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A PICK FUNCTION RELATED TO AN INEQUALITY FOR THE ENTROPY FUNCTION

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ABSTRACT. The function $\psi(z) = 2/(1+z) + 1/(\mathrm{Log}\,(1-z)/2)$, holomorphic in the cut plane $\mathbb{C} \setminus [1, \infty[$, is shown to be a Pick function. This leads to an integral representation of the coefficients in the power series expansion $\psi(z) = \sum_{n=0}^{\infty} \beta_n z^n$, $|z| < 1$. The representation shows that (β_n) decreases to zero as conjectured by F. Topsøe. Furthermore, (β_n) is completely monotone.

Key words and phrases: Pick functions, completely monotone sequences.

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1. INTRODUCTION AND STATEMENT OF RESULTS

In the paper [2] about bounds for entropy Topsøe considered the function

(1.1) $$\psi(x) = \frac{2}{1+x} + \frac{1}{\ln \frac{1-x}{2}}\,, \quad -1 < x < 1$$

with the power series expansion

(1.2) $$\psi(x) = \sum_{n=0}^{\infty} \beta_n x^n$$

and conjectured from numerical evidence that (β_n) decreases to zero.

The purpose of this note is to prove the conjecture by establishing the integral representation

(1.3) $$\beta_n = \int_1^{\infty} \frac{dt}{t^{n+1}(\pi^2 + \ln^2 \frac{t-1}{2})}\,, \quad n \geq 0\,.$$

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This formula clearly shows $\beta_0 > \beta_1 > \cdots > \beta_n \to 0$. Furthermore, by a change of variable we find

$$\beta_n = \int_0^1 s^n \frac{ds}{s(\pi^2 + \ln^2 \frac{1-s}{2s})}\,, \quad n \geq 0,$$

which shows that (β_n) is a completely monotone sequence, cf. [3].

The representation (1.3) follows from the observation that ψ is the restriction of a Pick function with the following integral representation

$$\psi(z) = \int_1^\infty \frac{dt}{(t-z)(\pi^2 + \ln^2 \frac{t-1}{2})}\,, \quad z \in \mathbb{C} \setminus [1, \infty[\,. \tag{1.4}$$

From (1.4) we immediately get (1.3) since $\beta_n = \psi^{(n)}(0)/n!$.

2. PROOFS

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ in the upper half-plane is called a Pick function, cf. [1], if $\operatorname{Im} f(z) \geq 0$ for all $z \in \mathbb{H}$. Pick functions are also called Nevanlinna functions or Herglotz functions. They have the integral representation

$$f(z) = az + b + \int_{-\infty}^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t)\,, \tag{2.1}$$

where $a \geq 0$, $b \in \mathbb{R}$ and μ is a non-negative Borel measure on $\mathbb{R}$ satisfying

$$\int \frac{d\mu(t)}{1+t^2} < \infty\,.$$

It is known that

$$a = \lim_{y\to\infty} f(iy)/iy\,,\ b = \operatorname{Re} f(i)\,,\ \mu = \lim_{y\to 0^+} \frac{1}{\pi} \operatorname{Im} f(t+iy)dt\,, \tag{2.2}$$

where the limit refers to the vague topology. Finally f has a holomorphic extension to $\mathbb{C}\setminus[1,\infty[$ if and only if $\operatorname{supp}(\mu) \subseteq [1, \infty[$.

Let $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ denote the principal logarithm in the cut plane $\mathbb{C}\setminus]-\infty, 0]$, with $\operatorname{Arg} z \in\,]-\pi, \pi[$. Hence $\operatorname{Log} \frac{1-z}{2}$ is holomorphic in $\mathbb{C} \setminus [1, \infty[$ with $z = -1$ as a simple zero. It is easily seen that

$$\psi(z) = \frac{2}{1+z} + \frac{1}{\operatorname{Log} \frac{1-z}{2}}\,, \quad z \in \mathbb{C} \setminus [1, \infty[\tag{2.3}$$

is a holomorphic extension of (1.1) with a removable singularity for $z = -1$ where $\psi(-1) = 1/2$. To see that $V(z) = \operatorname{Im} \psi(z) \geq 0$ for $z \in \mathbb{H}$ it suffices by the boundary minimum principle for harmonic functions to verify $\liminf_{z\to x} V(z) \geq 0$ for $x \in \mathbb{R}$ and $\liminf_{|z|\to\infty} V(z) \geq 0$, where in both cases $z \in \mathbb{H}$.

We find

$$\lim_{z\to x} \psi(z) = \begin{cases} \psi(x)\,, & x \leq 1 \text{ (with } \psi(1) = 1) \\ \frac{2}{1+x} + \frac{1}{\ln \frac{x-1}{2} - i\pi}\,, & x > 1 \end{cases}$$

hence

$$\lim_{z\to x} V(z) = \begin{cases} 0\,, & x \leq 1 \\ \frac{\pi}{\pi^2 + \ln^2 \frac{x-1}{2}}\,, & x > 1\,, \end{cases}$$

whereas $\lim_{|z|\to\infty} \psi(z) = 0$. This shows that ψ is a Pick function, and from (2.2) we see that $a = 0$ and μ has the following continuous density with respect to Lebesgue measure

$$d(x) = \begin{cases} 0\,, & x \le 1 \\ 1\Big/\left(\pi^2 + \ln^2 \frac{x-1}{2}\right)\,, & x > 1. \end{cases}$$

Therefore

$$\psi(z) = b + \int_1^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) \frac{dt}{\pi^2 + \ln^2 \frac{t-1}{2}}\,.$$

In this case we can integrate term by term, and since $\lim_{x\to-\infty} \psi(x) = 0$, we find

$$\psi(z) = \int_1^\infty \frac{dt}{(t-z)(\pi^2 + \ln^2 \frac{t-1}{2})}$$

and

$$b = \mathrm{Re}\,\psi(i) = 1 - \frac{8\ln\,2}{\pi^2 + 4\ln^2\,2} = \int_1^\infty \frac{tdt}{(1+t^2)(\pi^2 + \ln^2 \frac{t-1}{2})},$$

which establishes (1.4).

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