\ne k}\lambda_q u^{(q)}).$$

The proof will be complete once we have demonstrated that

$$V = \{\eta \ge 0 \ : \ f(\eta) \le g(\eta)\}$$

is an interval containing 0.

Using Proposition 2.5, we obtain $f(\eta) = \left(\frac{1}{d}\sum_{q\neq k}\lambda_q\right)\eta - r$. Thus, we see that
(i) f is linear, with $f(0) = -r$ and slope $\frac{1}{d}\sum_{q\neq k}\lambda_q$.
On the other hand, from the definition of simplex distance $g(\eta) = \max_{p=1,\dots,d+1} g_p(\eta)$, where

$$g_p(\eta) = \langle b - \eta\sum_{q\neq k}\lambda_q u^{(q)} - b^{(j)}, u^{(p)}\rangle - r^{(j)}.$$

Since

$$A_{b,r} = \bigcap_{p=1}^{d+1}\{x \in H \ : \ \langle x, u^{(p)}\rangle \leq \langle b, u^{(p)}\rangle + r\}$$

and

$$A_j = \bigcap_{p=1}^{d+1}\{x \in H \ : \ \langle x, u^{(p)}\rangle \leq \langle b^{(j)}, u^{(p)}\rangle + r^{(j)}\}$$

and $A_{b,r} \not\subseteq A_j$ it must be the case that

$$\langle b, u^{(p)}\rangle + r > \langle b^{(j)}, u^{(p)}\rangle + r^{(j)}$$

for some index p. This leads to the conclusion that
(ii) $g(0) = \max_{p=1,\dots,d+1}\langle b - b^{(j)}, u^{(p)}\rangle - r^{(j)} > -r = f(0)$
Finally, each function g_p is linear g is piecewise linear and convex.
In addition, the slope of g_p is given by $-\langle\sum_{q\neq k}\lambda_q u^{(q)}, u^{(p)}\rangle$, so the same calculation as in the proof of Proposition 2.5 shows that the maximum slope occurs for g_k, and this function has the same slope as f. We have therefore shown that
(iii) g is piecewise linear and convex (and continuous), and the maximum slope of g, where g is differentiable, is the same as the slope of f.
Using properties (i), (ii) and (iii), it is easy to see that $0 \in V$, and either the graphs of f and g do not cross, or they cross at a single point, or they meet in an interval of the form $[\eta^*, +\infty)$. In each case, the set V forms an interval containing 0. □

5. PROOFS OF PROPOSITIONS IN SECTION 3

Proof of Proposition 3.2 The orthant distance satisfies

$$\tilde{d}_{O_y}(x + c\mathbf{1}) = \tilde{d}_{O_y}(x) + c,$$

and consequently

$$\tilde{d}_{O_{y^{(i)}}}(x + c\mathbf{1}) \leq \tilde{d}_{O_{y^{(j)}}}(x + c\mathbf{1})$$

if and only if

$$\tilde{d}_{O_{y^{(i)}}}(x) \leq \tilde{d}_{O_{y^{(j)}}}(x).$$

Thus, each $\tilde{V}_i$ is the union of the set of lines of the form $\{x + c\mathbf{1} \ : \ c \in \mathbb{R}\}$ where $x \in \tilde{V}_i \cap H$. It follows immediately that the nerve of the $\{\tilde{V}_i, i = 1, \dots, n\}$ coincides with the nerve of the $\{\tilde{V}_i \cap H\}$. □

Proof of Proposition 3.3 For $x \in H$ a straightforward calculation shows that

$$d_{A_{b^{(i)},r^{(i)}}}(x) = \max_p\left\{\langle x, u^{(p)}\rangle - \langle b^{(i)}, u^{(p)}\rangle - r^{(i)}\right\} = \max_p\{y_p - x_p\}/\omega_d = \tilde{d}_{O_{y^{(i)}}}(x)/\omega_d.$$

Thus

$$d_{A_{b^{(i)},r^{(i)}}}(x) \leq d_{A_{b^{(j)},r^{(j)}}}(x),$$

if and only if

$$\tilde{d}_{O_{y^{(i)}}}(x) \le \tilde{d}_{O_{y^{(j)}}}(x),$$

for $x \in H$. The second claim follows from Proposition 3.2. $\square$

Proof of Proposition 3.4 We have $xp \in O_y$ if and only if $x - \overline{x}\mathbf{1} \in O_{y-\overline{x}\mathbf{1}}$, which is equivalent to

$$\tilde{d}_{O_{y-\overline{x}\mathbf{1}}}(x - \overline{x}\mathbf{1}) \le 0.$$

Since $x - \overline{x}\mathbf{1} \in H$ we can use the calculation in the proof of Proposition 3.3 to conclude that an equivalent condition is

$$d_{A_{y-\overline{y}\mathbf{1},(\overline{x}-\overline{y})/\omega_d}}(x - \overline{x}\mathbf{1}) \le 0,$$

and this gives the desired result. $\square$

Proof of Proposition 3.5 If $\bigcap_{i\in J} \tilde{V}_i \neq \emptyset$, fix $x \in \bigcap_{i\in J} \tilde{V}_i$. Let $d^* = \min_{j=1,\dots,n} d_{O_{y^{(j)}}}(x)$, so that $d_{O_{y^{(i)}}}(x) = d^*$ for $i \in J$ and $d_{O_{y^{(i)}}}(x) > d^*$ for $i \notin J$. If $x^* = x + d^*\mathbf{1}$ then we have $d_{O_{y^{(i)}}}(x^*) = 0$ for $i \in J$ and $d_{O_{y^{(i)}}}(x^*) > 0$ for $i \notin J$, thus $x^* \in \partial O_{y^{(i)}}$ for $i \in J$, and $x^* \notin \left\{\bigcup_{i=1}^n O_{y^{(i)}}\right\}^{int} = \bigcup_{i=1}^n O_{y^{(i)}}^{int}$. We conclude that $x^* \in \bigcap_{i\in J} B_i$. Conversely, if $x \in \bigcap_{i\in J} B_i$ then for $i \in J$ we have $x \in \partial O_{y^{(i)}}$ so $\tilde{d}_{O_{y^{(i)}}}(x) = 0$. Furthermore, for all i we have $x \notin O_{y^{(i)}}^{int}$ so $\tilde{d}_{O_{y^{(i)}}}(x) \ge 0$. We conclude therefore, that $x \in \bigcap_{i\in J} \tilde{V}_i$. $\square$

Proof of Corollary 3.6 We use the characterization of faces in Proposition 3.5. Fix a nonempty index set J and let $m = \max_{i\in J} y^{(i)}$.

Suppose $\bigcap_{i\in J} B_i \neq \emptyset$, and let $x \in \bigcap_{i\in J} B_i$. Then $x \in O_{y^{(i)}}$ for $i \in J$ and $x \notin O_{y^{(i)}}^{int}$ for all i. It follows that $x \succ y^{(i)}$ and hence $x \succ m$. Furthermore, we cannot have $m \succ y^{(j)}$ for some j since this would give $x \succ y^{(j)}$. This proves that m satisfies the stated condition.

Conversely, if $m \not\succ y^{(j)}$ for all j, then for $i \in J$ we have $m \in O_{y^{(i)}} \backslash \bigcup_{i=1}^n O_{y^{(i)}}^{int} = B_i$, so $\bigcap_{i\in J} B_i \neq \emptyset$. $\square$

Proof of Proposition 3.7 Suppose an index set J defines a face in $\mathcal{S}$, and let $m = \max_{i\in J} y^{(i)}$. By the general position assumption, for each coordinate index j there is a unique index $i_j \in J$ such that $m_j = y_j^{(i_j)}$. If $|J| \ge d+2$ then since $\{i_j, j = 1, \dots, d+1\}$ consists of at most $d+1$ elements, there must be some index $k \in J\backslash\{i_j, j = 1, \dots, d+1\}$. It follows that $m \succ y^{(k)}$, which contradicts the characterization in Corollary 3.6. $\square$

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