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ON KY FAN'S MINIMAX INEQUALITIES, MIXED EQUILIBRIUM PROBLEMS AND HEMIVARIATIONAL INEQUALITIES

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ABSTRACT. In this note, we present a generalization of the Ky Fan's minimax inequality theorem by means of a new version of the KKM lemma. Application is then given to establish existence of solutions for mixed equilibrium problems. Finally, we investigate the relationship between the latter problems and hemivariational inequalities.

Key words and phrases: Ky Fan's minimax inequalities, KKM Theorem, saddle points, mixed variational and hemivariational inequalities, transfer semicontinuity, Hemivariational inequalities, and T-pseudomonotonicity.

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1. INTRODUCTION

Blum-Oettli [2] understood by the so-called equilibrium problem, the following abstract variational inequality problem:

(EP) $$\text{find } \bar{x} \in C \text{ such that } f(\bar{x}, y) \geq 0 \text{ for all } y \in C,$$

where C is a given set and f is a given scalar valued bifunction on C. It is well-known that (EP) is closely related to Ky Fan's minimax inequalities [9]. When f is written as a sum of two real bifunctions, (EP) will be called a mixed equilibrium problem and we shall denote it by (MEP). Many interesting and sophisticated problems in nonlinear analysis can be cast into the form of (EP); say, for instance, optimization, saddle points, Nash equilibrium, fixed points, variational inequalities and complementarity problems.

The purpose of this paper is two-fold. First, we continuously study the existence problem of solutions for (EP) under some more general conditions, using a new version of the Fan KKM lemma. Then, to show the significance of the treatment of such problems, we investigate the relationship between hemivariational inequalities and mixed equilibrium problems. More precisely, the plan of our contribution is as follows. In Section 2, we state most of the material used in this paper. In Section 3, we present first a generalized Fan KKM lemma for transfer

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closed-valued maps. This result is then used as a tool for proving a new existence theorem for (EP). Some special cases are derived from this result; in particular, we give an application to saddle point problems. In Section 4, we confine ourselves to the study of mixed equilibrium problems. Indeed, we prove the existence of a mixed equilibrium by relaxing the upper semicontinuity condition on the nonmonotone part; then we apply this result to solve some mixed variational inequality problems. Finally, Section 5 indicates how the result of the previous section can be used to ensure the existence of solutions to hemivariational inequalities involving some topological pseudomonotone functionals.

2. Definitions and Notations

Before the formal discussion, we begin with some notations and definitions, which will be needed in the sequel. Let X be a topological vector space, and let X' be its topological dual. Let C be a nonempty convex subset in X. Denote by $\mathcal{F}(C)$ the set of all nonempty finite subsets of C. Let

$$F : C \to 2^C \quad \text{be a set-valued map,}$$
$$f : C \times C \to \mathbb{R} \quad \text{and} \quad \Phi : C \to \mathbb{R} \quad \text{two functions, and}$$
$$S : C \to 2^{X'} \quad \text{a set-valued operator.}$$

F is called a KKM map if for any subset $A \in \mathcal{F}(C)$, $coA \subset \bigcup_{x\in A} F(x)$. ϕ is said to be quasi convex if the strict lower level set $\{x \in C : \phi(x) < 0\}$ is convex. It is quasi concave if $-\phi$ is quasi convex. For $\lambda \in \mathbb{R}$, ϕ is λ-quasi convex (concave) if $\phi - \lambda$ is quasi convex (concave).

F is said to be transfer closed-valued [13] if, for any $x, y \in C$ with $y \notin F(x)$, there exists $x' \in C$ such that $y \notin cl_C F(x)$. It is clear that this definition is equivalent to say that $\bigcap_{x\in C} F(x) = \bigcap_{x\in C} cl_C F(x)$. We will say that F is transfer closed-valued on a subset B of C if the set-valued map $F_B : B \to 2^B$, defined by $F_B(x) := F(x) \cap B$ for all $x \in B$, is transfer closed-valued. Related to this concept, let us recall the definition of transfer semicontinuity. f is said to be transfer lower semicontinuous in y if, for each $x, y \in C$ with $f(x, y) > 0$, there exist $x' \in C$ and a neighborhood U_y of y in C such that $f(x', z) > 0$ for all $z \in U_y$. f is said to be transfer upper semicontinuous if $-f$ is transfer lower semicontinuous. It's easily seen that a lower (upper) semicontinuous bifunction in y is transfer lower (upper) semicontinuous in y. We will say that f is transfer lower semicontinuous on B if the restriction of f on $B\times B$ is transfer lower semicontinuous. For $\lambda \in \mathbb{R}$, f is λ-transfer lower (upper) semicontinuous in y if the bifunction $f - \lambda$ is transfer lower (upper) semicontinuous in y.

Φ is called upper hemicontinuous if, for each $x, y \in C$, the function $t \mapsto \Phi(tx + (1 - t)y)$, defined for $t \in [0, 1]$, is upper semicontinuous. The operator S is said to be upper hemicontinuous if $t \mapsto S(tx + (1 - t)y)$, defined for $t \in [0, 1]$, is upper semicontinuous as a set-valued map.

f is said to be monotone if $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$. f is pseudomonotone if, for every $x, y \in C$, $f(x, y) \ge 0$ implies $f(y, x) \le 0$. One can easily see that a monotone bifunction is pseudomonotone. S is said to be monotone if for all $x, y \in C$ and for all $s \in Sx$, $r \in Sy$, one has $\langle s - r, x - y\rangle \ge 0$.

f is pseudomonotone in the topological sense (T-pseudomonotone for short), whenever (x_α) is a net on C converging to $x \in C$ such that $\liminf f(x_\alpha, x) \ge 0$, then $\limsup f(x_\alpha, y) \le f(x, y)$ for all $y \in C$. Suppose now that X is a reflexive Banach space. Let $J : C \to I\!R$ be a locally Lipschitz function. Denote by J^0 its directional differential in the sense of Clarke. We know [7] that J^0 is upper semicontinuous and $J^0(x, .)$ is convex for every $x \in C$. We say that $J \in PM(C)$ if the bifunction f, defined by $f(x, y) := J^0(x, y - x)$ for all $x, y \in C$, is T-pseudomonotone. When this bifunction is only T-quasi pseudomonotone (that is, if for any sequence $(x_n) \in C$ such that $x_n \rightharpoonup x$ in C and $\liminf J^0(x_n, x - x_n) \geq 0$, then $\lim J^0(x_n, x - x_n) = 0$)), we shall say that $J \in QPM(C)$. A function belonging to the class $PM(C)$ (resp. $QPM(C)$) has the property that its Clarke's subdifferential is pseudomonotone in the sense of Browder and Hess (resp. quasi-pseudomonotone) (see [10, Proposition 2.13]). The operator S is said to be T-pseudomonotone [6] if so is the bifunction f, given by $f(x,y) := \sup_{s \in Sx} \langle s, y - x \rangle$. Suppose now that S is single-valued. We recall also that S satisfies the $(S)_+$ condition, if

$$x_n \rightharpoonup x \text{ in } C \text{ and } \limsup \langle Sx_n, x_n - x \rangle \leq 0 \text{ imply } x_n \to x \text{ in } C.$$

ϕ is said to be inf compact [1, p 318] if the set lower level set $cl_C\{x \in C : \phi(x) \leq 0\}$ is compact. ϕ is sup compact if $-\phi$ is inf compact. For $\lambda \in \mathbb{R}$, ϕ is λ-inf (sup) compact if $\phi - \lambda$ is inf (sup) compact. We say that S is x_0-coercive for some $x_0 \in C$ if there exists a real-valued function c on $\mathbb{R}_+$ with $\lim_{r \to +\infty} c(r) = +\infty$ such that for all $x \in C$ $\langle Sx, x - x_0 \rangle \geq c(\|x\|)\|x\|$.

3. KY FAN'S MINIMAX INEQUALITY

3.1. **A KKM Result.** It is interesting to note that the Fan KKM lemma [8] plays a crucial role to prove existence results for (EP). In [3], this result was improved by assuming the closedness condition only upper finite dimensional subspaces, with some topological pseudomonotone condition. In [6], Chowdhury and Tan, replacing finite dimensional subspaces by convex hulls of finite subsets, restated the Brézis-Nirenberg-Stampacchia result under weaker assumptions. On the other hand, Tian [13] introduced a new class of closedness conditions, namely the transfer closedness, and give the KKM conclusion for multifunctions satisfying this weak assumption. Here, using this class, we give another more refined version of the Fan KKM lemma containing Chowdhury-Tan result as a special case.

Lemma 3.1. *Suppose that C is convex. If*

(i) *$cl_C F(x_0)$ is compact for some $x_0 \in C$;*
(ii) *F is a KKM map;*
(iii) *for each $A \in \mathcal{F}(C)$ with $x_0 \in A$, F is transfer closed-valued on coA;*
(iv) *for every $A \in F(C)$ with $x_0 \in A$, we have*

$$[cl_C(\bigcap_{x \in coA} F(x))] \cap coA = [\bigcap_{x \in coA} F(x)] \cap coA,$$

then $\bigcap_{x \in C} F(x) \neq \emptyset$.

Proof. Let $A \in \mathcal{F}(C)$ with $x_0 \in A$. Consider a set-valued map $F_A : coA \to 2^{coA}$, defined by $F_A(x) := cl_C(F(x) \cap coA)$ for all $x \in coA$. F_A so defined satisfies the KKM conditions. Indeed, first F_A is nonempty and compact-valued since F is a KKM map ($x \in F(x)$ for all $x \in coA$) and coA is compact; then, for each $B \in \mathcal{F}(coA)$, we have $coB \subset \bigcup_{x \in B} F(x)$, but $coB \subset coA$, hence

$$coB \subset \bigcup_{x \in B} F(x) \cap coA \subset \bigcup_{x \in B} cl_C(F(x) \cap coA),$$

thus F_A is a KKM map. It follows that

$$\bigcap_{x \in coA} F_A(x) \neq \emptyset.$$

Hence by (iii), we obtain

$$\bigcap_{x \in coA} F(x) \cap coA \neq \emptyset.$$

Then we follow the same argument in [6, proof of Lemma 2] to get our assertion. □

3.2. **General Existence Results.** Now we are in position to give the following generalization of Ky Fan's minimax inequality theorem.

Theorem 3.2. *Suppose that ϕ and ψ are two scalar valued bifunctions on C such that*

(A1) $\psi(x,y) \leq 0$ *implies* $\phi(x,y) \leq 0$ *for all* $x, y \in C$*;*

(A2) *for each* $A \in \mathcal{F}(C)$*,* $\sup_{y \in coA} \min_{x \in A} \psi(x,y) \leq 0$*;*

(A3) *for each* $A \in \mathcal{F}(C)$*,* ϕ *is transfer lower semicontinuous in* y *on* coA*;*

(A4) *for each* $A \in \mathcal{F}(C)$*, whenever* $x, y \in coA$ *and* (y_α) *is a net on* C *converging to* y*, then*

$$\phi(tx + (1-t)y, y_\alpha) \leq 0 \ \forall t \in [0,1] \ \Rightarrow \ \phi(x,y) \leq 0\,;$$

(A5) *there is* $x_0 \in C$ *such that* $\phi(x_0, .)$ *is inf compact.*

Then there exists $\overline{y} \in C$ *such that* $\phi(x, \overline{y}) \leq 0$ *for all* $x \in C$*.*

Proof. It's a simple matter to see that all conditions of Lemma 3.1 are fulfilled if we take

$$F(x) = \{y \in C : \phi(x,y) \leq 0\} \ \forall x \in C.$$

Indeed, (i) follows from (A5), and (ii) from (A1) and (A2). It remains to show that (A3) implies (iii), and (A4) implies (iv). To do the former, fix $A \in F(C)$ and let $(x,y) \in coA \times coA$ with $y \notin F(x)$; that is $\phi(x,y) > 0$; hence, there exist $x' \in coA$ and a neighborhood U_y of y in coA such that $\phi(x', z) > 0$ for all $z \in U_y$; thus $x \notin cl_C(F(x) \cap coA)$. For the latter, fix also $A \in F(C)$ and let $y \in cl_C[\bigcap_{x \in coA} F(x)] \cap coA$; that is $y \in coA$ and there is a net (y_α) converging to y such that $\phi(x, y_\alpha) \leq 0$ for all $x \in coA$; it follows that $\phi(tx + (1-t)y, y_\alpha) \leq 0$ for all $x \in coA$ and all $t \in [0,1]$; hence, from (A4), we get $\phi(x,y) \leq 0$ for all $x \in coA$; we conclude that $y \in [\bigcap_{x \in coA} F(x)] \cap coA$. The proof is complete. □

Remark 3.3. It has to be observed that assumption (A2) holds provided that

(i) $\psi(x,x) \leq 0$ for all $x \in C$, and

(ii) for each $y \in C$, $\psi(., y)$ is quasi concave.

Remark 3.4. Assumption (A3) holds clearly when $\phi(x, .)$ is supposed to be lower semicontinuous on coA, for every $A \in \mathcal{F}(C)$ and every $x \in coA$. Moreover, both of assumptions (A3) and (A4) are satisfied when the classical assumption of lower semicontinuity of $\phi(x, .)$ is supposed to be true for every $x \in C$.

Remark 3.5. The compactness condition (A5) is satisfied if we suppose that there exists a compact subset B of C and $x_0 \in B$ such that $\phi(x_0, y) > 0$ for all $y \in C \setminus B$.

Theorem 3.2 is now a generalization of [6, Theorem 4]. It also improves [12, Theorem 1]. Let us single out some particular cases of this theorem. First, let $\phi = \psi$ and make use of Remark 3.3.

Theorem 3.6. *Suppose that* $\phi : C \to \mathbb{R}$ *satisfy*

(B1) $\phi(x,x) \leq 0$ *for all* $x \in C$*;*

(B2) *for each* $y \in C$*,* $\phi(., y)$ *is quasi concave;*

(B3) *for each* $A \in \mathcal{F}(C)$*,* ϕ *is transfer lower semicontinuous in* y *on* coA*;*

(B4) *for each* $A \in \mathcal{F}(C)$*, whenever* $x, y \in coA$ *and* (y_α) *is a net on* C *converging to* y*, then*

$$\phi(tx + (1-t)y, y_\alpha) \leq 0 \ \forall t \in [0,1] \ \Rightarrow \ \phi(x,y) \leq 0\,;$$

(B5) *there is* $x_0 \in C$ *such that* $\phi(x_0, .)$ *is inf compact.*

Then there exists $\overline{y} \in C$ *such that* $\phi(x, \overline{y}) \leq 0$ *for all* $x \in C$*.*

We can also derive the following generalization of a Ky Fan's minimax inequality theorem due to Yen [14, Theorem 1].

Theorem 3.7. *Let $f, g : C \to \mathbb{R}$. Suppose that $\lambda = \sup_{x \in C} g(x,x) < \infty$, and*

(C1) $f(x,y) \le g(x,y)$ *for all* $x, y \in C$*;*
(C2) *for each* $y \in C$*,* $g(.,y)$ *is* λ*-quasi concave;*
(C3) *for each* $A \in \mathcal{F}(C)$*,* f *is* λ*-transfer lower semicontinuous in* y *on* coA*;*
(C4) *for each* $A \in \mathcal{F}(C)$*, whenever* $x, y \in coA$ *and* (y_α) *is a net on* C *converging to* y*, then*

$$f(tx + (1-t)y, y_\alpha) \le \lambda \ \forall t \in [0,1] \ \Rightarrow \ f(x,y) \le \lambda;$$

(C5) *there exist a compact subset* B *of* C *and* $x_0 \in B$ *such that* $f(x_0, y) > \lambda$ *for all* $y \in C \backslash B$.

Then we have the following minimax inequality

$$\inf_{y \in B} \sup_{x \in C} f(x,y) \le \sup_{x \in C} g(x,x).$$

Proof. Set $\phi(x,y) = f(x,y) - \lambda$ and $\psi(x,y) = g(x,y) - \lambda$ for all $x, y \in C$. By virtue of Theorem 3.2 and taking into account assumption (C5), there exists $\overline{y} \in B$ such that $\sup_{x \in C} f(x, \overline{y}) \le \sup_{x \in C} g(x,x)$, which is our assertion. □

The following another particular case will be needed in the next section.

Theorem 3.8. *Let* C *be closed convex in* X*. Suppose that*

(D1) $\psi(x,y) \le 0$ *implies* $\phi(x,y) \le 0$ *for all* $x, y \in C$*;*
(D2) $\psi(x,x) \le 0$ *for all* $x \in C$*,*
(D3) *for every fixed* $y \in C$*,* $\psi(.,y)$ *is quasi concave.*
(D4) *for every fixed* $x \in C$*,* $\phi(x,.)$ *is lower semicontinuous on the intersection of finite dimensional subspaces of* X *with* C*;*
(D5) *whenever* $x, y \in C$ *and* (y_α) *is a net on* C *converging to* y*, then*

$$\phi(tx + (1-t)y, y_\alpha) \le 0 \ \forall t \in [0,1] \ \Rightarrow \ \phi(x,y) \le 0\,;$$

(D6) *there exist a compact subset* B *of* X *and* $x_0 \in C \cap B$ *such that* $\phi(x_0, y) > 0$ *for all* $y \in C \setminus B$.

Then there exists $\overline{y} \in C \cap B$ *such that* $\phi(x, \overline{y}) \le 0$ *for all* $x \in C$.

Proof. According to Theorem 3.2, there exists $\overline{y} \in C$ such that $\phi(x, \overline{y}) \le 0$ for all $x \in C$. The element $\overline{y}$ is in the compact B due to condition (D6). □

If we set $\phi = \psi$, we recover [3, Theorem 1].

We balance now the continuity requirements by assuming the algebraic pseudomonotonicity on the criterion bifuncion, and we get from Theorem 3.2 the following existence theorem for (EP).

Theorem 3.9. *Let* $f : C \times C \to \mathbb{R}$ *be a bifunction such as to satisfy*

(E1) f *is pseudomonotone,*
(E2) $f(x,x) \ge 0$ *for all* $x \in C$*,*
(E3) *for each* $x \in C$*,* $f(x,.)$ *is convex,*
(E4) *for each* $A \in \mathcal{F}(C)$*,* f *is transfer lower semicontinuous in* y *on* coA*,*
(E5) *for each* $A \in \mathcal{F}(C)$*, whenever* $x, y \in coA$ *and* (y_α) *is a net on* C *converging to* y*, then*

$$f(tx + (1-t)y, y_\alpha) \le 0 \ \forall t \in [0,1] \ \Rightarrow \ f(x,y) \le 0\,;$$

(E6) *for each* $y \in C$*,* $f(.,y)$ *is upper hemicontinuous,*
(E7) *there exist a compact subset* B *of* C *and* $x_0 \in B$ *such that* $\phi(x_0, y) > 0$ *for all* $y \in C \backslash B$.

Then (EP) has at least one solution, which is in B.

Proof. Set $\varphi(x,y) = f(x,y)$, $\psi(x,y) = -f(y,x)$ for all $x, y \in C$. All assumptions of Theorem 3.2 are clearly satisfied; hence there exists $\overline{x} \in B$ such that $f(y,\overline{x}) \leq 0$ for all $y \in C$. The conclusion holds if we show the following assertion: for every $x \in C$, one has

$$f(y,x) \leq 0 \ \ \forall y \in C \implies f(x,y) \geq 0 \ \ \forall y \in C$$

To do this, let $x \in C$ such that

$$f(y,x) \leq 0 \ \ \forall y \in C \tag{3.1}$$

Fix $y \in C$, and set $y_t = ty + (1-t)x$ for $t \in]0,1[$. Since $f(y_t,.)$ is convex and $f(y_t,y_t) \geq 0$, then $tf(y_t,y) + (1-t)f(y_t,x) \geq 0$. It follows clearly from (3.1) that $tf(y_t,y) \geq 0$; hence $f(y_t,y) \geq 0$. The upper hemicontinuity of $f(.,y)$ allow us to conclude that $f(x,y) \geq 0$. This completes our proof. □

3.3. **Saddle Points.** Let $F : C \times C \to \mathbb{R}$. A (strategy) pair $(\overline{x},\overline{y})$ is a saddle point of F if one has

$$F(\overline{x},y) \leq F(\overline{x},\overline{y}) \leq F(x,\overline{y}) \quad \forall x, y \in C.$$

This equivalent to writing

$$\inf_{x\in C}\sup_{y\in C} F(x,y) = \sup_{y\in C}\inf_{x\in C} F(x,y) = F(\overline{x},\overline{y}).$$

By making use of Theorem 3.6, we can formulate the following existence theorem for saddle points.

Theorem 3.10. *Assume that*

(i) $F(x,x) = 0$ *for all* $x \in C$,
(ii) F *is quasi convex (resp. concave) with respect to* x *(resp.* y*),*
(iii) *for each* $A \in \mathcal{F}(C)$, F *is transfer upper (resp. lower) semicontinuous in* y *(resp.* x*) on* coA,
(iv) *for each* $A \in \mathcal{F}(C)$, *whenever* $x, y \in coA$ *and* (y_α) *(resp.* (x_α)*) is a net on* C *converging to* y *(resp.* x*), then*

$$F(tx + (1-t)y, y_\alpha) \geq 0 \ \forall t \in [0,1] \ \Rightarrow \ F(x,y) \geq 0$$

(resp.

$$F(x_\alpha, tx + (1-t)y) \leq 0 \ \forall t \in [0,1] \ \Rightarrow \ \phi(x,y) \leq 0);$$

(v) *there exist* x_0 *(resp.* y_0*)* $\in C$ *such that* $F(x_0,.)$ *(resp.* $F(.,y_0)$*) is sup (resp. inf) compact.*

Then F *has at least one saddle point* $(\overline{x},\overline{y})$ *in* C, *which satisfies* $F(\overline{x},\overline{y}) = 0$.

Proof. We apply Theorem 3.6, first for $\phi = -F$, we get the existence of $\overline{y} \in C$ which satisfies $F(x,\overline{y}) \geq 0$ for all $x \in C$; then for $\phi(x,y) = F(y,x)$ for all $x, y \in C$, there exists $\overline{x} \in C$ such that $F(\overline{x},y) \leq 0$ for all $y \in C$. We conclude that $(\overline{x},\overline{y})$ is a saddle point for F, with $F(\overline{x},\overline{y}) = 0$. □

We can deduce easily the von Neumann's minimax theorem [1, p 319, Theorem 8] when the sets of strategies are the same.

4. MIXED EQUILIBRIUM PROBLEMS

Here we focus our attention on the existence of solutions for equilibrium problems. The criterion mapping is composed of two parts, a monotone bifunction and a nonmonotone perturbation. Our aim is to establish the existence of equilibria by relaxing the upper semicontinuity condition on the nonmonotone part.
From this point on, C is supposed to be closed and convex in X.

Theorem 4.1. *Consider $f, g : C \times C \to \mathbb{R}$ such that*

1. *g is monotone;*
2. *$f(x,x) = g(x,x) = 0$ for all $x \in C$;*
3. *for every fixed $x \in C$, $f(x,.)$ and $g(x,.)$ are convex;*
4. *for every fixed $x \in C$, $g(x,.)$ is lower semicontinuous;*
5. *for every fixed $y \in C$, $f(.,y)$ is upper semicontinuous on $C \cap F$, while F is a finite dimensional subspace in X;*
6. *f is T-pseudomonotone;*
7. *for every fixed $y \in C$, $g(.,y)$ is upper hemicontinuous;*
8. *there is a compact subset B of X and $x_0 \in C \cap B$ such that*

$$g(x_0, y) - f(y, x_0) > 0, \quad \textit{for all } y \in C \backslash B.$$

Then there exists at least one solution to (MEP) associated to f and g.

Proof. In order to apply Theorem 3.8, set $\phi(x,y) = g(x,y) - f(y,x)$ and $\psi(x,y) = -g(y,x) - f(y,x)$ for all $x, y \in C$.
Let us show that the assumptions of Theorem 3.8 are satisfied.
(D1), (D2) and (D6) are easily checked from (1), (2) and (8) respectively. For (D3), if $x,\ y,\ z \in C$ then for any $t \in]0,1[$ one has

$$g(y, tx + (1-t)z) + f(y, tx + (1-t)z) \leq t[g(y,x) + f(y,x)] + (1-t)[g(y,z) + f(y,z)].$$

In this way, the set

$$\{x \in C : f(x,y) + g(x,y) < 0\}$$

is convex for all $y \in C$.
Moreover, since $g(x,.)$ is lower semicontinuous and $f(.,x)$ is upper semicontinuous on every finite dimensional subspace for all $x \in C$, $g(x,.) - f(.,x)$ is lower semicontinuous on every finite dimensional subspace in C for all $x \in C$. This means that (D4) holds.
To check (D5). Let $x, y \in C$ and (y_α) a net on C which converges to y such that, for every $t \in [0,1]$,

$$g(tx + (1-t)y, y_\alpha) - f(y_\alpha, tx + (1-t)y) \leq 0. \tag{4.1}$$

For $t = 0$, one has

$$g(y, y_\alpha) - f(y_\alpha, y) \leq 0. \tag{4.2}$$

Since $g(y,.)$ is lower semicontinuous, it follows that $g(y,y) \leq \liminf f(y_\alpha, y)$. Combining with (2), one can write $0 \leq \liminf f(y_\alpha, y)$. Therefore

$$\limsup f(y_\alpha, x) \leq f(y,x) \tag{4.3}$$

since f is T-pseudomonotone.
Recall (4.1) and take $t = 1$, it follows that $g(x, y_\alpha) - f(y_\alpha, x) \leq 0$. By virtue of assumption (4) and relation (4.3), we get

$$g(x,y) - f(y,x) \leq 0.$$

Thus (D5) is satisfied.
We deduce that there exists $\overline{x} \in C$ such that

(4.4) $$g(y, \overline{x}) - f(\overline{x}, y) \leq 0 \text{ for all } y \in C,$$

Let $y \in C$ be a fixed point and set $y_t = ty + (1-t)\overline{x}$ for $t \in]0,1[$. Since $g(y_t, .)$ is convex and $g(y_t, y_t) = 0$, then

(4.5) $$tg(y_t, y) + (1-t)g(y_t, \overline{x}) \geq 0.$$

From relation (4.4), one has $(1-t)g(y_t, \overline{x}) - (1-t)f(\overline{x}, y_t) \leq 0$. Combining with (4.5), it follows $tg(y_t, y) + (1-t)f(\overline{x}, y_t) \geq 0$. Using the convexity of $f(\overline{x}, .)$, we can write $g(y_t, y) + (1-t)f(\overline{x}, y) \geq 0$ because $f(\overline{x}, \overline{x}) = 0$. The upper hemicontinuity of $g(., y)$ make it possible to write

$$g(\overline{x}, y) + f(\overline{x}, y) \geq 0.$$

The proof is complete. □

Remark 4.2. Since the upper semicontinuity of $f(., y)$, for all $y \in C$, implies that assumptions (5) and (6) are satisfied, the result of Blum-Oettli (Theorem 1 in [2]) is an immediate consequence of Theorem 4.1 when C is supposed to be compact. On the other hand, if $f = 0$ then Theorem 4.1 collapses to a Ky Fan's minimax inequality theorem for monotone bifunctions given in [1] (Theorem 9, p 332).

A consequence of this theorem is the following existence theorem for mixed variational inequalities. Suppose that the duality pairing $\langle ., . \rangle$ between X and X' is continuous.

Theorem 4.3. *Let h be a real convex lower semicontinuous function on C, and let S, T be two operators from C to $2^{X'}$ such as to satisfy :*
(i) S is T-pseudomonotone, upper semicontinuous on finite dimensional subspaces, and has convex weakly compact values;*
(ii) T is monotone, upper hemicontinuous, and has convex weakly compact values;*
(iii) there is a compact subset B of X and $x_0 \in C \cap B$ such that

$$\inf_{s \in Sy} \langle s, y - x_0 \rangle + \sup_{t \in Tx_0} \langle t, y - x_0 \rangle + h(y) - h(x_0) > 0 \ \forall y \in C \backslash B.$$

Then there exists $\overline{x} \in B$ solution to the following mixed variational inequality

(4.6) $$\exists \overline{s} \in S\overline{x}\,,\ \exists \overline{t} \in T\overline{x} : \langle \overline{s} + \overline{t}, y - \overline{x} \rangle + h(y) - h(\overline{x} \geq 0 \ \forall y \in C.$$

Proof. First of all, it has to be observed that inequality (4.6) is equivalent to write

$$\sup_{\substack{s \in S\overline{x} \\ t \in T\overline{x}}} \inf_{y \in C} [\langle s + t, y - \overline{x} \rangle + h(y) - h(\overline{x})] \geq 0.$$

Let $D := S\overline{x} \times T\overline{x}$ and $\varphi(d, y) := \langle s + t, y - \overline{x} \rangle + h(y) - h(\overline{x})$ for all $d = (s, t) \in D$ and all $y \in C$. It's easily seen that $\varphi(., y)$ is concave and upper semicontinuous for every $y \in C$, and that $\varphi(d, .)$ is convex for every $d \in D$. Moreover, D is a convex weakly* compact subset of X'. It follows, according to the Lopsided minimax theorem (see [1, p 319, Theorem 7]) that

$$\sup_{d \in D} \inf_{y \in C} \varphi(d, y) = \inf_{y \in C} \sup_{d \in D} \varphi(d, y).$$

Hence, if we set $f(x, y) = \sup_{s \in Sx} \langle s, y - x \rangle$ and $g(x, y) = \sup_{t \in Tx} \langle t, y - x \rangle + h(y) - h(x)$, then inequality (4.6) will be now equivalent to

$$\inf_{y \in C} (f(\overline{x}, y) + g(\overline{x}, y)) \geq 0.$$

Let us now check the assumptions of Theorem 4.1. Assumptions (2), (3) and (4) hold clearly. (iii) implies (8). By definition, the T-pseudomonotonicity of S implies that of f; hence (6) holds.

On the other hand, the finite dimensional upper semicontinuity of S together with the fact that S has weakly* compact values imply that (5) is satisfied (see [1, p 119, Proposition 21]. For (1), we have for each $x, y \in C$,

$$\begin{aligned} g(x,y) + g(y,x) &= \sup_{t\in Tx}\langle t, y-x\rangle + \sup_{r\in Ty}\langle r, x-y\rangle \\ &= \sup_{\substack{t \in Tx \\ r \in Ty}} \langle t-r, y-x\rangle \\ &\leq 0. \end{aligned}$$

Finally, by virtue of [1, p 373, Lemma 11], we have that the function $x \mapsto \sup_{s\in Sx}\langle s, y-x\rangle$ is upper hemicontinuous since S is upper hemicontinuous and has weakly* compact values; thus (7) holds. The conclusion follows the from the assertion of Theorem 4.1.

□

Remark 4.4. The coercivity condition (iii) is satisfied if there exists $x_0 \in C$ such that

$$\lim_{\substack{\|y\| \to \infty \\ y \in C}} \inf_{s\in Sy}\langle s, y-x_0\rangle + \sup_{t\in Tx_0}\langle t, y-x_0\rangle + h(y) - h(x_0) > 0.$$

Remark 4.5. When $h = 0$ and $S = 0$, Theorem 4.3 collapses to an existence result of a generalization of the Browder-Hartman-Stampacchia variational inequality [5, Theorem 4.1]. For $T = 0$ and S is a single-valued operator, it extends [3, Application 3]. Finally, under a minor change in the setting of Theorem 4.3, we can recover also [6, Theorem 7].

5. HEMIVARIATIONAL INEQUALITIES

When studying generalized mechanical problems that involve nonconvex energy functionals, Panagiotopoulos [11] introduced the hemivariational inequalities as a mathematical formulation. Since then, this theory has been proved very efficient for the treatment of certain as yet unsolved or partially solved problems in mechanic, engineering and economics.

The aim of this section is to show how that (MEP) can be an efficient tool for studying hemivariational inequalities that involve topological pseudomonotone functionals. More precisely, we shall use an existence result for (MEP) (Theorem 4.1) to get the existence of solutions to these inequalities without the hypothesis of quasi or strong quasi boundedness as in [10].

First, to illustrate the idea of hemivariational inequalities, we discuss an example concerning a body contact, which its variational formulation leads to a hemivariational inequality.

5.1. **An example.** Assume we are given a linear elastic body referred to a Cartesian orthogonal coordinate system $Ox_1x_2x_3$. This body is identified to an open bounded subset Ω of $\mathbb{R}^3$. We denote by Γ the boundary of Ω, which is supposed to be appropriately smooth. We denote also by $u = (u_i)_{1\leq i\leq 3}$ the displacement vector and by $S = (S_i)_{1\leq i\leq 3}$ the stress vector over Γ. We recall that $S_i = \sigma_{ij}n_j$, where $\sigma = (\sigma_{ij})$ is the stress tensor and $n = (n_i)$ is the outward unit normal vector on Γ. The vector S (resp., u) may be decomposed into a normal component S_N (resp., u_N) and a tangential one S_T (resp., u_T) with respect to Γ.

We begin first with the treatment of the case of monotone boundary conditions, which leads to variational inequalities as a formulation. Let β_N be a maximal monotone operator from $\mathbb{R}$ to $2^{\mathbb{R}}$. Then we may consider the following boundary condition in the normal direction:

$$-S_N \in \beta_N(u_N), \tag{5.1}$$

Similar conditions may be considered in the tangential direction $-S_T \in \beta_T(u_T)$, or generally $-S \in \beta(u)$.

One can formulate relations (5.1) otherwise by calling upon some proper convex and lower semicontinuous functional J_N that satisfies $\beta_N = \partial J_N$. Henceforth one can write

$$-S_N \in \partial J_N(u_N).$$

This law is multivalued and monotone. It includes many classical unilateral boundary conditions (e.g. $u_N = 0$ or $S_N = 0$). This kind of conditions have as variational formulation the following variational inequality:

$$J_N(v_N) - J_N(u_N) \geq -S_N(v_N - u_N), \quad \forall v_N \in \mathbb{R}.$$

However there are many other problems concerning the contact on an elastic body that may be expressed with multivalued boundary conditions which are nonmonotone. Consider an example which describes an adhesive contact with a rubber support. It may take the following form

$$\left\{ \begin{array}{lll} -S_N \in \widetilde{\beta}(u_N) & \text{if} & u_N < a \\ \widetilde{\beta}(a) \leq -S_N < +\infty & \text{if} & u_N = a \\ S_N = \emptyset & \text{if} & u_N > a \end{array} \right. \tag{5.2}$$

where $\widetilde{\beta}$ is defined as follows. Giving a function $\beta : \mathbb{R} \to \mathbb{R}$ in $L^\infty_{loc}\mathbb{R}$, consider two associated functions $\overline{\beta}_\rho$ and $\overline{\overline{\beta}}_\rho$ defined for $\rho > 0$ by

$$\overline{\beta}_\rho(t) := \operatorname*{ess\,sup}_{|t_1 - t| < \rho} \beta(t_1), \quad \forall t \in \mathbb{R}$$

and

$$\overline{\overline{\beta}}_\rho(t) := \operatorname*{ess\,inf}_{|t_1 - t| < \rho} \beta(t_1), \quad \forall t \in \mathbb{R}.$$

They are respectively decreasing and increasing with respect to ρ; hence their limits, when $\rho \to 0_+$, exist. We note

$$\overline{\beta}(t) := \lim_{\rho \to 0_+} \overline{\beta}_\rho(t), \quad \forall t \in \mathbb{R}$$

and

$$\overline{\overline{\beta}}(t) := \lim_{\rho \to 0_+} \overline{\overline{\beta}}_\rho(t), \quad \forall t \in \mathbb{R}.$$

At this stage, we define $\widetilde{\beta}$ by

$$\widetilde{\beta}(t) := [\overline{\beta}(t), \overline{\overline{\beta}}(t)], \quad \forall t \in \mathbb{R}.$$

In general, $\widetilde{\beta}$ so defined is not necessarily monotone.
Let us turn to (5.2). We have always $u_N \leq a$; while the case $u_N > a$ is impossible. Thus, for $u_N = a$, the relation my become infinite. (5.2) can be written as

$$-S_N \in \widetilde{\beta}(u_N) + N_C(u_N). \tag{5.3}$$

Here $C =]-\infty, a]$ and N_C is the normal cone of C. Moreover, Chang stated in [4] that, if $\beta(t_{\pm 0})$ exist for all $t \in \mathbb{R}$, then we can determine a locally Lipschitz function J by

$$J(t) = \int_0^t \beta(s)ds, \quad \forall t \in \mathbb{R}$$

so that

$$\partial J(t) = \widetilde{\beta}(t), \quad \forall t \in \mathbb{R}.$$

Here ∂ stands for the generalized gradient of Clarke (see [7]). Clearly (5.3) becomes

$$J^0(u_N, v_N - u_N) \geq -S_N(v_N - u_N), \quad \forall v_N \in C.$$

This is a simple hemivariational inequality. Panagiotopoulos called it so to point out its difference to the classical variational inequalities.

This example was summarized from [10], which is a comprehensive reference for the interested reader in the theory and applications of hemivariational inequalities.
We shall now turn our attention to the mathematical concepts of the theory by considering a general form.

5.2. **Problem Formulation.** Let X be a reflexive Banach space and C be a nonempty convex closed subset of X. Let $J : C \to I\!R$ be a locally Lipschitz function. Let also A be an operator from C to X', φ be a real lower semicontinuous convex function on C and $l \in X'$.

We are concerned with the following hemivariational inequality :
Find $\overline{x} \in C$ such that

$$\text{(HI)} \qquad \langle A\overline{x}, y - \overline{x}\rangle + J^0(\overline{x}, y - \overline{x}) + \varphi(y) - \varphi(\overline{x}) \geq \langle l, y - \overline{x}\rangle, \quad \forall y \in C.$$

Particular cases of this inequality arise, e.g in the variational formulation of the problem of a linear elastic body subjected to two- or three-dimensional friction law and also in the theory of laminated von Kármán plates.

Remark 5.1. Due to the presence of the monotone part corresponding to φ, (HI) was called in [10] variational-hemivariational inequality. The particular case of hemivariational inequalities of [10] corresponds to (HI) when $\varphi = 0$.

5.3. **Existence Theorem.** As an application of Theorem 4.1, we get the existence of solutions to the (HI) problem.

Theorem 5.2. *Assume that*

(i) *A is pseudomonotone and locally bounded on finite dimensional subspaces;*
(ii) *either $J \in PM(C)$, or $J \in QPM(C)$ and A satisfies the $(S)_+$ condition;*
(iii) *there exists $x_0 \in C$ such that A is x_0-coercive and*

$$J^0(y, x_0 - y) \leq k(1 + \|y\|) \text{ for all } y \in C,\ k = const. \tag{5.4}$$

Then the hemivariational inequality (HI) has at least one solution.

Proof. Assumption (i) implies that A is continuous from each finite dimensional subspace of X to the weak topology on X' (see [15, Proposition 27.7, (b)]). If we take X equipped with the weak topology, $f(x, y) = \langle Ax, y - x\rangle + J^0(x, y - x)$ and $g(x, y) = \varphi(y) - \varphi(x) - \langle l, y - x\rangle$, then it suffices according to Theorem 4.1 to prove two assertions: First that f is pseudomonotone, and second that assumption (8) of Theorem 4.1 is satisfied.

Let us begin with the proof of the first one. Suppose that $J \in PM(C)$ then f is pseudomonotone as a sum of two pseudomonotone mappings (see [15, Proposition 27.6, (e)]); the same proof can be used here).
Suppose on the other hand that $J \in QPM(C)$ and A has the $(S)_+$ property. Let (x_n) be a sequence in C converging weakly to $x \in C$ such that

$$\liminf[\langle Ax_n, x - x_n\rangle + J^0(x_n, x - x_n)] \geq 0. \tag{5.5}$$

It suffices to show that

$$\liminf\langle Ax_n, x - x_n\rangle \geq 0. \tag{5.6}$$

Indeed, if (5.6) holds then, by pseudomonotonicity of A, we can write

$$\limsup\langle Ax_n, y - x_n\rangle \leq \langle Ax, y - x\rangle \text{ for all } y \in C. \tag{5.7}$$

The $(S)_+$ condition of A implies that $x_n \to x$ in C. Therefore

$$\limsup J^0(x_n, y - x_n) \leq J^0(x, y - x) \text{ for all } y \in C \tag{5.8}$$

since J^0 is upper semicontinuous. Hence, combining (5.7) with (5.8), it follows

$$\limsup[\langle Ax_n, y - x_n\rangle + J^0(x_n, y - x_n)] \leq \langle Ax, y - x\rangle + J^0(x, y - x) \text{ for all } y \in C.$$

Now, let us show that (5.6) holds. Suppose on the contrary that there exist $r < 0$ and a subsequence of (x_n), which we note also (x_n), such that $\lim\langle Ax_n, x - x_n\rangle = r$. Hence, due to (5.5) we can write

$$\liminf J^0(x_n, x - x_n) \geq -r > 0. \tag{5.9}$$

Since $J \in \mathit{QPM}(C)$, it follows

$$\lim J^0(x_n, x - x_n) = 0,$$

which contradicts (5.9).
To show the second result, it suffices, since $g(x, .)$ is weakly lower semicontinuous for every $x \in C$ and following a remark of Blum and Oettli ([2, p. 131]), to prove that

$$(\langle Ay, x_0 - y\rangle + J^0(y, x_0 - y))/\|y - x_0\| \longrightarrow -\infty \text{ as } \|y - x_0\| \to +\infty.$$

This is ensured by assumption (iii). □

Remark 5.3.

(1) Estimation (5.4) is given in [10] with another form more relaxed. It can be omitted when the multivalued operator $A + \partial J$ is x_0-coercive.
(2) Observe that we have got here a solution of the variational-hemivariational inequalities problem without recourse to a condition of quasi or strong quasi boundedness on A or $\partial\varphi$ as it was made in [10].
(3) It is also interesting to note that we cannot make use of Theorem 1 in [2] to solve (HI) with the same conditions since the function $J^0(, , y)$ is not necessarily weakly upper semicontinuous which is the assumption of [2].

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