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REFINEMENTS OF CARLEMAN'S INEQUALITY

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Abstract

In this paper, we obtain a class of refined Carleman's Inequalities with the arithmetic-geometric mean inequality by decreasing their weight coefficient.

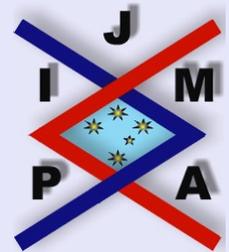
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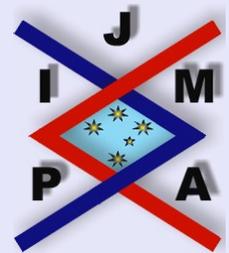
1. Introduction

Let $\{a_n\}_{n=1}^{+\infty}$ be a non-negative sequence such that $0 \leq \sum_{n=1}^{+\infty} a_n < +\infty$, then, we have

$$(1) \quad \sum_{n=1}^{+\infty} (a_1 a_2 \dots a_n)^{1/n} \leq e \sum_{n=1}^{+\infty} a_n.$$

The equality in (1) holds if and only if $a_n = 0, n = 1, 2, \dots$ the coefficient e is optimal.

Inequality (1) is called Carleman's inequality. For details please refer to [1, 2]. The Carleman's inequality has found many applications in mathematics, and the study of the Carleman's inequality has a rich literature, for details, please refer to [3, 4]. Though the coefficient e is optimal, we can refine its weight coefficient. In this article we give a class of improved Carleman's inequalities by decreasing the weight coefficient with the arithmetic-geometric mean inequality.



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2. Two Special Cases

In this section, we give two special cases of refined Carleman's inequality. First we prove two lemmas.

Lemma 2.1. For $m = 1, 2, \dots$, the inequality

$$(2) \quad \left(1 + \frac{1}{m}\right)^m \leq e \left(1 - \frac{1 - 2/e}{m}\right)$$

holds, where the constant $1 - \frac{2}{e} \approx 0.2642411$ is best possible.

Proof. Inequality

$$(3) \quad \left(1 + \frac{1}{m}\right)^m \leq e \left(1 - \frac{\beta}{m}\right)$$

is equivalent to $\beta \leq m - \frac{m}{e} \left(1 + \frac{1}{m}\right)^m$.

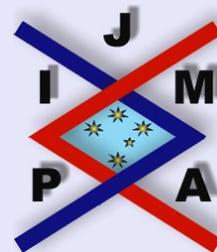
Let $f(x) = \frac{1}{x} - \frac{1}{ex} (1+x)^{\frac{1}{x}}$, $x \in (0, 1]$.

It is obvious that the function f is decreasing on the interval $(0, 1]$. Consequently, $\beta = f(1) = 1 - \frac{2}{e}$ is the optimal value satisfying inequality (3), so (2) holds. The proof of Lemma 2.1 follows. \square

Lemma 2.2. For $m = 1, 2, \dots$, the inequality

$$(4) \quad \left(1 + \frac{1}{m}\right)^m \leq \frac{e}{\left(1 + \frac{1}{m}\right)^{\frac{1}{\ln 2} - 1}}$$

holds, where the constant $\frac{1}{\ln 2} - 1 \approx 0.442695$ is the best possible.



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Proof. Inequality

$$(5) \quad \left(1 + \frac{1}{m}\right)^m \leq \frac{e}{\left(1 + \frac{1}{m}\right)^\alpha}$$

is equivalent to

$$\alpha \leq \frac{1}{\ln\left(1 + \frac{1}{m}\right)} - m.$$

Let

$$f(x) = \frac{1}{\ln(1+x)} - \frac{1}{x} \quad x \in (0, 1].$$

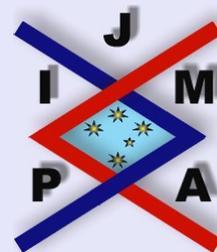
Since the function f is decreasing on the interval $(0, 1]$, $\alpha = f(1) = \frac{1}{\ln 2} - 1$ is the optimal value satisfying inequality (5), and thus (4) holds. The proof of Lemma 2.2 follows. \square

Theorem 2.3. Let $\{a_n\}_{n=1}^{+\infty}$ be a non-negative sequence such that $0 \leq \sum_{n=1}^{+\infty} a_n < +\infty$. Then the following inequalities hold:

$$(6) \quad \sum_{n=1}^{+\infty} (a_1 a_2 \dots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \left(1 - \frac{1-2/e}{m}\right) a_m,$$

and

$$(7) \quad \sum_{n=1}^{+\infty} (a_1 a_2 \dots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \frac{a_m}{\left(1 + \frac{1}{m}\right)^{\frac{1}{\ln 2} - 1}}.$$



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Proof. Let $c_i > 0$ ($i = 1, 2, \dots$). According to the arithmetic-geometric mean inequality, we have

$$(c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \leq \frac{1}{n} \sum_{m=1}^n c_m a_m.$$

Consequently,

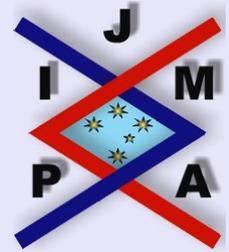
$$\begin{aligned} \sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} &= \sum_{n=1}^{+\infty} \left(\frac{c_1 a_1 c_2 a_2 \cdots c_n a_n}{c_1 c_2 \cdots c_n} \right)^{1/n} \\ &= \sum_{n=1}^{+\infty} (c_1 c_2 \cdots c_n)^{-1/n} (c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \\ &\leq \sum_{n=1}^{+\infty} (c_1 c_2 \cdots c_n)^{-1/n} \frac{1}{n} \sum_{m=1}^n c_m a_m \\ &= \sum_{m=1}^{+\infty} c_m a_m \sum_{n=m}^{+\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n}. \end{aligned}$$

Let $c_m = \frac{(m+1)^m}{m^{m-1}}$ ($m = 1, 2, \dots$). Then $c_1 c_2 \cdots c_n = (n+1)^n$, and

$$\sum_{n=m}^{+\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n} = \sum_{n=m}^{+\infty} \frac{1}{n(n+1)} = \frac{1}{m}.$$

Therefore

$$(8) \quad \sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq \sum_{m=1}^{+\infty} \frac{c_m}{m} a_m = \sum_{m=1}^{+\infty} \left(1 + \frac{1}{m} \right)^m a_m.$$



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According to Lemmas 2.1 and 2.2, and substituting for $(1 + \frac{1}{m})^m$ of inequality (8), so (6) and (7) follow from Lemmas 2.1 and 2.2.

The proof is complete. □



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3. A Class of Refined Carleman's Inequalities

In this section we give a class of refined Carleman's inequalities. First we have the following inequality

Lemma 3.1. For $m = 1, 2, \dots$, the inequality

$$(9) \quad \left(1 + \frac{1}{m}\right)^m \leq \frac{e \left(1 - \frac{\beta}{m}\right)}{\left(1 + \frac{1}{m}\right)^\alpha},$$

holds, where $0 \leq \alpha \leq \frac{1}{\ln 2} - 1$, $0 \leq \beta \leq 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$.

Proof. Inequality (9) is equivalent to

$$(10) \quad \beta \leq m - \frac{m}{e} \left(1 + \frac{1}{m}\right)^{m+\alpha}.$$

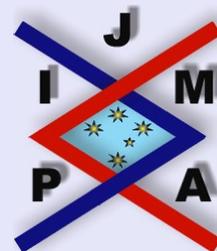
If

$$f(x) = \frac{1}{x} - \frac{1}{ex} (1+x)^{\frac{1}{x}+\alpha}, \quad x \in (0, 1], \quad 0 \leq \alpha \leq \frac{1}{\ln 2} - 1,$$

then f is decreasing on interval $(0, 1]$. Consequently, $\beta = f(1) = 1 - \frac{1}{e}2^{1+\alpha}$ is the optimal value satisfying inequality (10). Moreover, $0 \leq \beta \leq 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$. So (9) holds. The proof is complete. \square

Remark 3.1. If $\alpha = 0$, then $\beta = 1 - \frac{2}{e}$, and we obtain Lemma 2.1; if $\beta = 0$, then $\alpha = \frac{1}{\ln 2} - 1$, and we obtain Lemma 2.2.

Similar to Theorem 2.3, according to Lemma 3.1, we have



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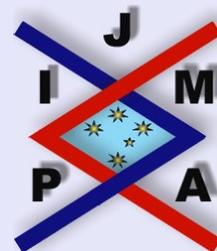
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Theorem 3.2. Let $a_n \geq 0$ ($n = 1, 2, \dots$), $0 \leq \sum_{n=1}^{+\infty} a_n < +\infty$, then

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \frac{(1 - \frac{\beta}{m})}{(1 + \frac{1}{m})^\alpha} a_m,$$

where α, β satisfy $0 \leq \alpha \leq \frac{1}{\ln 2} - 1$, $0 \leq \beta \leq 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$.

Remark 3.2. Theorem 2.3 gives two special cases of Theorem 3.2. If $\alpha = 0$, $\beta = 1 - \frac{2}{e}$, and $\alpha = \frac{1}{\ln 2} - 1$, $\beta = 0$, we can obtain (6) and (7) in Theorem 2.3 respectively.



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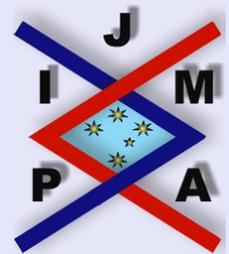
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