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SUBHARMONIC FUNCTIONS AND THEIR RIESZ MEASURE

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

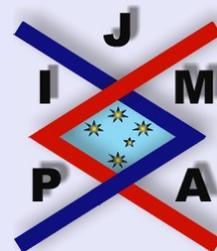
For subharmonic functions u in \mathbb{R}^N , of Riesz measure μ , the growth of the function $s \mapsto \mu(s) = \int_{|\zeta| \leq s} d\mu(\zeta)$ ($s \geq 0$) is described and compared with the growth of u . It is also shown that, if $\int_{\mathbb{R}^N} u^+(x) [-\varphi'(|x|^2)] dx < +\infty$ for some decreasing C^1 function $\varphi \geq 0$, then $\int_{\mathbb{R}^N} \frac{1}{|\zeta|^2} \varphi(|\zeta|^2 + 1) d\mu(\zeta) < +\infty$. Given two subharmonic functions u_1 and u_2 , of Riesz measures μ_1 and μ_2 , with a growth like $u_i(x) \leq A + B|x|^\gamma \forall x \in \mathbb{R}^N$ ($i = 1, 2$), it is proved that $\mu_1 + \mu_2$ is not necessarily the Riesz measure of a subharmonic function u with such a growth as $u(x) \leq A' + B'|x|^\gamma \forall x \in \mathbb{R}^N$ (here $A > 0$, $A' > 0$ and $0 < B' < 2B$).

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Key words: Subharmonic functions, order of growth, Riesz measure.

Contents

1	Introduction	4
2	Some Preliminaries	7
3	Estimations of the Riesz Measure	10
	3.1 Jensen–Privalov formula	10
	3.2 The case $N = 2$	10
	3.3 The case $N \geq 3$	12
4	Growth of the Repartition Function	15
	4.1 A measure on $[0, +\infty[$, image of μ	15
	4.2 The case $N = 2$	15
	4.3 Proof of Theorem 4.1 in the case of a continuous repartition function	16



Subharmonic Functions and their Riesz Measure

Raphaela Supper

Title Page

Contents



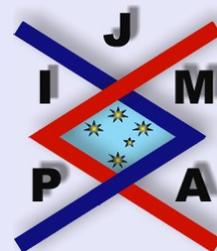
Go Back

Close

Quit

Page 2 of 33

4.4	Splitting measure μ	17
4.5	A reformulation of Theorem 4.1	19
4.6	The case $N \geq 3$	20
4.7	Proof of Theorem 4.3 in the case of a continuous repartition function	21
4.8	A reformulation of Theorem 4.3	23
5	Sum of Two Riesz Measures	25
6	Subharmonic Functions Subject to Conditions of L^1 Type ..	27
6.1	A weighted integral condition for subharmonic func- tions.	27
6.2	Proof of Theorem 6.1 in the case $N = 2$	28
6.3	Proof of Theorem 6.1 in the case $N \geq 3$	29
References		



**Subharmonic Functions and
their Riesz Measure**

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 3 of 33

1. Introduction

Let μ be the Riesz measure of some subharmonic function u in \mathbb{R}^N ($N \in \mathbb{N}$, $N \geq 2$ and u non identically $-\infty$, see [1, p. 104]) and $\mu(s) = \int_{|\zeta| \leq s} d\mu(\zeta)$ for any $s \geq 0$ (where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^N). The function $s \mapsto \mu(s)$ is non-decreasing since μ is a positive measure. The order of the function $s \mapsto s^{2-N} \mu(s)$ is known to coincide with the convergence exponent of μ :

$$\inf \left\{ c : \int_1^{+\infty} s^{2-N-c} d\mu(s) \right\} = \inf \left\{ c : \int_1^{+\infty} s^{1-N-c} \mu(s) ds \right\}$$

(see [2, p. 66]) and does not exceed γ if u has a growth of the kind:

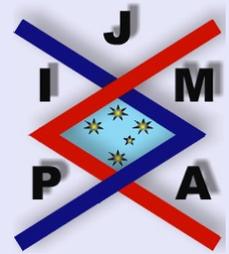
$$(1.1) \quad u(x) \leq A + B|x|^\gamma \quad \forall x \in \mathbb{R}^N$$

(with constants $A \in \mathbb{R}$, $B > 0$ and $\gamma > 0$). This estimation of the growth of $\mu(s)$ will be examined below, in Sections 3 and 4.

Definition 1.1. Given $\gamma > 0$ and $B > 0$, let $SH(\gamma, B)$ stand for the set of all subharmonic functions u in \mathbb{R}^N which are harmonic in some neighbourhood of the origin with $u(0) = 0$ and which satisfy estimate (1.1) for some constant $A \in \mathbb{R}$.

In Proposition 5.2 (see Section 5), a counterexample is produced to show that, given u_1 and u_2 two functions in this set $SH(\gamma, B)$ and $B' \in]0, 2B[$, the sum of their respective Riesz measures μ_1 and μ_2 is not necessarily the Riesz measure of a function of $SH(\gamma, B')$.

Of course $\mu_1 + \mu_2$ is the Riesz measure associated with $u_1 + u_2 \in SH(\gamma, 2B)$, but $\mu_1 + \mu_2$ is also the Riesz measure of $u_1 + u_2 - h$ for any harmonic function h



Title Page

Contents



Go Back

Close

Quit

Page 4 of 33

in \mathbb{R}^N . This proposition means that there does not necessarily exist a harmonic function h such that $u_1 + u_2 - h \in SH(\gamma, B')$.

Let μ denote the Riesz measure of some function of $SH(\gamma, B)$ with growth (1.1). Sections 3 and 4 are devoted to the growth of the repartition function $s \mapsto \mu(s)$. For instance, when $N = 2$, we obtain the inequality: $\mu(s) \leq B e^\gamma s^\gamma e^{\frac{A\gamma}{\mu(s)}}$ (see Theorem 3.1 and Corollary 3.2).

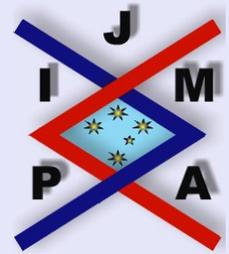
Notation. When $N \geq 3$, throughout the paper we set $C(\gamma, N) = \left(\frac{\gamma+N-2}{\gamma}\right)^{\frac{\gamma+N-2}{N-2}}$ and $D(B, \gamma, N) = \frac{\gamma+N-2}{\gamma} \left(\frac{B\gamma}{N-2}\right)^{\frac{N-2}{\gamma+N-2}}$, sometimes written merely D for brevity.

Note that

$$\begin{aligned} \frac{\gamma}{N-2} C(\gamma, N) &= \frac{\gamma}{N-2} \left(\frac{\gamma+N-2}{\gamma}\right)^{\frac{\gamma+N-2}{N-2}} \\ &= \frac{\gamma+N-2}{N-2} \left(1 + \frac{N-2}{\gamma}\right)^{\frac{\gamma}{N-2}} \\ &\leq e \frac{\gamma+N-2}{N-2}. \end{aligned}$$

For $N \geq 3$, we also obtain inequalities describing the growth of $s \mapsto \mu(s)$ and the constants involved in these estimations are given explicitly in terms of A, B and γ . For example:

$$\mu(s) \leq \frac{B\gamma}{N-2} C(\gamma, N) s^{\gamma+N-2} \left(1 + \frac{A}{D \cdot [\mu(s)]^{\frac{\gamma}{\gamma+N-2}}}\right)^{\frac{\gamma+N-2}{N-2}}$$



Subharmonic Functions and their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 5 of 33

(see Theorem 3.4 and Corollary 3.5).

It points out that $\limsup_{s \rightarrow +\infty} \frac{\mu(s)}{s^{\gamma+N-2}}$ is not greater than $B\gamma$ (when $N = 2$) or $\frac{B\gamma}{N-2} C(\gamma, N)$ (when $N \geq 3$). Moreover, $\liminf_{s \rightarrow +\infty} \frac{\mu(s)}{s^{\gamma+N-2}}$ does not exceed $B\gamma$ (if $N = 2$) or $\frac{B\gamma}{N-2}$ (if $N \geq 3$). This will follow from Theorems 4.1 and 4.3 which assert that the sets:

$$\left\{ s : \mu(s) < B\gamma s^\gamma e^{\frac{A\gamma}{\mu(s)}} \right\}$$

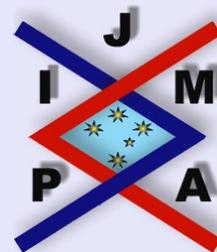
and

$$\left\{ s : \mu(s) < \frac{B\gamma}{N-2} s^{\gamma+N-2} \left(1 + \frac{A}{D \cdot [\mu(s)]^{\frac{\gamma}{\gamma+N-2}}} \right)^{\frac{\gamma+N-2}{N-2}} \right\}$$

are unbounded in the cases when $N = 2$ and $N \geq 3$ respectively.

The last section studies subharmonic functions u in \mathbb{R}^N (harmonic in some neighbourhood of the origin with $u(0) = 0$) such that the subharmonic function u^+ (defined by $u^+(x) = \max(u(x), 0) \forall x \in \mathbb{R}^N$) satisfies a L^1 condition, for example in Theorem 6.1:

$\int_{\mathbb{R}^N} u^+(x) [-\varphi'(|x|^2)] dx < +\infty$ (see Section 6.1 for more details on the decreasing function φ). The Riesz measure μ of u is then proved to verify: $\int_{\mathbb{R}^N} \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2} d\mu(\zeta) < +\infty$. Propositions 6.2 and 6.3 provide similar results under different L^1 conditions.



Subharmonic Functions and their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 6 of 33

2. Some Preliminaries

Lemma 2.1. *If $N = 2$, then*

$$\int_{|\zeta| \leq s} \log \frac{r}{|\zeta|} d\mu(\zeta) \leq \int_{|\zeta| \leq r} \log \frac{r}{|\zeta|} d\mu(\zeta)$$

for each $r > 0$ and each $s > 0$.

Proof. If $r \leq s$, then $h_r(\zeta) := \log \frac{r}{|\zeta|} \leq 0$ for $r < |\zeta| \leq s$, so that

$$\int_{|\zeta| \leq s} h_r(\zeta) d\mu(\zeta) = \int_{|\zeta| \leq r} h_r(\zeta) d\mu(\zeta) + \underbrace{\int_{r < |\zeta| \leq s} h_r(\zeta) d\mu(\zeta)}_{\leq 0} \leq \int_{|\zeta| \leq r} h_r(\zeta) d\mu(\zeta).$$

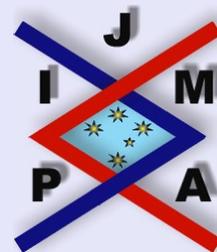
If $s < r$, then $h_r(\zeta) \geq 0$ for $|\zeta| \leq r$, hence

$$\int_{|\zeta| \leq r} h_r(\zeta) d\mu(\zeta) = \int_{|\zeta| \leq s} h_r(\zeta) d\mu(\zeta) + \underbrace{\int_{s < |\zeta| \leq r} h_r(\zeta) d\mu(\zeta)}_{\geq 0} \geq \int_{|\zeta| \leq s} h_r(\zeta) d\mu(\zeta).$$

□

Lemma 2.2. *When $N \geq 3$, the following majoration is valid for all $r > 0$ and $s > 0$:*

$$\int_{|\zeta| \leq s} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta) \leq \int_{|\zeta| \leq r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta).$$



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 7 of 33

Proof. As in the previous proof, with $h_r(\zeta) = \frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}}$ instead of $\log \frac{r}{|\zeta|}$. □

Lemma 2.3. *If $N = 2$, then:*

$$\int_0^r \frac{\mu(t)}{t} dt = \int_{|\zeta| \leq r} \log \frac{r}{|\zeta|} d\mu(\zeta),$$

for any $r > 0$.

Proof. It follows from Fubini's theorem that:

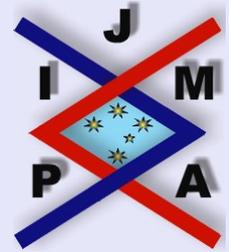
$$\begin{aligned} \int_0^r \frac{\mu(t)}{t} dt &= \int_0^r \frac{1}{t} \left(\int_{|\zeta| \leq t} d\mu(\zeta) \right) dt \\ &= \int_{|\zeta| \leq r} \left(\int_{|\zeta|}^r \frac{dt}{t} \right) d\mu(\zeta) \\ &= \int_{|\zeta| \leq r} \log \frac{r}{|\zeta|} d\mu(\zeta). \end{aligned}$$

□

Lemma 2.4. *When $N \geq 3$, then*

$$(N-2) \int_0^r \frac{\mu(t)}{t^{N-1}} dt = \int_{|\zeta| \leq r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta),$$

for any $r > 0$



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

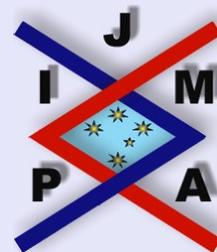
Quit

Page 8 of 33

Proof. As in the previous proof:

$$\begin{aligned}
 (N - 2) \int_0^r \frac{\mu(t)}{t^{N-1}} dt &= \int_0^r \frac{N-2}{t^{N-1}} \left(\int_{|\zeta| \leq t} d\mu(\zeta) \right) dt \\
 &= \int_{|\zeta| \leq r} \left(\int_{|\zeta|}^r \frac{N-2}{t^{N-1}} dt \right) d\mu(\zeta) \\
 &= \int_{|\zeta| \leq r} \left[\frac{-1}{t^{N-2}} \right]_{|\zeta|}^r d\mu(\zeta) \\
 &= \int_{|\zeta| \leq r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta).
 \end{aligned}$$

□



**Subharmonic Functions and
their Riesz Measure**

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 9 of 33

3. Estimations of the Riesz Measure

3.1. Jensen–Privalov formula.

For any function u , subharmonic in \mathbb{R}^N , harmonic in some neighbourhood of the origin, the Jensen–Privalov formula (see [2, p. 44]) holds for every $r > 0$:

$$\frac{1}{2\pi} \int_0^{2\pi} u(r e^{i\theta}) d\theta = \int_0^r \frac{\mu(t)}{t} dt + u(0) \quad \text{if } N = 2$$

$$\frac{1}{\sigma_N} \int_{S_N} u(rx) d\sigma_x = (N - 2) \int_0^r \frac{\mu(t)}{t^{N-1}} dt + u(0) \quad \text{if } N \geq 3$$

with S_N the unit sphere in \mathbb{R}^N , $d\sigma$ the area element on S_N and $\sigma_N = \int_{S_N} d\sigma = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ (see [1, p. 29]). In all statements of both Sections 3 and 4, it will be assumed that $u \in SH(\gamma, B)$ and that its growth is indicated by (1.1).

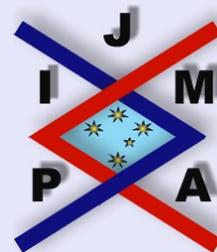
3.2. The case $N = 2$

Theorem 3.1. *When $N = 2$, the following inequality holds for each $s > 0$:*

$$\frac{\mu(s)}{\gamma} \log \left(\frac{\mu(s)}{Be\gamma} \right) \leq A + \int_{|\zeta| \leq s} \log |\zeta| d\mu(\zeta).$$

Proof. For each $r > 0$ and each $s > 0$, it follows from Lemmas 2.1 and 2.3 that

$$\int_{|\zeta| \leq s} \log \frac{r}{|\zeta|} d\mu(\zeta) \leq \frac{1}{2\pi} \int_0^{2\pi} u(r e^{i\theta}) d\theta \leq A + B r^\gamma,$$



Title Page

Contents



Go Back

Close

Quit

Page 10 of 33

so that

$$\int_{|\zeta| \leq s} \log \frac{1}{|\zeta|} d\mu(\zeta) \leq A + B r^\gamma - \mu(s) \log r = A + \frac{\mu(s)}{\gamma} \left(\frac{B\gamma}{\mu(s)} r^\gamma - \log r^\gamma \right) \\ := \varphi(r).$$

Consider s constant, the minimum of φ is attained when $B\gamma r^\gamma = \mu(s)$, since $\varphi'(r) = \frac{1}{r}(B\gamma r^\gamma - \mu(s))$. Finally, for each $s > 0$:

$$\int_{|\zeta| \leq s} \log \frac{1}{|\zeta|} d\mu(\zeta) \leq A + \frac{\mu(s)}{\gamma} \left[1 - \log \left(\frac{\mu(s)}{B\gamma} \right) \right] = A - \frac{\mu(s)}{\gamma} \log \left(\frac{\mu(s)}{Be\gamma} \right)$$

□

In Corollaries 3.2, 3.3 and 3.5, we set $\varepsilon > 0$ such that $\mu(s) > 0 \forall s > \varepsilon$.

Corollary 3.2. *If $N = 2$, then $\mu(s) \leq Be\gamma s^\gamma e^{\frac{A\gamma}{\mu(s)}}$ for any $s > \varepsilon$.*

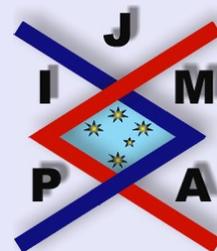
Proof. Theorem 3.1 may be rewritten as:

$$(3.1) \quad \log \left(\frac{\mu(s)}{Be\gamma} \right) \leq \frac{A\gamma}{\mu(s)} + \int_{|\zeta| \leq s} \log(|\zeta|^\gamma) \frac{d\mu(\zeta)}{\mu(s)}.$$

The previous integral being $\leq \log s^\gamma$, Corollary 3.2 results. □

Corollary 3.3. *When $N = 2$, we have for every $s > \varepsilon$:*

$$[\mu(s)]^2 \leq Be\gamma \exp \left(\frac{A\gamma}{\mu(s)} \right) \int_{|\zeta| \leq s} |\zeta|^\gamma d\mu(\zeta).$$



Title Page

Contents



Go Back

Close

Quit

Page 11 of 33

Proof. Jensen's inequality applies to (3.1) since $\int_{|\zeta| \leq s} \frac{d\mu(\zeta)}{\mu(s)} = 1$, hence:

$$\begin{aligned} \frac{\mu(s)}{B e^\gamma} &\leq \exp\left(\frac{A\gamma}{\mu(s)}\right) \cdot \exp\left(\int_{|\zeta| \leq s} \log(|\zeta|^\gamma) \frac{d\mu(\zeta)}{\mu(s)}\right) \\ &\leq \exp\left(\frac{A\gamma}{\mu(s)}\right) \int_{|\zeta| \leq s} |\zeta|^\gamma \frac{d\mu(\zeta)}{\mu(s)}. \end{aligned}$$

□

3.3. The case $N \geq 3$

Theorem 3.4. *When $N \geq 3$, the following estimation is valid for each $s > 0$:*

$$\int_{|\zeta| \leq s} \frac{1}{|\zeta|^{N-2}} d\mu(\zeta) \leq A + D [\mu(s)]^{\frac{\gamma}{\gamma+N-2}}.$$

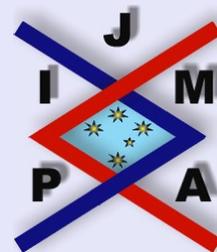
Proof. For all $r > 0$ and $s > 0$, Lemmas 2.2 and 2.4 lead to:

$$\int_{|\zeta| \leq s} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta) \leq \frac{1}{\sigma_N} \int_{S_N} u(rx) d\sigma_x \leq A + B r^\gamma,$$

that is

$$\int_{|\zeta| \leq s} \frac{1}{|\zeta|^{N-2}} d\mu(\zeta) \leq A + B r^\gamma + \frac{\mu(s)}{r^{N-2}}$$

whose minimum (with s constant) is attained when $B\gamma r^\gamma = (N-2) \frac{\mu(s)}{r^{N-2}}$. In other words, this minimum is $A + \left(\frac{N-2}{\gamma} + 1\right) \frac{\mu(s)}{r^{N-2}}$ with $\frac{1}{r^{N-2}} = \left(\frac{B\gamma}{N-2} \frac{1}{\mu(s)}\right)^{\frac{N-2}{\gamma+N-2}}$.



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 12 of 33

Finally:

$$\int_{|\zeta| \leq s} \frac{1}{|\zeta|^{N-2}} d\mu(\zeta) \leq A + \left(\frac{N-2}{\gamma} + 1 \right) \left(\frac{B\gamma}{N-2} \right)^{\frac{N-2}{\gamma+N-2}} (\mu(s))^{\frac{\gamma}{\gamma+N-2}}.$$

□

Corollary 3.5. *When $N \geq 3$, the following estimation holds for every $s > \varepsilon$:*

$$\mu(s) \leq \frac{B\gamma}{N-2} C(\gamma, N) s^{\gamma+N-2} \left(1 + \frac{A}{D \cdot [\mu(s)]^{\frac{\gamma}{\gamma+N-2}}} \right)^{\frac{\gamma+N-2}{N-2}}.$$

Proof. Let $\alpha = \frac{N-2}{\gamma+N-2}$. According to Theorem 3.4, for any $s > \varepsilon$ we have

$$\frac{1}{s^{N-2}} \leq \frac{1}{\mu(s)} \int_{|\zeta| \leq s} \frac{1}{|\zeta|^{N-2}} d\mu(\zeta) \leq \frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha} = \frac{D}{\mu(s)^\alpha} \left(1 + \frac{A}{D} \frac{\mu(s)^\alpha}{\mu(s)} \right).$$

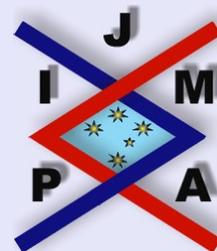
Hence

$$[\mu(s)]^\alpha \leq D s^{N-2} \left(1 + \frac{A}{D} \frac{1}{[\mu(s)]^{1-\alpha}} \right).$$

Now, it is obvious that $1 - \alpha = \frac{\gamma}{\gamma+N-2}$ and $D^{1/\alpha} = \frac{B\gamma}{N-2} C(\gamma, N)$. □

Corollary 3.6. *With $N \geq 3$ and $\alpha = \frac{N-2}{\gamma+N-2}$, the following holds for each $s > 0$:*

$$\mu(s) \log \left(\frac{\mu(s)^\alpha}{D} \right) - \frac{A}{D} \mu(s)^\alpha \leq (N-2) \int_{|\zeta| \leq s} \log |\zeta| d\mu(\zeta)$$



Title Page

Contents



Go Back

Close

Quit

Page 13 of 33

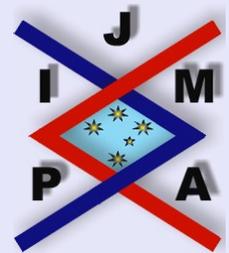
Proof. It follows from Jensen's inequality that:

$$\begin{aligned} \exp \left(\int_{|\zeta| \leq s} \left(\log \frac{1}{|\zeta|^{N-2}} \right) \frac{d\mu(\zeta)}{\mu(s)} \right) &\leq \int_{|\zeta| \leq s} \exp \left(\log \frac{1}{|\zeta|^{N-2}} \right) \frac{d\mu(\zeta)}{\mu(s)} \\ &= \int_{|\zeta| \leq s} \frac{1}{|\zeta|^{N-2}} \frac{d\mu(\zeta)}{\mu(s)} \\ &\leq \frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha}, \end{aligned}$$

so that:

$$\begin{aligned} -(N-2) \int_{|\zeta| \leq s} \log |\zeta| \frac{d\mu(\zeta)}{\mu(s)} &\leq \log \left(\frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha} \right) \\ &\leq \log \left(\frac{D}{\mu(s)^\alpha} \right) + \frac{A}{D} \frac{\mu(s)^\alpha}{\mu(s)}. \end{aligned}$$

□



Subharmonic Functions and their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 14 of 33

4. Growth of the Repartition Function

4.1. A measure on $[0, +\infty[$, image of μ

Let $\Phi : \mathbb{R}^N \rightarrow [0, +\infty[$ be the measurable map defined by $\Phi(\zeta) = \mu(|\zeta|)$ (the function $s \mapsto \mu(s)$ is increasing hence measurable on $[0, +\infty[$). Let $\nu = \Phi_*\mu = \mu \circ \Phi^{-1}$ denote the measure image of μ under Φ (see [3, p. 80]):

$$\int_0^{+\infty} f(t) d\nu(t) = \int_{\mathbb{R}^N} f(\Phi(\zeta)) d\mu(\zeta)$$

holds for any nonnegative measurable function f on $[0, +\infty[$ (and for any ν -integrable f)

Remark 4.1. If $s \mapsto \mu(s)$ is continuous on some interval $[a, +\infty[$ with $a \geq 0$, then $\nu(I) = c - b$ for any interval I with bounds b and c ($c > b > \mu(a)$).

4.2. The case $N = 2$

Up to the end of Section 4, μ stands for the Riesz measure associated with a function of $SH(\gamma, B)$ with growth (1.1).

Theorem 4.1. If $N = 2$ and $A > \frac{2}{\gamma}$, then the set of those $s > 0$ which satisfy $\mu(s) < B\gamma s^\gamma e^{\frac{A\gamma}{\mu(s)}}$ is unbounded.

A proof is required only in the case where $\lim_{s \rightarrow +\infty} \mu(s) = +\infty$ (otherwise, Theorem 4.1 is obvious). When the function $s \mapsto \mu(s)$ is continuous, at least on some interval $[a, +\infty[$ with $a > 0$, there is a direct proof which is quoted below in Subsection 4.3. In this case, the assumption $A > \frac{2}{\gamma}$ is no longer required. The proof in the general case is the subject of Subsection 4.5.



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 15 of 33

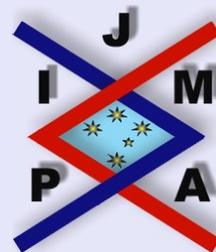
4.3. Proof of Theorem 4.1 in the case of a continuous repartition function

Proof. Let us suppose that the set $\left\{s > 0 : \mu(s) < B\gamma s^\gamma e^{\frac{A\gamma}{\mu(s)}}\right\}$ is bounded and let s_0 be one of its majorants, chosen in such a way that $s \mapsto \mu(s)$ is continuous on some neighbourhood of $[s_0, +\infty[$.

Thus $\mu(s) \geq B\gamma s^\gamma e^{\frac{A\gamma}{\mu(s)}}$ for all $s \geq s_0$, that is: $\log s \leq \frac{1}{\gamma} \log \left(\frac{\mu(s)}{B\gamma}\right) - \frac{A}{\mu(s)}$, such that:

$$\begin{aligned} \int_{s_0 \leq |\zeta| \leq s} \log |\zeta| d\mu(\zeta) &\leq \int_{s_0 \leq |\zeta| \leq s} \left(\frac{1}{\gamma} \log \left(\frac{\mu(|\zeta|)}{B\gamma} \right) - \frac{A}{\mu(|\zeta|)} \right) d\mu(\zeta) \\ &= \int_{\mu(s_0)}^{\mu(s)} \left(\frac{1}{\gamma} \log \left(\frac{t}{B\gamma} \right) - \frac{A}{t} \right) d\nu(t) \\ &= \int_{\mu(s_0)}^{\mu(s)} \left(\frac{1}{\gamma} \log \left(\frac{t}{B\gamma} \right) - \frac{A}{t} \right) dt \\ &= B \left[x \log \left(\frac{x}{e} \right) \right]_{\mu(s_0)/B\gamma}^{\mu(s)/B\gamma} - A [\log t]_{\mu(s_0)}^{\mu(s)} \\ &= \frac{\mu(s)}{\gamma} \log \left(\frac{\mu(s)}{Be\gamma} \right) - A \log \mu(s) + K(s_0), \end{aligned}$$

where $K(s_0)$ stands for $A \log \mu(s_0) - \frac{\mu(s_0)}{\gamma} \log \left(\frac{\mu(s_0)}{Be\gamma} \right)$. It follows from Theo-



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 16 of 33

rem 3.1 that:

$$\frac{\mu(s)}{\gamma} \log \left(\frac{\mu(s)}{Be\gamma} \right) \leq A + \int_{|\zeta| < s_0} \log |\zeta| d\mu(\zeta) + \frac{\mu(s)}{\gamma} \log \left(\frac{\mu(s)}{Be\gamma} \right) - A \log \mu(s) + K(s_0).$$

Finally: $A \log \mu(s) \leq A + K(s_0) + \mu(s_0) \log s_0$ for all $s \geq s_0$. When s tends to $+\infty$, a contradiction arises. \square

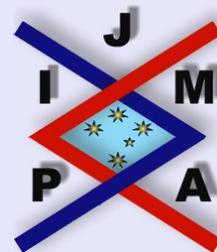
4.4. Splitting measure μ

Now, in order to prove Theorem 4.1 in the general case, we will introduce some notations which will also be useful in proving Theorem 4.3 (where $N \geq 3$). That is why these notations are already given in \mathbb{R}^N for any $N \in \mathbb{N}$, $N \geq 2$.

It is still assumed that $\lim_{s \rightarrow +\infty} \mu(s) = +\infty$. Let $(s_n)_n$ be the non-decreasing sequence defined by: $s_n = \inf\{s > 0 : \mu(s) \geq n\}$. As the function $s \mapsto \mu(s)$ is right-continuous, we have $\mu(s_n) \geq n$ for all $n \in \mathbb{N}$. If this function is continuous at some point s_n , then $\mu(s_n) = n$.

If $s_n < s_{n+1}$, then $\mu(s_n) < n + 1$. There are infinitely many integers n such that $s_n < s_{n+1}$ because the measure $d\mu$ is finite on compact subsets of \mathbb{R}^N (see [1, p. 81]).

For any $s > 0$, let $\mu^-(s) = \int_{|\zeta| < s} d\mu(\zeta)$. The discontinuity points of $s \mapsto \mu(s)$ are thus characterized by $\mu(s) > \mu^-(s)$. For every $n \in \mathbb{N}$, let $c_n = 0$ if the function $s \mapsto \mu(s)$ is continuous at point s_n , and $c_n = \frac{\mu(s_n) - n}{\mu(s_n) - \mu^-(s_n)}$ if



Title Page

Contents



Go Back

Close

Quit

Page 17 of 33

this function is discontinuous at s_n . Note that $1 - c_n = \frac{n - \mu^-(s_n)}{\mu(s_n) - \mu^-(s_n)}$ in case of discontinuity at s_n .

For all $0 < t < s$, let I_t and $I_{t,s}$ be defined in \mathbb{R}^N by:

$$I_t(\zeta) = \begin{cases} 1 & \text{if } |\zeta| = t \\ 0 & \text{otherwise} \end{cases} \quad I_{t,s}(\zeta) = \begin{cases} 1 & \text{if } t < |\zeta| < s \\ 0 & \text{otherwise} \end{cases}$$

Let us write $\mu = \mu_1 + \mu_2 + \dots + \mu_n + \dots$, where measures μ_k are defined such that

$$\int_{\mathbb{R}^N} d\mu_k(\zeta) = \int_{s_{k-1} \leq |\zeta| \leq s_k} d\mu_k(\zeta) = 1$$

in the following way:

$$d\mu_k = (c_{k-1} I_{s_{k-1}} + I_{s_{k-1}, s_k} + (1 - c_k) I_{s_k}) d\mu \quad \text{if } s_{k-1} < s_k$$

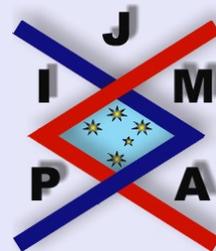
$$d\mu_k = \frac{1}{\mu(s_k) - \mu^-(s_k)} I_{s_k} d\mu \quad \text{if } s_{k-1} = s_k.$$

Remark 4.2. If $s_{k-1} < s_k = s_{k+1} = \dots = s_{k+l} < s_{k+l+1}$, then $\mu^-(s_k) \leq k < k + l \leq \mu(s_k)$ and it is easy to check that

$$(1 - c_k) I_{s_k} + \sum_{j=k+1}^{k+l} \frac{1}{\mu(s_j) - \mu^-(s_j)} I_{s_j} + c_{k+l} I_{s_{k+l}} = I_{s_k}.$$

In addition, notice that $\sum_{k=1}^n \mu_k(s) = \min[n, \mu(s)]$ and that, for any integrable function $h \geq 0$:

$$\int_{|\zeta| \leq s_n} h(\zeta) d\mu \geq \sum_{k=1}^n \int h(\zeta) d\mu_k$$



Title Page

Contents



Go Back

Close

Quit

Page 18 of 33

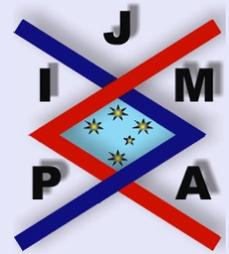
$$\int_{|\zeta| \leq s_n} h(\zeta) d\mu \leq \sum_{k=1}^{n+1} \int h(\zeta) d\mu_k \quad \text{if } s_n < s_{n+1}$$

4.5. A reformulation of Theorem 4.1

Proposition 4.2. *If $N = 2$ and $A > \frac{2}{\gamma}$, then $n < B\gamma(s_n)^\gamma e^{\frac{A\gamma}{n}}$ for infinitely many $n \in \mathbb{N}^*$.*

Proof. Suppose that there exists some integer $m \in \mathbb{N}^*$ such that $n \geq B\gamma(s_n)^\gamma e^{\frac{A\gamma}{n}}$ for each $n \geq m$. It may be assumed that $s_m > s_{m-1} \geq 1$. For any $n \geq m$ satisfying $s_n < s_{n+1}$, we have:

$$\begin{aligned} \int_{s_m \leq |\zeta| \leq s_n} \log |\zeta| d\mu(\zeta) &\leq \sum_{k=m}^{n+1} \int \log |\zeta| d\mu_k(\zeta) \\ &\leq \sum_{k=m}^{n+1} \log s_k \\ &\leq \sum_{k=m}^{n+1} \left(\frac{1}{\gamma} \log \left(\frac{k}{B\gamma} \right) - \frac{A}{k} \right) \\ &\leq \int_m^{n+2} \left(\frac{1}{\gamma} \log \left(\frac{t}{B\gamma} \right) - \frac{A}{t} \right) dt \\ &= \frac{n+2}{\gamma} \log \left(\frac{n+2}{Be\gamma} \right) - A \log(n+2) + K_m \end{aligned}$$



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 19 of 33

with a constant K_m independent from n . Since $\mu(s_n) \geq n$, Theorem 3.1 leads to:

$$\frac{n}{\gamma} \log \left(\frac{n}{Be\gamma} \right) \leq A + (\log s_m)\mu(s_m) + \frac{n+2}{\gamma} \log \left(\frac{n+2}{Be\gamma} \right) - A \log(n+2) + K_m$$

hence

$$\left(A - \frac{2}{\gamma} \right) \log(n+2) \leq A + \underbrace{\frac{n}{\gamma} \log \left(\frac{n+2}{n} \right)}_{\leq \frac{2}{\gamma}} - \frac{2}{\gamma} \log(Be\gamma) + K_m + (\log s_m)\mu(s_m)$$

The contradiction stems from the fact that there exists infinitely many $n > m$ with $s_n < s_{n+1}$. \square

Proof of Theorem 4.1 in the general case. Obviously, function $s \mapsto B\gamma s^\gamma$ is increasing. Thus, for any n such that $ne^{-\frac{A\gamma}{n}} < B\gamma(s_n)^\gamma$, there exists an open non-empty interval J_n (with upper bound s_n) such that $ne^{-\frac{A\gamma}{n}} < B\gamma s^\gamma < B\gamma(s_n)^\gamma \forall s \in J_n$. Moreover $\mu(s)e^{-\frac{A\gamma}{\mu(s)}} < ne^{-\frac{A\gamma}{n}} \forall s \in J_n$ (because $\mu(s) < n$ for every $s < s_n$). Hence Theorem 4.1. \square

4.6. The case $N \geq 3$

Theorem 4.3. *When $N \geq 3$, the set of those $s > 0$ such that*

$$(4.1) \quad \mu(s) < \frac{B\gamma}{N-2} s^{\gamma+N-2} \left(1 + \frac{A}{D \cdot [\mu(s)]^{\frac{\gamma}{\gamma+N-2}}} \right)^{\frac{\gamma+N-2}{N-2}}$$

is unbounded.



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 20 of 33

Inequalities (4.1) and (4.2) are equivalent, with

$$(4.2) \quad \frac{1}{s^{N-2}} < \frac{\gamma}{\gamma + N - 2} \left(\frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha} \right)$$

and $\alpha = \frac{N-2}{\gamma+N-2}$ as in Section 3.3. Indeed, (4.2) may be rewritten

$$\mu(s)^\alpha < s^{N-2} \frac{\gamma D}{\gamma + N - 2} \left(1 + \frac{A}{D[\mu(s)]^{1-\alpha}} \right).$$

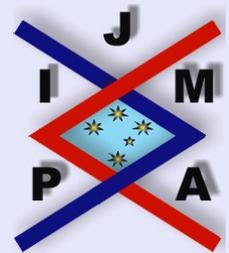
Now $\frac{\gamma D}{\gamma+N-2} = \left(\frac{B\gamma}{N-2}\right)^\alpha$ so that formula (4.1) arises.

To prove Theorem 4.3, we can still assume $\lim_{s \rightarrow +\infty} \mu(s) = +\infty$. The case where function $s \mapsto \mu(s)$ is continuous (at least on some interval $[a, +\infty[$ with $a > 0$) is proved in Subsection 4.7 and the general case is proved in Subsection 4.8.

4.7. Proof of Theorem 4.3 in the case of a continuous repartition function

Proof. Let us assume that there exists some $s_0 > 0$ such that $s \mapsto \mu(s)$ is continuous on some neighbourhood of $[s_0, +\infty[$ and that

$$\frac{1}{s^{N-2}} \geq \frac{\gamma}{\gamma + N - 2} \left(\frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha} \right)$$



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 21 of 33

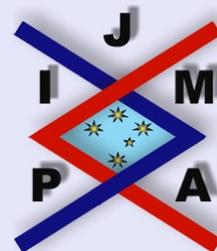
for all $s \geq s_0$. It follows that:

$$\begin{aligned}
 \int_{|\zeta| \leq s} \frac{d\mu(\zeta)}{|\zeta|^{N-2}} &\geq \int_{s_0 \leq |\zeta| \leq s} \frac{d\mu(\zeta)}{|\zeta|^{N-2}} \\
 &\geq \frac{\gamma}{\gamma + N - 2} \int_{s_0 \leq |\zeta| \leq s} \left(\frac{A}{\mu(|\zeta|)} + \frac{D}{\mu(|\zeta|)^\alpha} \right) d\mu(\zeta) \\
 &= \frac{\gamma}{\gamma + N - 2} \int_{\mu(s_0)}^{\mu(s)} \left(\frac{A}{t} + \frac{D}{t^\alpha} \right) d\nu(t) \\
 &= \frac{\gamma}{\gamma + N - 2} \int_{\mu(s_0)}^{\mu(s)} \left(\frac{A}{t} + \frac{D}{t^\alpha} \right) dt \\
 &= \frac{\gamma}{\gamma + N - 2} \left[A \log t + \frac{D}{1 - \alpha} t^{1-\alpha} \right]_{\mu(s_0)}^{\mu(s)} \\
 &= \frac{A\gamma \log \mu(s)}{\gamma + N - 2} + D \mu(s)^{1-\alpha} - K'(s_0),
 \end{aligned}$$

with

$$K'(s_0) = \frac{A\gamma}{\gamma + N - 2} \log \mu(s_0) + D \mu(s_0)^{1-\alpha}.$$

The majoration of $\int_{|\zeta| \leq s} \frac{1}{|\zeta|^{N-2}} d\mu(\zeta)$ (Theorem 3.4) leads, after cancellation of $D \mu(s)^{1-\alpha} = D \mu(s)^{\frac{\gamma}{\gamma+N-2}}$, to: $\frac{A\gamma \log \mu(s)}{\gamma+N-2} \leq A + K'(s_0)$ for any $s \geq s_0$. A contradiction arises as $s \rightarrow +\infty$. \square



Subharmonic Functions and their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 22 of 33

4.8. A reformulation of Theorem 4.3

Proposition 4.4. With $N \geq 3$ and $\alpha = \frac{N-2}{\gamma+N-2}$, infinitely many $n \in \mathbb{N}^*$ satisfy:

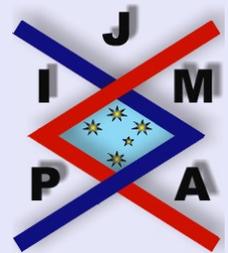
$$(4.3) \quad \frac{1}{s_n^{N-2}} < \frac{\gamma}{\gamma + N - 2} \left(\frac{A}{n} + \frac{D}{n^\alpha} \right).$$

Proof. Suppose that there exists some $m \in \mathbb{N}$ such that $\frac{1}{s_n^{N-2}} \geq \frac{\gamma}{\gamma+N-2} \left(\frac{A}{n} + \frac{D}{n^\alpha} \right) \forall n > m$. It then follows for all $n > m$:

$$\begin{aligned} \int_{s_m \leq |\zeta| \leq s_n} \frac{1}{|\zeta|^{N-2}} d\mu(\zeta) &\geq \sum_{k=m+1}^n \int \frac{1}{|\zeta|^{N-2}} d\mu_k(\zeta) \\ &\geq \sum_{k=m+1}^n \frac{1}{s_k^{N-2}} \\ &\geq \frac{\gamma}{\gamma + N - 2} \sum_{k=m+1}^n \left(\frac{A}{k} + \frac{D}{k^\alpha} \right) \\ &\geq \frac{\gamma}{\gamma + N - 2} \int_{m+1}^{n+1} \left(\frac{A}{t} + \frac{D}{t^\alpha} \right) dt \\ &= \frac{\gamma A \log(n+1)}{\gamma + N - 2} + D(n+1)^{1-\alpha} - K'_m \end{aligned}$$

where the constant K'_m does not depend on n . For those $n > m$ such that $s_n < s_{n+1}$ we have $\mu(s_n) < n + 1$ and Theorem 3.4 provides us with:

$$\int_{s_m \leq |\zeta| \leq s_n} \frac{1}{|\zeta|^{N-2}} d\mu(\zeta) \leq \int_{|\zeta| \leq s_n} \frac{1}{|\zeta|^{N-2}} d\mu(\zeta) \leq A + D(n+1)^{1-\alpha}$$



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 23 of 33

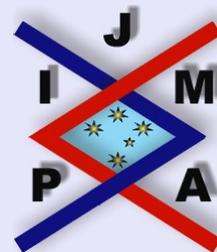
hence $\frac{\gamma A \log(n+1)}{\gamma+N-2} \leq A + K'_m$. A contradiction arises as $n \rightarrow +\infty$. \square

Proof of Theorem 4.3 in the general case. Since the function $s \mapsto \frac{1}{s^{N-2}}$ is decreasing, for each $n \in \mathbb{N}^*$ satisfying (4.3) there exists an open interval $J_n \neq \emptyset$ (with right bound s_n) where

$$\frac{1}{s_n^{N-2}} < \frac{1}{s^{N-2}} < \frac{\gamma}{\gamma+N-2} \left(\frac{A}{n} + \frac{D}{n^\alpha} \right) \quad (\forall s \in J_n).$$

Now, $\mu(s) < n$ for each $s < s_n$, so that $\frac{A}{n} + \frac{D}{n^\alpha} < \frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha}$. Hence

$\frac{1}{s^{N-2}} < \frac{\gamma}{\gamma+N-2} \left(\frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha} \right) \quad \forall s \in J_n$ and Theorem 4.3 follows. \square



Subharmonic Functions and their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 24 of 33

5. Sum of Two Riesz Measures

Lemma 5.1. Given $\gamma > 0$, $B > 0$ and $\varepsilon \in]0, 1[$, let u_ε be defined in \mathbb{R}^N by :

$$u_\varepsilon(x) = \max\{0, \varphi_\varepsilon(|x|)\} \quad \forall x \in \mathbb{R}^N$$

with $\varphi_\varepsilon(r) = B r^\gamma - B \varepsilon^\gamma \forall r \geq 0$. Then $u_\varepsilon \in SH(\gamma, B)$. Let μ_ε denote its Riesz measure, then: $\mu_\varepsilon(s) = \frac{B\gamma}{\tau_N} s^{\gamma+N-2} + k_\varepsilon \forall s \geq 1$, where $\tau_N = \max(1, N - 2)$ and k_ε is a constant depending only on B, γ, N and ε .

Proof. Subharmonicity of $u_\varepsilon = \max(u_1, u_2)$ will follow (see [1, p. 41]) from the subharmonicity of both functions u_1 and u_2 defined in \mathbb{R}^N by $u_1(x) = \varphi_\varepsilon(|x|)$ and $u_2(x) \equiv 0$: it is easy to verify that $\Delta u_1(x) = \varphi_\varepsilon''(r) + \frac{N-1}{r} \varphi_\varepsilon'(r) = B\gamma r^{\gamma-2}(\gamma + N - 2) \geq 0$ (see [1, p. 26]). Obviously, u_ε has a growth of the kind (1.1), $u_\varepsilon(0) = 0$ and u_ε is harmonic in the neighbourhood $\{x \in \mathbb{R}^N : |x| < \varepsilon\}$ of the origin. \square

Let $\theta_N = (N - 2)\sigma_N$ when $N \geq 3$ and $\theta_2 = 2\pi$ (see [2, p. 43]), since $d\mu_\varepsilon = \frac{1}{\theta_N} \Delta u_\varepsilon dx = \frac{1}{\theta_N} \Delta u_\varepsilon r^{N-1} dr d\sigma$, it is possible for all $s \geq 1$ to compute

$$\mu_\varepsilon(s) = \mu_\varepsilon(1) + \int_1^s \frac{\sigma_N}{\theta_N} B\gamma(\gamma + N - 2)r^{\gamma+N-3} dr = \mu_\varepsilon(1) + \frac{1}{\tau_N} B\gamma [r^{\gamma+N-2}]_1^s$$

Proposition 5.2. Given $\gamma > 0$, $B > 0$ and $0 < B' < 2B$, let μ_1 and μ_2 be the Riesz measures of two functions, respectively u_1 and u_2 , belonging to $SH(\gamma, B)$. Then $\mu_1 + \mu_2$ is not necessarily the Riesz measure associated with a function of $SH(\gamma, B')$.



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

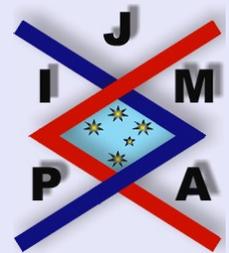
Close

Quit

Page 25 of 33

Proof. Given ε_1 and $\varepsilon_2 \in]0, 1[$, let u_{ε_1} and $u_{\varepsilon_2} \in SH(\gamma, B)$ be defined as in the previous lemma and $\mu = \mu_{\varepsilon_1} + \mu_{\varepsilon_2}$ be the sum of their Riesz measures. Thus $\mu(s) = \frac{2B\gamma}{\tau_N} s^{\gamma+N-2} + k_{\varepsilon_1} + k_{\varepsilon_2} \quad \forall s \geq 1$. Note that $\lim_{s \rightarrow +\infty} \frac{\mu(s)}{s^{\gamma+N-2}} = \frac{2B\gamma}{\tau_N}$.

Suppose that μ is the Riesz measure of some function $u \in SH(\gamma, B')$ with an estimate such as: $u(x) \leq A + B'|x|^\gamma$ ($\forall x \in \mathbb{R}^N$) for some constant $A \in \mathbb{R}$. In Theorems 4.1 and 4.3, one asserts that $\liminf_{s \rightarrow +\infty} \frac{\mu(s)}{s^{\gamma+N-2}} \leq \frac{B'\gamma}{\tau_N}$, which leads to $2B \leq B'$, hence a contradiction. \square



**Subharmonic Functions and
their Riesz Measure**

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 26 of 33

6. Subharmonic Functions Subject to Conditions of L^1 Type

6.1. A weighted integral condition for subharmonic functions.

Theorem 6.1. *Given $N \in \mathbb{N}$ ($N \geq 2$) and a positive non-increasing C^1 function φ on $[0, +\infty[$ such that $\lim_{s \rightarrow +\infty} (\log s)\varphi(s) = 0$ (when $N = 2$) or $\lim_{s \rightarrow +\infty} s^{\frac{N}{2}-1}\varphi(s) = 0$ (when $N \geq 3$), let u be a subharmonic function in \mathbb{R}^N , harmonic in some neighbourhood of the origin with $u(0) = 0$, such that:*

$$\int_{\mathbb{R}^N} u^+(x) [-\varphi'(|x|^2)] dx < +\infty$$

where the subharmonic function u^+ is defined by $u^+(x) = \max(u(x), 0) \forall x \in \mathbb{R}^N$. Then the Riesz measure μ of u verifies:

$$\int_{\mathbb{R}^N} \frac{\varphi(|\zeta|^2 + 1)}{|\zeta|^2} d\mu(\zeta) < +\infty.$$

Example 1. With $N \geq 2$, $\beta > 0$ and φ defined by $\varphi(s) = e^{-\beta s} \forall s > 0$, obviously

$$\lim_{s \rightarrow +\infty} (\log s)\varphi(s) = \lim_{s \rightarrow +\infty} s^{\frac{N}{2}-1}\varphi(s) = 0.$$

If a subharmonic function u in \mathbb{R}^N (harmonic in some neighbourhood of the origin, with $u(0) = 0$) satisfies $\int_{\mathbb{R}^N} u^+(x) e^{-\beta|x|^2} dx < +\infty$ then its Riesz measure μ verifies $\int_{\mathbb{R}^N} \frac{e^{-\beta|\zeta|^2}}{|\zeta|^2} d\mu(\zeta) < +\infty$. One thus encounters a result of [4, p. 88] for holomorphic functions in \mathbb{C} .



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 27 of 33

6.2. Proof of Theorem 6.1 in the case $N = 2$

Proof. Abiding by Jensen's formula (Subsection 3.1) and by Lemma 2.3:

$$\int_{|\zeta| \leq r} \log \frac{r}{|\zeta|} d\mu(\zeta) \leq \frac{1}{2\pi} \int_0^{2\pi} u^+(r e^{i\theta}) d\theta \quad \forall r > 0.$$

Since $-\varphi'(r^2) \geq 0$, it follows that:

$$\int_0^{+\infty} \left(\int_{|\zeta| \leq r} \log \frac{r}{|\zeta|} d\mu(\zeta) \right) [-\varphi'(r^2)] r dr < +\infty.$$

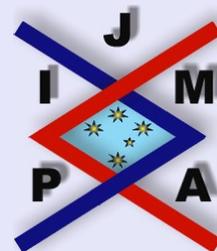
Fubini's theorem transforms the above integral into:

$$\int_{\mathbb{R}^2} \underbrace{\left(\int_{|\zeta|}^{+\infty} \log \frac{r}{|\zeta|} [-\varphi'(r^2)] r dr \right)}_{:=I(\zeta) \geq 0} d\mu(\zeta).$$

Now,

$$I(\zeta) = \frac{1}{4} \int_{|\zeta|^2}^{+\infty} \log \frac{s}{|\zeta|^2} [-\varphi'(s)] ds$$

for any $\zeta \in \mathbb{R}^2$ and an integration by parts leads to: $4I(\zeta) = \int_{|\zeta|^2}^{+\infty} \frac{\varphi(s)}{s} ds$ since $\lim_{s \rightarrow +\infty} (\log s) \varphi(s) = 0$ and $\lim_{s \rightarrow +\infty} \varphi(s) = 0$ as well. The positive function $f : s \mapsto \frac{\varphi(s)}{s}$ decreases for $s > 0$ so that $\int_b^{+\infty} f(s) ds \geq f(b+1)$ for all $b > 0$, hence: $4I(\zeta) \geq \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2+1}$ for all $\zeta \in \mathbb{R}^2$. If $|\zeta| \geq 1$, then $\frac{1}{|\zeta|^2+1} \geq \frac{1}{2|\zeta|^2}$ and $8I(\zeta) \geq \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2} \geq 0$. Because of the harmonicity of u in a neighbourhood of the origin, $\int_{|\zeta| < 1} \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2} d\mu(\zeta) < +\infty$. The conclusion follows from $\int_{|\zeta| \geq 1} I(\zeta) d\mu(\zeta) < +\infty$. \square



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 28 of 33

6.3. Proof of Theorem 6.1 in the case $N \geq 3$

Proof. Jensen–Privalov formula together with Lemma 2.4 lead to:

$$\int_{|\zeta| \leq r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta) \leq \frac{1}{\sigma_N} \int_{S_N} u(rx) d\sigma_x \quad \forall r > 0.$$

Hence:

$$\int_0^{+\infty} \left(\int_{|\zeta| \leq r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta) \right) [-\varphi'(r^2)] r^{N-1} dr < +\infty.$$

Taking Fubini's theorem into account, this integral becomes:

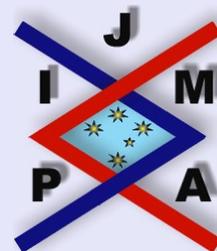
$$\int_{\mathbb{R}^N} \underbrace{\left(\int_{|\zeta|}^{+\infty} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) [-\varphi'(r^2)] r^{N-1} dr \right)}_{:=J(\zeta)} d\mu(\zeta).$$

Now, for any $\zeta \in \mathbb{R}^N$:

$$\begin{aligned} 0 &\leq J(\zeta) \\ &= \int_{|\zeta|}^{+\infty} \left(\frac{r^{N-2}}{|\zeta|^{N-2}} - 1 \right) [-\varphi'(r^2)] r dr \\ &= \frac{1}{2} \int_{|\zeta|^2}^{+\infty} \left(\frac{s^{\frac{N}{2}-1}}{|\zeta|^{N-2}} - 1 \right) [-\varphi'(s)] ds. \end{aligned}$$

Since $\lim_{s \rightarrow +\infty} \left(s^{\frac{N}{2}-1} - |\zeta|^{N-2} \right) \varphi(s) = 0$, an integration by parts leads to:

$$2J(\zeta) = \frac{N-2}{2} \int_{|\zeta|^2}^{+\infty} \frac{s^{\frac{N}{2}-2}}{|\zeta|^{N-2}} \varphi(s) ds.$$



Subharmonic Functions and
their Riesz Measure

Raphaela Supper

Title Page

Contents



Go Back

Close

Quit

Page 29 of 33

Obviously, $s^{\frac{N}{2}-2} \geq |\zeta|^{N-4}$ for all $s \geq |\zeta|^2$, so that:

$$\frac{4}{N-2} J(\zeta) \geq \frac{1}{|\zeta|^2} \int_{|\zeta|^2}^{+\infty} \varphi(s) ds \geq \frac{\varphi(|\zeta|^2 + 1)}{|\zeta|^2} \geq 0.$$

□

Propositions 6.2 and 6.3 will be proved by using the same method.

Proposition 6.2. *Let φ be a positive C^1 non-increasing function on $[0, +\infty[$ such that*

$\lim_{r \rightarrow +\infty} r \varphi(r) \log r = 0$. If a subharmonic function u in \mathbb{R}^2 (harmonic in some neighbourhood of the origin with $u(0) = 0$) verifies:

$$\int_{\mathbb{R}^2} u^+(x) [-\varphi'(|x|)] dx < +\infty$$

then its Riesz measure μ satisfies: $\int_{\mathbb{R}^2} \varphi(|\zeta| + 1) d\mu(\zeta) < +\infty$ and

$$\int_{|\zeta| \geq 1} \varphi(|\zeta|^\alpha + 1) \log |\zeta| d\mu(\zeta) < +\infty$$

holds for each $\alpha > 1$.

Proof. As in Section 6.2: $\int_{\mathbb{R}^2} I(\zeta) d\mu(\zeta) < +\infty$, here with

$$I(\zeta) = \int_{|\zeta|}^{+\infty} r \log \frac{r}{|\zeta|} [-\varphi'(r)] dr$$



Title Page

Contents



Go Back

Close

Quit

Page 30 of 33

which turns into $I(\zeta) = \int_{|\zeta|}^{+\infty} \varphi(r) \log \frac{er}{|\zeta|} dr$ after an integration by parts which uses

$\lim_{r \rightarrow +\infty} r \varphi(r) \log r = 0$ (this guarantees that $\lim_{r \rightarrow +\infty} r \varphi(r) = 0$ as well). Since φ is non-increasing and $\log \frac{er}{|\zeta|} \geq 1$ for each $r \geq |\zeta|$, it follows that $I(\zeta) \geq \varphi(|\zeta| + 1) \forall \zeta \in \mathbb{R}^2$.

Given $\alpha > 1$, obviously $|\zeta|^\alpha \geq |\zeta|$ as soon as $|\zeta| \geq 1$, so that

$$\begin{aligned} I(\zeta) &\geq \int_{|\zeta|^\alpha}^{+\infty} \varphi(r) \log \frac{er}{|\zeta|} dr \\ &\geq (\alpha - 1) \int_{|\zeta|^\alpha}^{+\infty} \varphi(r) \log |\zeta| dr \geq (\alpha - 1) \varphi(|\zeta|^\alpha + 1) \log |\zeta| \\ &\geq 0. \end{aligned}$$

The conclusion proceeds from $\int_{|\zeta| \geq 1} I(\zeta) d\mu(\zeta) < +\infty$. □

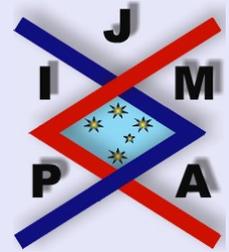
Proposition 6.3. *Given $N \in \mathbb{N}$, $N \geq 3$, let φ be a positive non-increasing C^1 function in $[0, +\infty[$ such that $\lim_{r \rightarrow +\infty} r^{N-1} \varphi(r) = 0$. If a subharmonic function u in \mathbb{R}^N (harmonic in some neighbourhood of the origin with $u(0) = 0$) verifies:*

$$\int_{\mathbb{R}^N} u^+(x) [-\varphi'(|x|)] dx < +\infty$$

then its Riesz measure μ satisfies

$$\int_{\mathbb{R}^N} \varphi(|\zeta|^\alpha + 1) |\zeta|^{(\alpha-1)(N-2)} d\mu(\zeta) < +\infty$$

for any $\alpha \geq 1$.



Title Page

Contents



Go Back

Close

Quit

Page 31 of 33

Remark 6.1. When $\alpha = 1$, we encounter $\int_{\mathbb{R}^N} \varphi(|\zeta| + 1) d\mu(\zeta) < +\infty$ again.

Proof. As in Section 6.3: $\int_{\mathbb{R}^N} J(\zeta) d\mu(\zeta) < +\infty$, here with

$$\begin{aligned} J(\zeta) &= \int_{|\zeta|}^{+\infty} \left(\frac{r^{N-1}}{|\zeta|^{N-2}} - r \right) [-\varphi'(r)] dr \\ &= \int_{|\zeta|}^{+\infty} \left((N-1) \frac{r^{N-2}}{|\zeta|^{N-2}} - 1 \right) \varphi(r) dr \end{aligned}$$

after an integration by parts. Obviously, $\frac{r^{N-2}}{|\zeta|^{N-2}} \geq 1$ for every $r \geq |\zeta|$, so that:

$$(N-1) \frac{r^{N-2}}{|\zeta|^{N-2}} - 1 \geq (N-2) \frac{r^{N-2}}{|\zeta|^{N-2}}$$

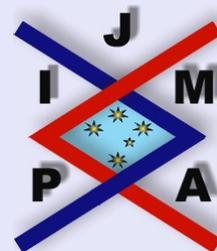
and

$$J(\zeta) \geq (N-2) \int_{|\zeta|}^{+\infty} \frac{r^{N-2}}{|\zeta|^{N-2}} \varphi(r) dr \quad \forall \zeta \in \mathbb{R}^N.$$

If $|\zeta| \geq 1$, then $|\zeta|^\alpha \geq |\zeta|$ since $\alpha \geq 1$, hence

$$\begin{aligned} J(\zeta) &\geq (N-2) \int_{|\zeta|^\alpha}^{+\infty} \frac{r^{N-2}}{|\zeta|^{N-2}} \varphi(r) dr \\ &\geq (N-2) |\zeta|^{(\alpha-1)(N-2)} \int_{|\zeta|^\alpha}^{+\infty} \varphi(r) dr \\ &\geq (N-2) |\zeta|^{(\alpha-1)(N-2)} \varphi(|\zeta|^\alpha + 1). \end{aligned}$$

□



Subharmonic Functions and
their Riesz Measure

Raphaele Supper

Title Page

Contents



Go Back

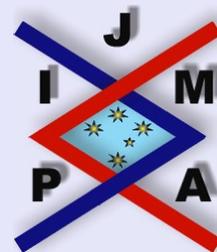
Close

Quit

Page 32 of 33

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Subharmonic Functions and
their Riesz Measure

Raphaele Supper

Title Page

Contents



Go Back

Close

Quit

Page 33 of 33