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SOME INEQUALITIES FOR THE DISPERSION OF A RANDOM VARIABLE WHOSE PDF IS DEFINED ON A FINITE INTERVAL

NEIL S. BARNETT, PIETRO CERONE, SEVER S. DRAGOMIR AND JOHN ROUMELIOTIS

School of Communications and Informatics
Victoria University of Technology
PO Box 14428, Melbourne City MC
8001 Victoria, Australia

EMail: neil@matilda.vu.edu.au

EMail: pc@matilda.vu.edu.au

EMail: sever@matilda.vu.edu.au

EMail: johnr@matilda.vu.edu.au

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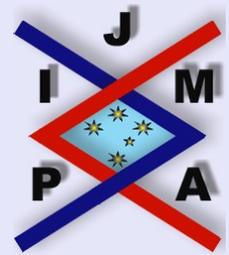


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Abstract

Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval and applications are given.

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Key words: Random variable, Expectation, Variance, Dispersion, Grüss Inequality, Chebychev's Inequality, Lupaş Inequality.

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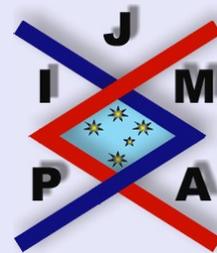
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1. Introduction

In this note we obtain some inequalities for the dispersion of a continuous random variable X having the probability density function (p.d.f.) f defined on a finite interval $[a, b]$.

Tools used include: Korkine's identity, which plays a central role in the proof of Chebychev's integral inequality for synchronous mappings [24], Hölder's weighted inequality for double integrals and an integral identity connecting the variance $\sigma^2(X)$ and the expectation $E(X)$. Perturbed results are also obtained by using Grüss, Chebyshev and Lupaş inequalities. In Section 4, results from an identity involving a double integral are obtained for a variety of norms.



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2. Some Inequalities for Dispersion

Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$ be the p.d.f. of the random variable X and

$$E(X) := \int_a^b t f(t) dt$$

its *expectation* and

$$\sigma(X) = \left[\int_a^b (t - E(X))^2 f(t) dt \right]^{\frac{1}{2}} = \left[\int_a^b t^2 f(t) dt - [E(X)]^2 \right]^{\frac{1}{2}}$$

its *dispersion* or *standard deviation*.

The following theorem holds.

Theorem 2.1. *With the above assumptions, we have*

$$(2.1) \quad 0 \leq \sigma(X) \leq \begin{cases} \frac{\sqrt{3}(b-a)^2}{6} \|f\|_{\infty}, & \text{provided } f \in L_{\infty}, [a, b]; \\ \frac{\sqrt{2}(b-a)^{1+\frac{1}{q}}}{2[(q+1)(2q+1)]^{\frac{1}{q}}} \|f\|_p, & \text{provided } f \in L_p[a, b] \\ & \text{and } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\sqrt{2}(b-a)}{2}. \end{cases}$$

Proof. Korkine's identity [24], is

$$(2.2) \quad \int_a^b p(t) dt \int_a^b p(t) g(t) h(t) dt - \int_a^b p(t) g(t) dt \cdot \int_a^b p(t) h(t) dt \\ = \frac{1}{2} \int_a^b \int_a^b p(t) p(s) (g(t) - g(s)) (h(t) - h(s)) dt ds,$$



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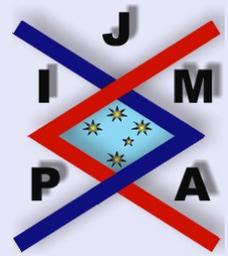


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which holds for the measurable mappings $p, g, h : [a, b] \rightarrow \mathbb{R}$ for which the integrals involved in (2.2) exist and are finite. Choose in (2.2) $p(t) = f(t)$, $g(t) = h(t) = t - E(X)$, $t \in [a, b]$ to get

$$(2.3) \quad \sigma^2(X) = \frac{1}{2} \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds.$$

It is obvious that

$$(2.4) \quad \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds \leq \sup_{(t,s) \in [a,b]^2} |f(t) f(s)| \int_a^b \int_a^b (t-s)^2 dt ds = \frac{(b-a)^4}{6} \|f\|_\infty^2$$

and then, by (2.3), we obtain the first part of (2.1).

For the second part, we apply Hölder's integral inequality for double integrals to obtain

$$\begin{aligned} \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds &\leq \left(\int_a^b \int_a^b f^p(t) f^p(s) dt ds \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b (t-s)^{2q} dt ds \right)^{\frac{1}{q}} \\ &= \|f\|_p^2 \left[\frac{(b-a)^{2q+2}}{(q+1)(2q+1)} \right]^{\frac{1}{q}}, \end{aligned}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the second inequality in (2.1) is proved.

For the last part, observe that

$$\begin{aligned} \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds &\leq \sup_{(t,s) \in [a,b]^2} (t-s)^2 \int_a^b \int_a^b f(t) f(s) dt ds \\ &= (b-a)^2 \end{aligned}$$

as

$$\int_a^b \int_a^b f(t) f(s) dt ds = \int_a^b f(t) dt \int_a^b f(s) ds = 1.$$

□

Using a finer argument, the last inequality in (2.1) can be improved as follows.

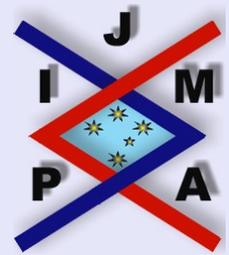
Theorem 2.2. *Under the above assumptions, we have*

$$(2.5) \quad 0 \leq \sigma(X) \leq \frac{1}{2}(b-a).$$

Proof. We use the following Grüss type inequality:

$$(2.6) \quad 0 \leq \frac{\int_a^b p(t) g^2(t) dt}{\int_a^b p(t) dt} - \left(\frac{\int_a^b p(t) g(t) dt}{\int_a^b p(t) dt} \right)^2 \leq \frac{1}{4}(M-m)^2,$$

provided that p, g are measurable on $[a, b]$ and all the integrals in (2.6) exist and are finite, $\int_a^b p(t) dt > 0$ and $m \leq g \leq M$ a.e. on $[a, b]$. For a proof of this inequality see [19].



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Choose in (2.6), $p(t) = f(t)$, $g(t) = t - E(X)$, $t \in [a, b]$. Observe that in this case $m = a - E(X)$, $M = b - E(X)$ and then, by (2.6) we deduce (2.5). \square

Remark 2.1. The same conclusion can be obtained for the choice $p(t) = f(t)$ and $g(t) = t$, $t \in [a, b]$.

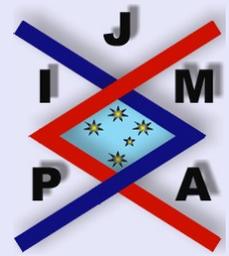
The following result holds.

Theorem 2.3. *Let X be a random variable having the p.d.f. given by $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$. Then for any $x \in [a, b]$ we have the inequality:*

$$(2.7) \quad \sigma^2(X) + (x - E(X))^2 \leq \begin{cases} (b-a) \left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 \right] \|f\|_\infty, & \text{provided } f \in L_\infty[a, b]; \\ \left[\frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} \right]^{\frac{1}{q}} \|f\|_p, & \text{provided } f \in L_p[a, b], p > 1, \\ & \text{and } \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right| \right)^2. \end{cases}$$

Proof. We observe that

$$(2.8) \quad \int_a^b (x-t)^2 f(t) dt = \int_a^b (x^2 - 2xt + t^2) f(t) dt \\ = x^2 - 2xE(X) + \int_a^b t^2 f(t) dt$$



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and as

$$(2.9) \quad \sigma^2(X) = \int_a^b t^2 f(t) dt - [E(X)]^2,$$

we get, by (2.8) and (2.9),

$$(2.10) \quad [x - E(X)]^2 + \sigma^2(X) = \int_a^b (x - t)^2 f(t) dt,$$

which is of interest in itself too.

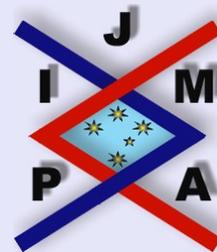
We observe that

$$\begin{aligned} \int_a^b (x - t)^2 f(t) dt &\leq \operatorname{ess\,sup}_{t \in [a,b]} |f(t)| \int_a^b (x - t)^2 dt \\ &= \|f\|_\infty \frac{(b - x)^3 + (x - a)^3}{3} \\ &= (b - a) \|f\|_\infty \left[\frac{(b - a)^2}{12} + \left(x - \frac{a + b}{2}\right)^2 \right] \end{aligned}$$

and the first inequality in (2.7) is proved.

For the second inequality, observe that by Hölder's integral inequality,

$$\begin{aligned} \int_a^b (x - t)^2 f(t) dt &\leq \left(\int_a^b f^p(t) dt \right)^{\frac{1}{p}} \left(\int_a^b (x - t)^{2q} dt \right)^{\frac{1}{q}} \\ &= \|f\|_p \left[\frac{(b - x)^{2q+1} + (x - a)^{2q+1}}{2q + 1} \right]^{\frac{1}{q}}, \end{aligned}$$



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and the second inequality in (2.7) is established.

Finally, observe that,

$$\begin{aligned} \int_a^b (x-t)^2 f(t) dt &\leq \sup_{t \in [a,b]} (x-t)^2 \int_a^b f(t) dt \\ &= \max \{ (x-a)^2, (b-x)^2 \} \\ &= (\max \{ x-a, b-x \})^2 \\ &= \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^2, \end{aligned}$$

and the theorem is proved. □

The following corollaries are easily deduced.

Corollary 2.4. *With the above assumptions, we have*

$$(2.11) \quad 0 \leq \sigma(X) \leq \begin{cases} (b-a)^{\frac{1}{2}} \left[\frac{(b-a)^2}{12} + (E(X) - \frac{a+b}{2})^2 \right]^{\frac{1}{2}} \|f\|_{\infty}^{\frac{1}{2}}, \\ \quad \text{provided } f \in L_{\infty}[a, b]; \\ \\ \left[\frac{(b-E(X))^{2q+1} + (E(X)-a)^{2q+1}}{2q+1} \right]^{\frac{1}{2q}} \|f\|_p^{\frac{1}{2}}, \\ \quad \text{if } f \in L_p[a, b], p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{b-a}{2} + \left| E(X) - \frac{a+b}{2} \right|. \end{cases}$$



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Remark 2.2. The last inequality in (2.12) is worse than the inequality (2.5), obtained by a technique based on Grüss' inequality.

The best inequality we can get from (2.7) is that one for which $x = \frac{a+b}{2}$, and this applies for all the bounds since

$$\min_{x \in [a,b]} \left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2} \right)^2 \right] = \frac{(b-a)^2}{12},$$

$$\min_{x \in [a,b]} \frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} = \frac{(b-a)^{2q+1}}{2^{2q}(2q+1)},$$

and

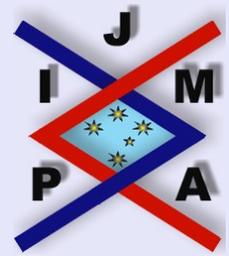
$$\min_{x \in [a,b]} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] = \frac{b-a}{2}.$$

Consequently, we can state the following corollary as well.

Corollary 2.5. *With the above assumptions, we have the inequality:*

$$(2.12) \quad 0 \leq \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2$$

$$\leq \begin{cases} \frac{(b-a)^3}{12} \|f\|_\infty, & \text{provided } f \in L_\infty[a, b]; \\ \frac{(b-a)^{2q+1}}{4(2q+1)^{\frac{1}{q}}} \|f\|_p, & \text{if } f \in L_p[a, b], p > 1, \\ \frac{(b-a)^2}{4}. & \text{and } \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$



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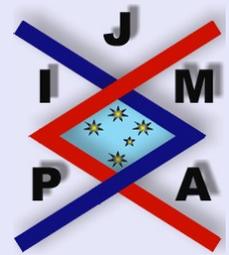
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Remark 2.3. From the last inequality in (2.12), we obtain

$$(2.13) \quad 0 \leq \sigma^2(X) \leq (b - E(X))(E(X) - a) \leq \frac{1}{4}(b - a)^2,$$

which is an improvement on (2.5).



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3. Perturbed Results Using Grüss Type inequalities

In 1935, G. Grüss (see for example [26]) proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of the integrals.

Theorem 3.1. *Let $h, g : [a, b] \rightarrow \mathbb{R}$ be two integrable mappings such that $\phi \leq h(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$, where $\phi, \Phi, \gamma, \Gamma$ are real numbers. Then,*

$$(3.1) \quad |T(h, g)| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma),$$

where

$$(3.2) \quad T(h, g) = \frac{1}{b-a} \int_a^b h(x) g(x) dx - \frac{1}{b-a} \int_a^b h(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this as well as for extensions, generalisations, discrete variants and other associated material, see [25], and [1]-[21] where further references are given.

A ‘premature’ Grüss inequality is embodied in the following theorem which was proved in [23]. It provides a sharper bound than the above Grüss inequality.



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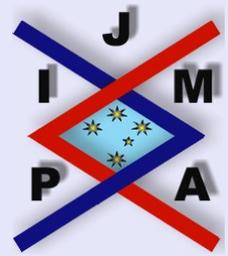


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Theorem 3.2. Let h, g be integrable functions defined on $[a, b]$ and let $d \leq g(t) \leq D$. Then

$$(3.3) \quad |T(h, g)| \leq \frac{D - d}{2} |T(h, h)|^{\frac{1}{2}},$$

where $T(h, g)$ is as defined in (3.2).

Theorem 3.2 will now be used to provide a perturbed rule involving the variance and mean of a p.d.f.

3.1. Perturbed Results Using ‘Premature’ Inequalities

In this subsection we develop some perturbed results.

Theorem 3.3. Let X be a random variable having the p.d.f. given by $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$. Then for any $x \in [a, b]$ and $m \leq f(x) \leq M$ we have the inequality

$$(3.4) \quad |P_V(x)| := \left| \sigma^2(X) + (x - E(X))^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \leq \frac{M-m}{2} \cdot \frac{(b-a)^2}{\sqrt{45}} \left[\left(\frac{b-a}{2}\right)^2 + 15 \left(x - \frac{a+b}{2}\right) \right]^{\frac{1}{2}} \leq (M-m) \frac{(b-a)^3}{\sqrt{45}}.$$

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Proof. Applying the ‘premature’ Grüss result (3.3) by associating $g(t)$ with $f(t)$ and $h(t) = (x - t)^2$, gives, from (3.1)-(3.3)

$$(3.5) \quad \left| \int_a^b (x - t)^2 f(t) dt - \frac{1}{b - a} \int_a^b (x - t)^2 dt \cdot \int_a^b f(t) dt \right| \leq (b - a) \frac{M - m}{2} [T(h, h)]^{\frac{1}{2}},$$

where from (3.2)

$$(3.6) \quad T(h, h) = \frac{1}{b - a} \int_a^b (x - t)^4 dt - \left[\frac{1}{b - a} \int_a^b (x - t)^2 dt \right]^2.$$

Now,

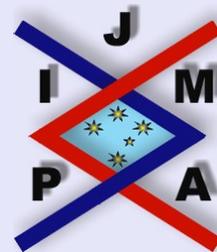
$$(3.7) \quad \begin{aligned} \frac{1}{b - a} \int_a^b (x - t)^2 dt &= \frac{(x - a)^3 + (b - x)^3}{3(b - a)} \\ &= \frac{1}{3} \left(\frac{b - a}{2} \right)^2 + \left(x - \frac{a + b}{2} \right)^2 \end{aligned}$$

and

$$\frac{1}{b - a} \int_a^b (x - t)^4 dt = \frac{(x - a)^5 + (b - x)^5}{5(b - a)}$$

giving, for (3.6),

$$(3.8) \quad 45T(h, h) = 9 \left[\frac{(x - a)^5 + (b - x)^5}{b - a} \right] - 5 \left[\frac{(x - a)^3 + (b - x)^3}{b - a} \right]^2.$$



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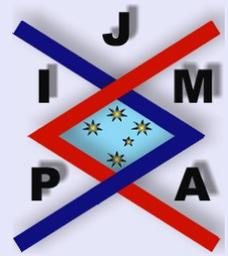


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Let $A = x - a$ and $B = b - x$ in (3.8) to give

$$\begin{aligned}
 45T(h, h) &= 9 \left(\frac{A^5 + B^5}{A + B} \right) - 5 \left(\frac{A^3 + B^3}{A + B} \right)^2 \\
 &= 9 [A^4 - A^3B + A^2B^2 - AB^3 + B^4] - 5 [A^2 - AB + B^2]^2 \\
 &= (4A^2 - 7AB + 4B^2) (A + B)^2 \\
 &= \left[\left(\frac{A + B}{2} \right)^2 + 15 \left(\frac{A - B}{2} \right)^2 \right] (A + B)^2.
 \end{aligned}$$

Using the facts that $A + B = b - a$ and $A - B = 2x - (a + b)$ gives

$$(3.9) \quad T(h, h) = \frac{(b - a)^2}{45} \left[\left(\frac{b - a}{2} \right)^2 + 15 \left(x - \frac{a + b}{2} \right)^2 \right]$$

and from (3.7)

$$\begin{aligned}
 \frac{1}{b - a} \int_a^b (x - t)^2 dt &= \frac{A^3 + B^3}{3(A + B)} \\
 &= \frac{1}{3} [A^2 - AB + B^2] \\
 &= \frac{1}{3} \left[\left(\frac{A + B}{2} \right)^2 + 3 \left(\frac{A - B}{2} \right)^2 \right],
 \end{aligned}$$

giving

$$(3.10) \quad \frac{1}{b - a} \int_a^b (x - t)^2 dt = \frac{(b - a)^2}{12} + \left(x - \frac{a + b}{2} \right)^2.$$

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Hence, from (3.5), (3.9) (3.10) and (2.10), the first inequality in (3.4) results. The coarsest uniform bound is obtained by taking x at either end point. Thus the theorem is completely proved. \square

Remark 3.1. The best inequality obtainable from (3.4) is at $x = \frac{a+b}{2}$ giving

$$(3.11) \quad \left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{M-m}{12} \frac{(b-a)^3}{\sqrt{5}}.$$

The result (3.11) is a tighter bound than that obtained in the first inequality of (2.12) since $0 < M - m < 2 \|f\|_\infty$.

For a symmetric p.d.f. $E(X) = \frac{a+b}{2}$ and so the above results would give bounds on the variance.

The following results hold if the p.d.f $f(x)$ is differentiable, that is, for $f(x)$ absolutely continuous.

Theorem 3.4. *Let the conditions on Theorem 3.1 be satisfied. Further, suppose that f is differentiable and is such that*

$$\|f'\|_\infty := \sup_{t \in [a,b]} |f'(t)| < \infty.$$

Then

$$(3.12) \quad |P_V(x)| \leq \frac{b-a}{\sqrt{12}} \|f'\|_\infty \cdot I(x),$$

where $P_V(x)$ is given by the left hand side of (3.4) and,

$$(3.13) \quad I(x) = \frac{(b-a)^2}{\sqrt{45}} \left[\left(\frac{b-a}{2} \right)^2 + 15 \left(x - \frac{a+b}{2} \right)^2 \right]^{\frac{1}{2}}.$$



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Proof. Let $h, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous and h', g' be bounded. Then Chebychev's inequality holds (see [23])

$$|T(h, g)| \leq \frac{(b-a)^2}{12} \sup_{t \in [a, b]} |h'(t)| \cdot \sup_{t \in [a, b]} |g'(t)|.$$

Matić, Pečarić and Ujević [23] using a 'premature' Grüss type argument proved that

$$(3.14) \quad |T(h, g)| \leq \frac{(b-a)}{\sqrt{12}} \sup_{t \in [a, b]} |g'(t)| \sqrt{T(h, h)}.$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x - \cdot)^2$ with $h(\cdot)$ in (3.13) gives, from (3.5) and (3.9), $I(x) = (b-a)[T(h, h)]^{\frac{1}{2}}$, which simplifies to (3.13) and the theorem is proved. \square

Theorem 3.5. *Let the conditions of Theorem 3.3 be satisfied. Further, suppose that f is locally absolutely continuous on (a, b) and let $f' \in L_2(a, b)$. Then*

$$(3.15) \quad |P_V(x)| \leq \frac{b-a}{\pi} \|f'\|_2 \cdot I(x),$$

where $P_V(x)$ is the left hand side of (3.4) and $I(x)$ is as given in (3.13).

Proof. The following result was obtained by Lupaş (see [23]). For $h, g : (a, b) \rightarrow \mathbb{R}$ locally absolutely continuous on (a, b) and $h', g' \in L_2(a, b)$, then

$$|T(h, g)| \leq \frac{(b-a)^2}{\pi^2} \|h'\|_2^\dagger \|g'\|_2^\dagger,$$



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where

$$\|k\|_2^\dagger := \left(\frac{1}{b-a} \int_a^b |k(t)|^2 \right)^{\frac{1}{2}} \quad \text{for } k \in L_2(a, b).$$

Matić, Pečarić and Ujević [23] further show that

$$(3.16) \quad |T(h, g)| \leq \frac{b-a}{\pi} \|g'\|_2^\dagger \sqrt{T(h, h)}.$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x - \cdot)^2$ with h in (3.16) gives (3.15), where $I(x)$ is as found in (3.13), since from (3.5) and (3.9), $I(x) = (b-a)[T(h, h)]^{\frac{1}{2}}$. \square

3.2. Alternate Grüss Type Results for Inequalities Involving the Variance

Let

$$(3.17) \quad S(h(x)) = h(x) - \mathcal{M}(h)$$

where

$$(3.18) \quad \mathcal{M}(h) = \frac{1}{b-a} \int_a^b h(u) du.$$

Then from (3.2),

$$(3.19) \quad T(h, g) = \mathcal{M}(hg) - \mathcal{M}(h)\mathcal{M}(g).$$



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Dragomir and McAndrew [19] have shown, that

$$(3.20) \quad T(h, g) = T(S(h), S(g))$$

and proceeded to obtain bounds for a trapezoidal rule. Identity (3.20) is now applied to obtain bounds for the variance.

Theorem 3.6. *Let X be a random variable having the p.d.f. $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$. Then for any $x \in [a, b]$ the following inequality holds, namely,*

$$(3.21) \quad |P_V(x)| \leq \frac{8}{3} \nu^3(x) \left\| f(\cdot) - \frac{1}{b-a} \right\|_\infty \quad \text{if } f \in L_\infty[a, b],$$

where $P_V(x)$ is as defined by the left hand side of (3.4), and $\nu = \nu(x) = \frac{1}{3} \left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2$.

Proof. Using identity (3.20), associate with $h(\cdot)$, $(x - \cdot)^2$ and $f(\cdot)$ with $g(\cdot)$. Then

$$(3.22) \quad \int_a^b (x-t)^2 f(t) dt - \mathcal{M}((x-\cdot)^2) \\ = \int_a^b \left[(x-t)^2 - \mathcal{M}((x-\cdot)^2) \right] \left[f(t) - \frac{1}{b-a} \right] dt,$$

where from (3.18),

$$\mathcal{M}((x-\cdot)^2) = \frac{1}{b-a} \int_a^b (x-t)^2 dt = \frac{1}{3(b-a)} [(x-a)^3 + (b-x)^3]$$



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and so

$$(3.23) \quad 3\mathcal{M}((x - \cdot)^2) = \left(\frac{b-a}{2}\right)^2 + 3\left(x - \frac{a+b}{2}\right)^2.$$

Further, from (3.17),

$$S((x - \cdot)^2) = (x - t)^2 - \mathcal{M}((x - \cdot)^2)$$

and so, on using (3.23)

$$(3.24) \quad S((x - \cdot)^2) = (x - t)^2 - \frac{1}{3}\left(\frac{b-a}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2.$$

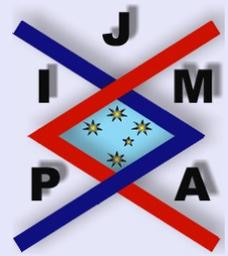
Now, from (3.22) and using (2.10), (3.23) and (3.24), the following identity is obtained

$$(3.25) \quad \sigma^2(X) + [x - E(X)]^2 - \frac{1}{3}\left[\left(\frac{b-a}{2}\right)^2 + 3\left(x - \frac{a+b}{2}\right)^2\right] \\ = \int_a^b S((x - t)^2) \left(f(t) - \frac{1}{b-a}\right) dt,$$

where $S(\cdot)$ is as given by (3.24). Taking the modulus of (3.25) gives

$$(3.26) \quad |P_V(x)| = \left| \int_a^b S((x - t)^2) \left(f(t) - \frac{1}{b-a}\right) dt \right|.$$

Observe that under different assumptions with regard to the norms of the p.d.f. $f(x)$ we may obtain a variety of bounds.



For $f \in L_\infty [a, b]$ then

$$(3.27) \quad |P_V(x)| \leq \left\| f(\cdot) - \frac{1}{b-a} \right\|_\infty \int_a^b |S((x-t)^2)| dt.$$

Now, let

$$(3.28) \quad S((x-t)^2) = (t-x)^2 - \nu^2 = (t-X_-)(t-X_+),$$

where

$$(3.29) \quad \begin{aligned} \nu^2 &= \mathcal{M}((x-\cdot)^2) \\ &= \frac{(x-a)^3 + (b-x)^3}{3(b-a)} \\ &= \frac{1}{3} \left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2, \end{aligned}$$

and

$$(3.30) \quad X_- = x - \nu, \quad X_+ = x + \nu.$$

Then,

$$(3.31) \quad \begin{aligned} H(t) &= \int S((x-t)^2) dt \\ &= \int [(t-x)^2 - \nu^2] dt \\ &= \frac{(t-x)^3}{3} - \nu^2 t + k \end{aligned}$$

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and so from (3.31) and using (3.28) - (3.29) gives,

$$\begin{aligned}
 (3.32) \quad & \int_a^b |S((x-t)^2)| dt \\
 &= H(X_-) - H(a) - [H(X_+) - H(X_-)] + [H(b) - H(X_+)] \\
 &= 2[H(X_-) - H(X_+)] + H(b) - H(a) \\
 &= 2 \left\{ -\frac{\nu^3}{3} - \nu^2 X_- - \frac{\nu^3}{3} + \nu^2 X_+ \right\} \\
 &\quad + \frac{(b-x)^3}{3} - \nu^2 b + \frac{(x-a)^3}{3} + \nu^2 a \\
 &= 2 \left[2\nu^3 - \frac{2}{3}\nu^3 \right] + \frac{(b-x)^3 + (x-a)^3}{3} - \nu^2(b-a) \\
 &= \frac{8}{3}\nu^3.
 \end{aligned}$$

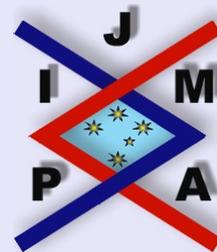
Thus, substituting into (3.27), (3.26) and using (3.29) readily produces the result (3.21) and the theorem is proved. \square

Remark 3.2. Other bounds may be obtained for $f \in L_p[a, b]$, $p \geq 1$ however obtaining explicit expressions for these bounds is somewhat intricate and will not be considered further here. They involve the calculation of

$$\sup_{t \in [a, b]} |(t-x)^2 - \nu^2| = \max \{ |(x-a)^2 - \nu^2|, \nu^2, |(b-x)^2 - \nu^2| \}$$

for $f \in L_1[a, b]$ and

$$\left(\int_a^b |(t-x)^2 - \nu^2|^q dt \right)^{\frac{1}{q}}$$



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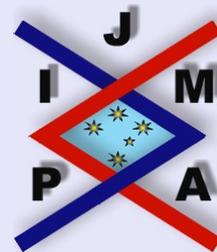
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for $f \in L_p [a, b]$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, where ν^2 is given by (3.29).



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4. Some Inequalities for Absolutely Continuous P.D.F's

We start with the following lemma which is interesting in itself.

Lemma 4.1. *Let X be a random variable whose probability density function $f : [a, b] \rightarrow \mathbb{R}_+$ is absolutely continuous on $[a, b]$. Then we have the identity*

$$(4.1) \quad \sigma^2(X) + [E(X) - x]^2 = \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 + \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt,$$

where the kernel $p : [a, b]^2 \rightarrow \mathbb{R}$ is given by

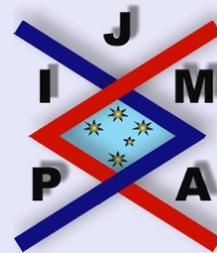
$$p(t, s) := \begin{cases} s - a, & \text{if } a \leq s \leq t \leq b, \\ s - b, & \text{if } a \leq t < s \leq b, \end{cases}$$

for all $x \in [a, b]$.

Proof. We use the identity (see (2.10))

$$(4.2) \quad \sigma^2(X) + [E(X) - x]^2 = \int_a^b (x-t)^2 f(t) dt$$

for all $x \in [a, b]$.



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On the other hand, we know that (see for example [22] for a simple proof using integration by parts)

$$(4.3) \quad f(t) = \frac{1}{b-a} \int_a^b f(s) ds + \frac{1}{b-a} \int_a^b p(t,s) f'(s) ds$$

for all $t \in [a, b]$.

Substituting (4.3) in (4.2) we obtain

$$(4.4) \quad \begin{aligned} \sigma^2(X) + [E(X) - x]^2 &= \int_a^b (t-x)^2 \left[\frac{1}{b-a} \int_a^b f(s) ds + \frac{1}{b-a} \int_a^b p(t,s) f'(s) ds \right] dt \\ &= \frac{1}{b-a} \cdot \frac{1}{3} [(x-a)^3 + (b-x)^3] \\ &\quad + \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt. \end{aligned}$$

Taking into account the fact that

$$\frac{1}{3} [(x-a)^3 + (b-x)^3] = \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2, \quad x \in [a, b],$$

then, by (4.4) we deduce the desired result (4.1). \square

The following inequality for P.D.F.s which are absolutely continuous and have the derivatives essentially bounded holds.



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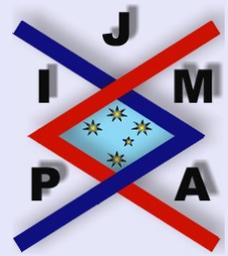
Theorem 4.2. *If $f : [a, b] \rightarrow \mathbb{R}_+$ is absolutely continuous on $[a, b]$ and $f' \in L_\infty [a, b]$, i.e., $\|f'\|_\infty := \text{ess sup}_{t \in [a, b]} |f'(t)| < \infty$, then we have the inequality:*

$$(4.5) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \leq \frac{(b-a)^2}{3} \left[\frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2}\right)^2 \right] \|f'\|_\infty$$

for all $x \in [a, b]$.

Proof. Using Lemma 4.1, we have

$$\begin{aligned} & \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ &= \frac{1}{b-a} \left| \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt \right| \\ &\leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| ds dt \\ &\leq \frac{\|f'\|_\infty}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| ds dt. \end{aligned}$$



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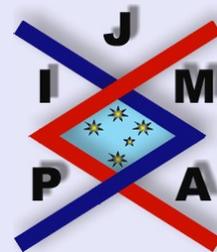
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We have

$$\begin{aligned}
 I &:= \int_a^b \int_a^b (t-x)^2 |p(t,s)| ds dt \\
 &= \int_a^b (t-x)^2 \left[\int_a^t (s-a) ds + \int_t^b (b-s) ds \right] dt \\
 &= \int_a^b (t-x)^2 \left[\frac{(t-a)^2 + (b-t)^2}{2} \right] dt \\
 &= \frac{1}{2} \left[\int_a^b (t-x)^2 (t-a)^2 dt + \int_a^b (t-x)^2 (b-t)^2 dt \right] \\
 &= \frac{I_a + I_b}{2}.
 \end{aligned}$$

Let $A = x - a$, $B = b - x$ then

$$\begin{aligned}
 I_a &= \int_a^b (t-x)^2 (t-a)^2 dt \\
 &= \int_0^{b-a} (u^2 - 2Au + A^2) u^2 du \\
 &= \frac{(b-a)^3}{3} \left[A^2 - \frac{3}{2}A(b-a) + \frac{3}{5}(b-a)^2 \right]
 \end{aligned}$$



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and

$$\begin{aligned}
 I_b &= \int_a^b (t-x)^2 (b-t)^2 dt \\
 &= \int_0^{b-a} (u^2 - 2Bu + B^2) u^2 du \\
 &= \frac{(b-a)^3}{3} \left[B^2 - \frac{3}{2}B(b-a) + \frac{3}{5}(b-a)^2 \right]
 \end{aligned}$$

Now,

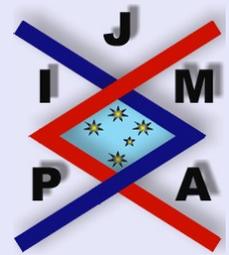
$$\begin{aligned}
 \frac{I_a + I_b}{2} &= \frac{(b-a)^3}{3} \left[\frac{A^2 + B^2}{2} - \frac{3}{4}(A+B)(b-a) + \frac{3}{5}(b-a)^2 \right] \\
 &= \frac{(b-a)^3}{3} \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 - 3 \frac{(b-a)^2}{20} \right] \\
 &= \frac{(b-a)^3}{3} \left[\frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2} \right)^2 \right]
 \end{aligned}$$

and the theorem is proved. \square

The best inequality we can get from (4.5) is embodied in the following corollary.

Corollary 4.3. *If f is as in Theorem 4.2, then we have*

$$(4.6) \quad \left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{(b-a)^4}{30} \|f'\|_\infty.$$



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We now analyze the case where f' is a Lebesgue p -integrable mapping with $p \in (1, \infty)$.

Remark 4.1. The results of Theorem 4.2 may be compared with those of Theorem 3.4. It may be shown that both bounds are convex and symmetric about $x = \frac{a+b}{2}$. Further, the bound given by the ‘premature’ Chebychev approach, namely from (3.12)-(3.13) is tighter than that obtained by the current approach (4.5) which may be shown from the following. Let these bounds be described by B_p and B_c so that, neglecting the common terms

$$B_p = \frac{b-a}{2\sqrt{15}} \left[\left(\frac{b-a}{2} \right)^2 + 15Y \right]^{\frac{1}{2}}$$

and

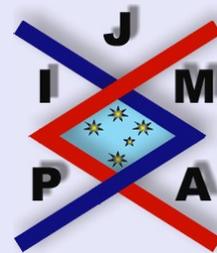
$$B_c = \frac{(b-a)^2}{100} + Y,$$

where

$$Y = \left(x - \frac{a+b}{2} \right)^2.$$

It may be shown through some straightforward algebra that $B_c^2 - B_p^2 > 0$ for all $x \in [a, b]$ so that $B_c > B_p$.

The current development does however have the advantage that the identity (4.1) is satisfied, thus allowing bounds for $L_p[a, b]$, $p \geq 1$ rather than the infinity norm.



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Theorem 4.4. If $f : [a, b] \rightarrow \mathbb{R}_+$ is absolutely continuous on $[a, b]$ and $f' \in L_p$, i.e.,

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} < \infty, \quad p \in (1, \infty)$$

then we have the inequality

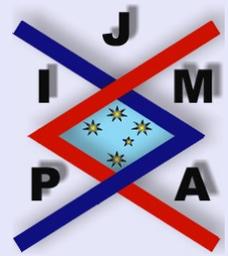
$$(4.7) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{\|f'\|_p}{(b-a)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}} \left[(x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2\right) \right. \\ \left. + (b-x)^{3q+2} \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right) \right]$$

for all $x \in [a, b]$, when $\frac{1}{p} + \frac{1}{q} = 1$ and $\tilde{B}(\cdot, \cdot, \cdot)$ is the quasi incomplete Euler's Beta mapping:

$$\tilde{B}(z; \alpha, \beta) := \int_0^z (u-1)^{\alpha-1} u^{\beta-1} du, \quad \alpha, \beta > 0, \quad z \geq 1.$$

Proof. Using Lemma 4.1, we have, as in Theorem 4.2, that

$$(4.8) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| ds dt.$$



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Using Hölder's integral inequality for double integrals, we have

$$\begin{aligned}
 (4.9) \quad & \int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| ds dt \\
 & \leq \left(\int_a^b \int_a^b |f'(s)|^p ds dt \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b (t-x)^{2q} |p(t,s)|^q ds dt \right)^{\frac{1}{q}} \\
 & = (b-a)^{\frac{1}{p}} \|f'\|_p \left(\int_a^b \int_a^b (t-x)^{2q} |p(t,s)|^q ds dt \right)^{\frac{1}{q}},
 \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

We have to compute the integral

$$\begin{aligned}
 (4.10) \quad D & := \int_a^b \int_a^b (t-x)^{2q} |p(t,s)|^q ds dt \\
 & = \int_a^b (t-x)^{2q} \left[\int_a^t (s-a)^q ds + \int_t^b (b-s)^q ds \right] dt \\
 & = \int_a^b (t-x)^{2q} \left[\frac{(t-a)^{q+1} + (b-t)^{q+1}}{q+1} \right] dt \\
 & = \frac{1}{q+1} \left[\int_a^b (t-x)^{2q} (t-a)^{q+1} dt \right. \\
 & \quad \left. + \int_a^b (t-x)^{2q} (b-t)^{q+1} dt \right].
 \end{aligned}$$

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Define

$$(4.11) \quad E := \int_a^b (t-x)^{2q} (t-a)^{q+1} dt.$$

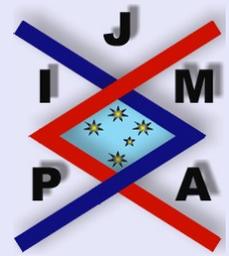
If we consider the change of variable $t = (1-u)a + ux$, we have $t = a$ implies $u = 0$ and $t = b$ implies $u = \frac{b-a}{x-a}$, $dt = (x-a) du$ and then

$$(4.12) \quad E = \int_0^{\frac{b-a}{x-a}} [(1-u)a + ux - x]^{2q} [(1-u)a + ux - a] (x-a) du$$
$$= (x-a)^{3q+2} \int_0^{\frac{b-a}{x-a}} (u-1)^{2q} u^{q+1} du$$
$$= (x-a)^{3q+2} \tilde{B} \left(\frac{b-a}{x-a}, 2q+1, q+2 \right).$$

Define

$$(4.13) \quad F := \int_a^b (t-x)^{2q} (b-t)^{q+1} dt.$$

If we consider the change of variable $t = (1-v)b + vx$, we have $t = b$ implies



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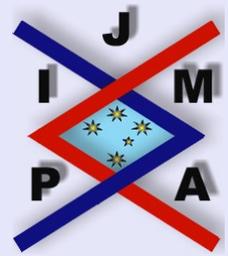


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$v = 0$, and $t = a$ implies $v = \frac{b-a}{b-x}$, $dt = (x - b) dv$ and then

$$\begin{aligned}
 (4.14) \quad F &= \int_{\frac{b-a}{b-x}}^0 [(1-v)b + vx - x]^{2q} \\
 &\quad \times [b - (1-v)b - vx]^{q+1} (x-b) dv \\
 &= (b-x)^{3q+2} \int_0^{\frac{b-a}{b-x}} (v-1)^{2q} v^{q+1} dv \\
 &= (b-x)^{3q+2} \tilde{B} \left(\frac{b-a}{b-x}, 2q+1, q+2 \right).
 \end{aligned}$$

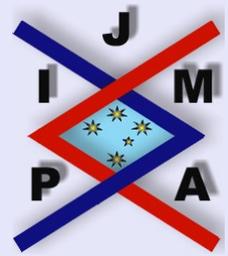
Now, using the inequalities (4.8)-(4.9) and the relations (4.10)-(4.14), since $D = \frac{1}{q+1} (E + F)$, we deduce the desired estimate (4.7). \square

The following corollary is natural to be considered.

Corollary 4.5. *Let f be as in Theorem 4.4. Then, we have the inequality:*

$$\begin{aligned}
 (4.15) \quad &\left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \\
 &\leq \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} [B(2q+1, q+1) + \Psi(2q+1, q+2)]^{\frac{1}{q}},
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and $B(\cdot, \cdot)$ is Euler's Beta mapping and $\Psi(\alpha, \beta) := \int_0^1 u^{\alpha-1} (u+1)^{\beta-1} du$, $\alpha, \beta > 0$.



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Proof. In (4.7) put $x = \frac{a+b}{2}$. The left side is clear. Now

$$\begin{aligned} \tilde{B}(2q+1, q+2) &= \int_0^2 (u-1)^{2q} u^{q+1} du \\ &= \int_0^1 (u-1)^{2q} u^{q+1} du + \int_1^2 (u-1)^{2q} u^{q+1} du \\ &= B(2q+1, q+2) + \Psi(2q+1, q+2). \end{aligned}$$

The right hand side of (4.7) is thus:

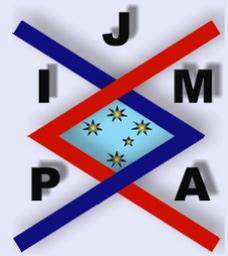
$$\begin{aligned} \frac{\|f'\|_p \left(\frac{b-a}{2}\right)^{\frac{3q+2}{q}}}{(b-a)^{\frac{1}{p}} (q+1)^{\frac{1}{q}}} [2B(2q+1, q+2) + 2\Psi(2q+1, q+2)]^{\frac{1}{q}} \\ = \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} [B(2q+1, q+2) + \Psi(2q+1, q+2)]^{\frac{1}{q}} \end{aligned}$$

and the corollary is proved. □

Finally, if f is absolutely continuous, $f' \in L_1[a, b]$ and $\|f'\|_1 = \int_a^b |f'(t)| dt$, then we can state the following theorem.

Theorem 4.6. *If the p.d.f., $f : [a, b] \rightarrow \mathbb{R}_+$ is absolutely continuous on $[a, b]$, then*

$$\begin{aligned} (4.16) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \|f'\|_1 (b-a) \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right| \right]^2 \end{aligned}$$



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for all $x \in [a, b]$.

Proof. As above, we can state that

$$\begin{aligned} & \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ & \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| ds dt \\ & \leq \sup_{(t,s) \in [a,b]^2} [(t-x)^2 |p(t,s)|] \frac{1}{b-a} \int_a^b \int_a^b |f'(s)| ds dt \\ & = \|f'\|_1 G \end{aligned}$$

where

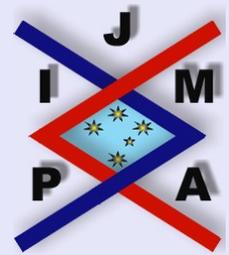
$$\begin{aligned} G & := \sup_{(t,s) \in [a,b]^2} [(t-x)^2 |p(t,s)|] \\ & \leq (b-a) \sup_{t \in [a,b]} (t-x)^2 \\ & = (b-a) [\max(x-a, b-x)]^2 \\ & = (b-a) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^2, \end{aligned}$$

and the theorem is proved. \square

It is clear that the best inequality we can get from (4.16) is the one when $x = \frac{a+b}{2}$, giving the following corollary.

Corollary 4.7. *With the assumptions of Theorem 4.6, we have:*

$$(4.17) \quad \left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{(b-a)^3}{4} \|f'\|_1.$$



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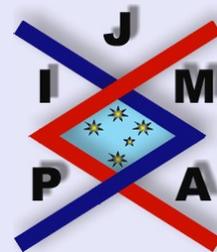
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References

- [1] P. CERONE AND S.S. DRAGOMIR, Three point quadrature rules involving, at most, a first derivative, *submitted*, *RGMIA Res. Rep. Coll.*, **2**(4) (1999), Article 8. [ONLINE] <http://rgmia.vu.edu.au/v2n4.html>.
- [2] P. CERONE AND S.S. DRAGOMIR, Trapezoidal type rules from an inequalities point of view, Accepted for publication in *Analytic-Computational Methods in Applied Mathematics*, G.A. Anastassiou (Ed), CRC Press, New York (2000), 65–134.
- [3] P. CERONE AND S.S. DRAGOMIR, Midpoint type rules from an inequalities point of view, Accepted for publication in *Analytic-Computational Methods in Applied Mathematics*, G.A. Anastassiou (Ed), CRC Press, New York (2000), 135–200.
- [4] P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS, An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications, *East Asian Math. J.*, **15**(1) (1999), 1–9. *Preprint. RGMIA Res. Rep Coll.*, **1**(1) (1998), Article 4, 1998. [ONLINE] <http://rgmia.vu.edu.au/v1n1.html>.
- [5] P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS, An inequality of Ostrowski-Grüss type for twice differentiable mappings and applications, *Kyungpook Math. J.*, **39**(2) (1999), 331–341. *Preprint. RGMIA Res. Rep Coll.*, **1**(2) (1998), Article 8, 1998. [ONLINE] <http://rgmia.vu.edu.au/v1n2.html>.



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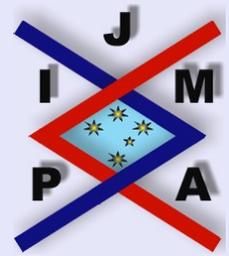


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- [6] P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS, An Ostrowski type inequality for mappings whose second derivatives belong to $L_p(a, b)$ and applications, *Preprint. RGMIA Res. Rep Coll.*, **1**(1) (1998), Article 5. [ONLINE] <http://rgmia.vu.edu.au/v1n1.html>.
- [7] P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS, On Ostrowski type for mappings whose second derivatives belong to $L_1(a, b)$ and applications, *Honam Math. J.*, **21**(1) (1999), 127–137. *Preprint. RGMIA Res. Rep Coll.*, **1**(2) (1998), Article 7. [ONLINE] <http://rgmia.vu.edu.au/v1n2.html>.
- [8] P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS, Some Ostrowski type inequalities for n -time differentiable mappings and applications, *Demonstratio Math.*, **32**(2) (1999), 697–712. *Preprint. RGMIA Res. Rep Coll.*, **1**(2) (1998), 51–66. [ONLINE] <http://rgmia.vu.edu.au/v1n2.html>
- [9] P. CERONE, S.S. DRAGOMIR, J. ROUMELIOTIS AND J. SUNDE, A new generalization of the trapezoid formula for n -time differentiable mappings and applications, *Demonstratio Math.*, **33**(4) (2000), 719–736. *RGMIA Res. Rep. Coll.*, **2**(5) (1999), Article 7. [ONLINE] <http://rgmia.vu.edu.au/v2n5.html>.
- [10] S.S. DRAGOMIR, Grüss type integral inequality for mappings of r -Hölder's type and applications for trapezoid formula, *Tamkang J. Math.*, **31**(1) (2000), 43–47.
- [11] S.S. DRAGOMIR, A Taylor like formula and application in numerical integration, *submitted*.

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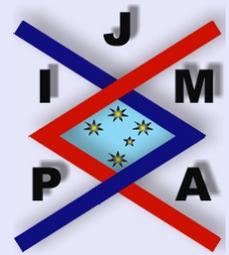
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- [12] S.S. DRAGOMIR, Grüss inequality in inner product spaces, *Austral. Math. Soc. Gaz.*, **26**(2) (1999), 66–70.
- [13] S.S. DRAGOMIR, New estimation of the remainder in Taylor’s formula using Grüss’ type inequalities and applications, *Math. Inequal. Appl.*, **2**(2) (1999), 183–194.
- [14] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.*, **31**(4) (2000), 397–415.
- [15] S.S. DRAGOMIR AND N. S. BARNETT, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications , *J. Indian Math. Soc.*, **66**(1-4) (1999), 237–245. *Preprint. RGMIA Res. Rep. Coll.*, **1**(2) (1998), Article 9. [ONLINE] <http://rgmia.vu.edu.au/v1n2.html>
- [16] S.S. DRAGOMIR, P. CERONE AND A. SOFO, Some remarks on the midpoint rule in numerical integration, *Studia Math. “Babeş-Bolyai” Univ.*, (in press).
- [17] S.S. DRAGOMIR, P. CERONE AND A. SOFO, Some remarks on the trapezoid rule in numerical integration, *Indian J. Pure Appl. Math.*, **31**(5) (2000), 475–494. *Preprint: RGMIA Res. Rep. Coll.*, **2**(5) (1999), Article 1. [ONLINE] <http://rgmia.vu.edu.au/v2n5.html>.
- [18] S.S. DRAGOMIR, Y.J. CHO AND S.S. KIM, Some remarks on the Milovanović-Pečarić Inequality and in Applications for special means and numerical integration, *Tamkang J. Math.*, **30**(3) (1999), 203–211.



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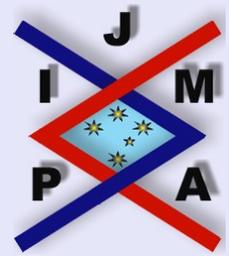
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- [19] S.S. DRAGOMIR AND A. McANDREW, On Trapezoid inequality via a Grüss type result and applications, *Tamkang J. Math.*, **31**(3) (2000), 193–201. *RGMIA Res. Rep. Coll.*, **2**(2) (1999), Article 6. [ONLINE] <http://rgmia.vu.edu.au/v2n2.html>.
- [20] S.S. DRAGOMIR, J.E. PEČARIĆ AND S. WANG, The unified treatment of trapezoid, Simpson and Ostrowski type inequality for monotonic mappings and applications, *Math. and Comp. Modelling*, **31** (2000), 61–70. *Preprint: RGMIA Res. Rep. Coll.*, **2**(4) (1999), Article 3. [ONLINE] <http://rgmia.vu.edu.au/v2n4.html>.
- [21] S.S. DRAGOMIR AND A. SOFO, An integral inequality for twice differentiable mappings and applications, *Preprint: RGMIA Res. Rep. Coll.*, **2**(2) (1999), Article 9. [ONLINE] <http://rgmia.vu.edu.au/v2n2.html>.
- [22] S.S. DRAGOMIR AND S. WANG, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Comput. Math. Appl.*, **33** (1997), 15–22.
- [23] M. MATIĆ, J.E. PEČARIĆ AND N. UJEVIĆ, On New estimation of the remainder in Generalised Taylor’s Formula, *Math. Inequal. Appl.*, **2**(3) (1999), 343–361.
- [24] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.



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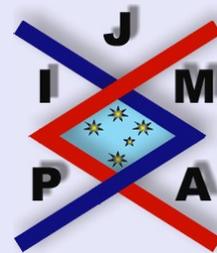
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- [25] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1994.
- [26] J.E. PEČARIĆ, F. PROSCHAN AND Y.L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, 1992.



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