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SHARP BOUNDS ON QUASICONVEX MOMENTS OF GENERALIZED ORDER STATISTICS

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ABSTRACT. Sharp lower and upper bounds for quasiconvex moments of generalized order statistics are proven by the use of the rearranged Moriguti's inequality. Even in the second moment case, the method yields improvements of known quantile and moment bounds for the expectation of order and record statistics based on independent identically distributed random variables. The bounds are attainable providing new characterizations of three-point and two-point distributions.

Key words and phrases: Generalized order statistics, quasiconvex moments, Moriguti inequality, Steffensen inequality.

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1. INTRODUCTION

Let $X, X_1, X_2, \dots$ be i.i.d. random variables with a common distribution function F. Define the quantile function $F^{-1}(t) = \inf\{s \in \mathbb{R}; F(s) \geq t\}$, $t \in (0,1)$. Let $X_{r,n}$ denote the r-th order statistic (OS, for short) from the sample $X_1, \dots, X_n$, and let $Y_r^{(k)}$ stand for the k-th record statistics (RS's, for short) from the sequence $X_1, X_2, \dots$, according to the definition of Dziubdziela and Kopociński [4], i.e.

$$Y_r^{(k)} = X_{L_k(r),L_k(r)+k-1}, \quad r = 1, 2, \dots, \quad k = 1, 2, \dots,$$

where $L_k(1) = 1$, $L_k(r+1) = \min\{j;\ X_{L_k(r),L_k(r)+k-1} < X_{j,j+k-1}\}$ for $r = 1, 2, \dots$.

The generalized order statistics are defined by Kamps [8] as follows:

Definition 1.1. Let $r, n \in \mathbb{N}$, $k, m \in \mathbb{R}$ be parameters such that $\eta_r = k+(n-r)(m+1) \geq 1$ for all $r \in \{1, ..., n\}$. If the random variables $U(r, n, m, k)$, $r = 1, \dots, n$, possess a joint density function of the form

$$f^{U(1,n,m,k),\dots,U(n,n,m,k)}(u_1, \dots, u_n) = k \left(\prod_{j=1}^{n-1} \eta_j \right) \left(\prod_{i=1}^{n-1} (1-u_i)^m \right) (1-u_n)^{k-1}$$

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on the cone $0 \leq u_1 \leq \ldots \leq u_n < 1$ of $\mathbb{R}^n$, then they are called uniform generalized order statistics. The random variables

$$X(r,n,m,k) = F^{-1}(U(r,n,m,k)),\ r = 1,\ldots,n,$$

are called generalized order statistics (g OS's, for short) based on the distribution function F.

In the case of $m = 0$ and $k = 1$ the g OS $X(r,n,m,k)$ reduces to the OS $X_{r,n}$ from the sample $X_1,\ldots,X_n$, while for a continuous F, $m = -1$ and $k \in \mathbb{N}$ we obtain the RS $Y_r^{(k)}$ based on the sequence $X_1, X_2, \ldots$.

Let $H : \mathbb{R} \to \mathbb{R}$ be a given measurable function. The generalized H-moment (H-moment, for short) of $X(r,n,m,k)$ is defined in Kamps [8] as follows

$$\mathrm{E}H\left(X(r,n,m,k)\right) = \int_0^1 H\left(F^{-1}(t)\right)\varphi_{r,n}(t)dt,$$

where the density function $\varphi_{r,n}$ of $U(r,n,m,k)$ is given by

$$\varphi_{r,n}(t) = \frac{a_{r-1}}{(r-1)!}(1-t)^{\eta_r - 1} g_m^{r-1}(t), \quad t \in [0,1), \tag{1.1}$$

with

$$a_{r-1} = \prod_{i=1}^{r} \eta_i,\ r = 1,\ldots,n,$$

$$g_m(t) = h_m(t) - h_m(0), \quad t \in [0,1),$$

$$h_m(t) = \begin{cases} -\frac{1}{m+1}(1-t)^{m+1}, & m \neq -1, \\ -log(1-t), & m = -1, \end{cases} t \in [0,1).$$

The aim of this paper is to present some new moment and quantile lower and upper bounds for the H-moment of the generalized order statistics $X(r,n,m,k)$ in the case H is quasiconvex. Recall that $f : \mathbb{R} \to \mathbb{R}$ is quasiconvex if for every $t \in \mathbb{R}$ the set $\{x \in \mathbb{R};\ f(x) \leq t\}$ is convex. The bounds of Proposition 3.1 are derived by the use of the rearranged Moriguti inequality (Lemma 2.1) i.e. applying a similar method as in Gajek and Okolewski [6] for $H \equiv id$. In Gajek and Okolewski [5] some bounds for OS's and RS's were obtained for $H(t) = t^\alpha$, $\alpha = 2s$, $s \in \mathbb{N}$, via the Steffensen inequality. Somewhat surprisingly, the present approach, which is equivalent to applying the Moriguti inequality first and the Steffensen inequality afterwards, provides better bounds (see Remarks 3.8 and 3.9). The bounds are attainable, which gives a new characterization of some three-point and two-point distributions (see Remarks 3.5, 3.6 and 3.7). Similar bounds on expectations of order statistics from possibly dependent identically distributed random variables were obtained by Rychlik [11] and independently by Caraux and Gascuel [2].

From Proposition 3.1 we can get sharp H-moment bounds for $\mathrm{E}H(X(r,n,m,k))$ (see Proposition 3.13), which generalize the result of Papadatos [10, Theorem 2.1].

In Proposition 3.16 quantile bounds for $\mathrm{E}H(X(r,n,m,k))$ are given under additional restrictions on the underlying distribution function. Some other quantile inequalities for moments of generalized order statistics from a particular restricted family of distributions were obtained by Gajek and Okolewski [7], via the Steffensen inequality.

A summary of known bounds for g OS's is presented in Kamps [8]. The results for OS's and RS's are presented e.g. in David [3] and Arnold and Balakrishnan [1].

2. AUXILIARY RESULTS

We reformulate Moriguti's result - [9, Theorem 1] - to the form which we shall use.

Lemma 2.1. *Let* Φ, $\underline{\Phi}$ *and* $\overline{\Phi} : [a,b] \to \mathbb{R}$ *be continuous, nondecreasing functions such that* $\Phi(a) = \underline{\Phi}(a) = \overline{\Phi}(a)$, $\Phi(b) = \underline{\Phi}(b) = \overline{\Phi}(b)$ *and* $\underline{\Phi}(t) \le \Phi(t) \le \overline{\Phi}(t)$ *for every* $t \in [a,b]$. *Then the following inequalities hold*

(i) $\int_a^b x(t)d\Phi(t) \le \int_a^b x(t)d\underline{\Phi}(t)$,

(ii) $\int_a^b x(t)d\Phi(t) \ge \int_a^b x(t)d\overline{\Phi}(t)$

for any nondecreasing function $x : (a,b) \to \mathbb{R}$ *for which the corresponding integrals exist. The equality in (i) holds iff either both sides are equal to* $+\infty$ $(-\infty)$ *or both are finite and* x *is constant on each connected interval from the set* $\{t \in (a,b);\ \underline{\Phi}(t) < \Phi(t)\}$. *The equality in (ii) holds iff either both sides are equal to* $+\infty$ $(-\infty)$ *or both are finite and* x *is constant on each connected interval from the set* $\{t \in (a,b);\ \overline{\Phi}(t) > \Phi(t)\}$.

Corollary 2.2. *If* x *is nonincreasing then the signs of inequalities (i) and (ii) are opposite.*

Remark 2.3. Part (i) of Lemma 2.1 follows from the proof of Moriguti's result. Replacing Φ by $\overline{\Phi}$ and $\underline{\Phi}$ by Φ in Lemma 2.1 (i) gives Lemma 2.1 (ii). Applying Lemma 2.1 to the function $-x$ instead of to x gives Corollary 2.2.

3. INEQUALITIES FOR GENERALIZED ORDER STATISTICS

Let us introduce the notation: $\overline{w} = (r,n,m,k)$,

$$W = \{\overline{w} \in \mathbb{N}\times\mathbb{N}\times\mathbb{R}\times\mathbb{R}; 1 \le r \le n, \forall_{1\le r\le n}\eta_r = k + (n-r)(m+1) \ge 1\},$$

$$W_1 = \{\overline{w} \in W; r = 1 \wedge \eta_1 = 1\},$$

$$W_2 = \{\overline{w} \in W; r = 1 \wedge \eta_1 > 1\},$$

$$W_3 = \{\overline{w} \in W; r \ge 2 \wedge \eta_r > 1 \wedge [m \ge -1 \vee (m < -1 \wedge \eta_1 > 1)]\},$$

$$W_4 = \{\overline{w} \in W; r \ge 2 \wedge [(m > -1 \wedge \eta_r = 1) \vee (m < -1 \wedge \eta_1 = 1 \wedge \eta_r > 1)]\},$$

$$W_5 = \{\overline{w} \in W; r \ge 2 \wedge m \le -1 \wedge \eta_r = 1\}.$$

Observe that $\forall_{i,j\in\{1,\dots,5\}}\ i \ne j \Rightarrow W_i \cap W_j = \emptyset$ and $W = W_1 \cup \ldots \cup W_5$.

Let

$$\Phi_{r,n}(t) = \int_0^t \varphi_{r,n}(x)dx, \quad t \in [0,1],$$

where the function $\varphi_{r,n}$ is defined by (1.1). In this notation parameters m and k are suppressed for brevity.

Moreover, let us put $b_n^r = 0$ for $\overline{w} \in W_1 \cup W_2$, $b_n^r = 1$ for $\overline{w} \in W_4 \cup W_5$ and

$$b_n^r = \begin{cases} 1 - \exp[-(r-1)/(\eta_r - 1)], & \text{for } \overline{w} \in W_3 \text{ such that } m = -1, \\ 1 - [(\eta_r - 1)/(\eta_1 - 1)]^{1/(m+1)}, & \text{for } \overline{w} \in W_3 \text{ such that } m \ne -1. \end{cases} \tag{3.1}$$

Additionally, let us define

(3.2)

$$\beta_{r,n} = \begin{cases} 1, & \text{for } \overline{w} \in W_1 \cup W_2, \\ \varphi_{r,n}(c_n^r -), & \text{for } \overline{w} \in W_3 \cup W_4, \end{cases} \quad \gamma_{r,n} = \begin{cases} 1, & \text{for } \overline{w} \in W_1 \cup W_4 \cup W_5, \\ \varphi_{r,n}(d_n^r), & \text{for } \overline{w} \in W_2 \cup W_3, \end{cases}$$

where $c_n^r = 0$ for $\overline{w} \in W_1 \cup W_2$, $c_n^r = 1$ for $\overline{w} \in W_4 \cup W_5$, $d_n^r = 0$ for $\overline{w} \in W_2$, $d_n^r = 1$ for $\overline{w} \in W_1 \cup W_4 \cup W_5$, and c_n^r and d_n^r, for $\overline{w} \in W_3$, are the unique zeros in $[0, b_n^r]$ and $[b_n^r, 1]$ of the functions

$$(1-t)\varphi_{r,n}(t) + \Phi_{r,n}(t) - 1 \quad \text{and} \quad t\varphi_{r,n}(t) - \Phi_{r,n}(t), \tag{3.3}$$

respectively. In the notation b_n^r, c_n^r, d_n^r, $\beta_{r,n}$ and $\gamma_{r,n}$ the constants m and k are suppressed for brevity. Note that $\beta_{r,n}$ is not defined for any $\overline{w} \in W_5$.

Now let us put $A = \{s \in \mathbb{R};\ \forall_{\epsilon>0}\ H(s-\epsilon) \geq H(s)\}$,

$$a = \begin{cases} \sup A, & \text{for } A \neq \emptyset, \\ -\infty, & \text{for } A = \emptyset, \end{cases} \tag{3.4}$$

and

$$z_a = \begin{cases} 0, & \text{for } a = -\infty, \\ F(a), & \text{for } a \in \mathbb{R}, \\ 1, & \text{for } a = +\infty. \end{cases} \tag{3.5}$$

Observe that if $z_a = 0$ or $z_a = 1$, then the function $H|_{I_F}$, where $I_F = J_F \cup (\inf J_F, \sup J_F)$ with J_F denoting the image of $(0,1)$ under F^{-1}, is monotone and corresponding bounds follow from Proposition 1 of Gajek and Okolewski [6]. Therefore, we shall present the inequalities for $\mathrm{E}H\left(X(r,n,m,k)\right)$ when H is quasiconvex and $z_a \in (0,1)$.

Let us define

$$\mu_{r,n} = \begin{cases} z_a^{-1}\Phi_{r,n}(z_a), & \text{for } \overline{w} \in W_1 \cup W_2, \\ \varphi_{r,n}(\bar{c}_n^r), & \text{for } \overline{w} \in W_3 \cup W_4 \cup W_5, \end{cases} \tag{3.6}$$

and

$$\nu_{r,n} = \begin{cases} \varphi_{r,n}(\bar{d}_n^r), & \text{for } \overline{w} \in W_1 \cup W_2 \cup W_3, \\ (1-z_a)^{-1}(1-\Phi_{r,n}(z_a)), & \text{for } \overline{w} \in W_4 \cup W_5, \end{cases} \tag{3.7}$$

where $z_a \in (0,1)$, $\bar{c}_n^r = z_a$ for $\overline{w} \in W_4 \cup W_5$, $\bar{d}_n^r = z_a$ for $\overline{w} \in W_1 \cup W_2$, and $\bar{c}_n^r$ and $\bar{d}_n^r$, for $\overline{w} \in W_3$, are the unique zeros of the function

$$\Phi_{r,n}(z_a) - \Phi_{r,n}(t) - \varphi_{r,n}(t)(z_a - t) \tag{3.8}$$

in the intervals $[0, b_n^r]$ and $[b_n^r, 1]$, respectively. In the notation $\mu_{r,n}$ and $\nu_{r,n}$ the constants m and k are suppressed for brevity. It is easily seen that $\bar{c}_n^r = z_a$ and $\bar{d}_n^r = z_a$ for these $\overline{w} \in W$ for which $z_a \in (0, b_n^r]$ and $z_a \in [b_n^r, 1)$, respectively.

Further, let us define

$$\lambda = z_a \mathbb{I}_{(0,d_n^r]}(z_a) + (\gamma_{r,n})^{-1}\Phi_{r,n}(z_a)\mathbb{I}_{(d_n^r,1)}(z_a), \tag{3.9}$$

$$\kappa = (\beta_{r,n})^{-1}(1-\Phi_{r,n}(z_a))\mathbb{I}_{(0,c_n^r]}(z_a) + (1-z_a)\mathbb{I}_{(c_n^r,1)}(z_a), \tag{3.10}$$

$$\chi = (\mu_{r,n})^{-1}\Phi_{r,n}(z_a), \tag{3.11}$$

$$\psi = (\nu_{r,n})^{-1}(1-\Phi_{r,n}(z_a)), \tag{3.12}$$

with c_n^r and d_n^r such as in (3.2), $\beta_{r,n}$, $\gamma_{r,n}$, $\mu_{r,n}$ and $\nu_{r,n}$ defined by (3.2), (3.6) and (3.7). In the notation λ, κ, χ and ψ the constants r, n, m, k and z_a are suppressed for brevity.

Throughout the paper we shall assume that the integrals appearing in the propositions exist and are finite.

Proposition 3.1. *Let z_a, λ, κ, χ and ψ be defined by (3.5), (3.9), (3.10), (3.11) and (3.12), respectively. Let $H : \mathbb{R} \to \mathbb{R}$ be an arbitrary quasiconvex function such that $z_a \in (0,1)$.*
(i) *If $\overline{w} \in W \setminus W_5$, then*

$$\mathrm{E}H\left(X(r,n,m,k)\right) \leq \frac{\Phi_{r,n}(z_a)}{\lambda}\int_0^\lambda H\left(F^{-1}(t)\right)dt + \frac{1-\Phi_{r,n}(z_a)}{\kappa}\int_{1-\kappa}^1 H\left(F^{-1}(t)\right)dt.$$

(ii) *If* $\overline{w} \in W$, *then*

$$\mathrm{E}H\left(X(r,n,m,k)\right) \geq \frac{\Phi_{r,n}(z_a)}{\chi}\int_{z_a-\chi}^{z_a} H\left(F^{-1}(t)\right)dt + \frac{1-\Phi_{r,n}(z_a)}{\psi}\int_{z_a}^{z_a+\psi} H\left(F^{-1}(t)\right)dt.$$

Proof. It is easy to check that: $\overline{w} \in W_1 \Rightarrow \varphi_{r,n} \equiv 1$ on $[0,1)$;
$\overline{w} \in W_2 \Rightarrow {\varphi_{r,n}}' < 0$ on $(0,1)$, $\varphi_{r,n}(0) < +\infty$, $\varphi_{r,n}(1-) = 0$;
$\overline{w} \in W_3 \Rightarrow {\varphi_{r,n}}' > 0$ on $(0, b_n^r)$, ${\varphi_{r,n}}' < 0$ on $(b_n^r, 1)$, $\varphi_{r,n}(0) = 0$, $\varphi_{r,n}(1-) = 0$;
$\overline{w} \in W_4 \Rightarrow {\varphi_{r,n}}' > 0$ on $(0,1)$, $\varphi_{r,n}(0) = 0$, $\varphi_{r,n}(1-) < +\infty$;
$\overline{w} \in W_5 \Rightarrow {\varphi_{r,n}}' > 0$ on $(0,1)$, $\varphi_{r,n}(0) = 0$, $\varphi_{r,n}(1-) = +\infty$.

For $\overline{w} \in W_1$, (i)-(ii) are obvious identities. So let us consider the other cases. From Kamps [8] we have

$$\mathrm{E}H\left(X(r,n,m,k)\right) = \int_0^{z_a} H\left(F^{-1}(t)\right)d\Phi_{r,n}(t) + \int_{z_a}^{1} H\left(F^{-1}(t)\right)d\Phi_{r,n}(t), \tag{3.13}$$

where z_a is given by (3.5). We shall apply Corollary 2.2 and Lemma 2.1 with the functions $x \equiv H \circ F^{-1}$, $\Phi \equiv \Phi_{r,n}$, $\overline{\Phi} \equiv \overline{\Phi}_{r,n}^{u}$ and $\underline{\Phi} \equiv \underline{\Phi}_{r,n}^{u}$; $\overline{\Phi}_{r,n}^{u}$ and $\underline{\Phi}_{r,n}^{u}$ are defined on $[0, z_a]$ and $[z_a, 1]$, respectively, as follows

$$\overline{\Phi}_{r,n}^{u}(t) = \begin{cases} z_a^{-1}\Phi_{r,n}(z_a)t, & \text{if } z_a \in (0, d_n^r], \\ \gamma_{r,n} t\mathbb{I}_{[0,\lambda]}(t) + \Phi_{r,n}(z_a)\mathbb{I}_{(\lambda,z_a]}(t), & \text{if } z_a \in (d_n^r, 1), \end{cases}$$

and

$$\underline{\Phi}_{r,n}^{u}(t) = \begin{cases} \Phi_{r,n}(z_a)\mathbb{I}_{[z_a,1-\kappa]}(t) + (\beta_{r,n}(t-1)+1)\mathbb{I}_{(1-\kappa,1]}(t), & \text{if } z_a \in (0, c_n^r], \\ (1-z_a)^{-1}[1-\Phi_{r,n}(z_a)](t-1)+1, & \text{if } z_a \in (c_n^r, 1), \end{cases}$$

where $\beta_{r,n}$ and $\gamma_{r,n}$ are given by (3.2). Moreover, let us observe that

$$\overline{\Phi}_{r,n}^{u}(t) = \int_0^t \overline{\varphi}_{r,n}^{u}(s)ds \quad \text{and} \quad \underline{\Phi}_{r,n}^{u}(t) = \underline{\Phi}_{r,n}^{u}(z_a) + \int_{z_a}^{t} \underline{\varphi}_{r,n}^{u}(s)ds, \tag{3.14}$$

where

$$\overline{\varphi}_{r,n}^{u}(s) = \begin{cases} {z_a}^{-1}\Phi_{r,n}(z_a), & \text{if } z_a \in (0, d_n^r], \\ \gamma_{r,n}\mathbb{I}_{[0,\lambda]}(s), & \text{if } z_a \in (d_n^r, 1), \end{cases} \tag{3.15}$$

and

$$\underline{\varphi}_{r,n}^{u}(s) = \begin{cases} \beta_{r,n}\mathbb{I}_{(1-\kappa,1]}(s), & \text{if } z_a \in (0, c_n^r], \\ \left[1-\Phi_{r,n}(z_a)\right](1-z_a)^{-1}, & \text{if } z_a \in (c_n^r, 1). \end{cases} \tag{3.16}$$

By Corollary 2.2, Lemma 2.1, (3.14), (3.15) and (3.16) we get

$$\begin{aligned} \mathrm{E}H\left(X(r,n,m,k)\right) &\leq \int_0^{z_a} H\left(F^{-1}(t)\right)d\overline{\Phi}_{r,n}^{u}(t) + \int_{z_a}^{1} H\left(F^{-1}(t)\right)d\underline{\Phi}_{r,n}^{u}(t) \\ &= \overline{\varphi}_{r,n}^{u}(0)\int_0^{\lambda} H\left(F^{-1}(t)\right)dt + \underline{\varphi}_{r,n}^{u}(1)\int_{1-\kappa}^{1} H\left(F^{-1}(t)\right)dt, \end{aligned}$$

which leads to (i).

In order to prove (ii) we shall use Corollary 2.2 and Lemma 2.1 with the functions $x \equiv H \circ F^{-1}$, $\Phi \equiv \Phi_{r,n}$, $\underline{\Phi} \equiv \underline{\Phi}_{r,n}^{l}$ and $\overline{\Phi} \equiv \overline{\Phi}_{r,n}^{l}$; $\underline{\Phi}_{r,n}^{l}$ and $\overline{\Phi}_{r,n}^{l}$ are defined on $[0, z_a]$ and $[z_a, 1]$, respectively, as follows

$$\underline{\Phi}^l_{r,n}(t)=\begin{cases} z_a^{-1}\Phi_{r,n}(z_a)t, & \text{for } \overline{w}\in W_2,\\ (\varphi_{r,n}(\bar{c}^r_n)(t-z_a)+\Phi_{r,n}(z_a))\mathbb{I}_{(z_a-\chi,z_a]}(t), & \text{for } \overline{w}\in W_3\cup W_4\cup W_5,\end{cases}$$

and

$$\overline{\Phi}^l_{r,n}(t)=\begin{cases} (1-z_a)^{-1}(1-\Phi_{r,n}(z_a))(t-1)+1, & \text{for } \overline{w}\in W_4\cup W_5,\\ (\varphi_{r,n}(\bar{d}^r_n)(t-z_a)+\Phi_{r,n}(z_a))\mathbb{I}_{[z_a,z_a+\psi]}(t)+\mathbb{I}_{(z_a+\psi,1]}(t), & \text{for } \overline{w}\in W_2\cup W_3,\end{cases}$$

where $\bar{c}^r_n$ and $\bar{d}^r_n$ are such as in (3.6) and (3.7).

Let us note that

$$\underline{\Phi}^l_{r,n}(t)=\int_0^t \underline{\varphi}^l_{r,n}(s)ds \quad \text{and} \quad \overline{\Phi}^l_{r,n}(t)=\overline{\Phi}^l_{r,n}(z_a)+\int_{z_a}^t \overline{\varphi}^u_{r,n}(s)ds, \tag{3.17}$$

where

$$\underline{\varphi}^l_{r,n}(s)=\begin{cases} z_a^{-1}\Phi_{r,n}(z_a), & \text{for } \overline{w}\in W_2,\\ \varphi_{r,n}(\bar{c}^r_n)\mathbb{I}_{(z_a-\chi,z_a]}(s), & \text{for } \overline{w}\in W_3\cup W_4\cup W_5,\end{cases} \tag{3.18}$$

and

$$\overline{\varphi}^l_{r,n}(s)=\begin{cases} (1-z_a)^{-1}(1-\Phi_{r,n}(z_a)), & \text{for } \overline{w}\in W_4\cup W_5,\\ \varphi_{r,n}(\bar{d}^r_n)\mathbb{I}_{[z_a,z_a+\psi]}(s), & \text{for } \overline{w}\in W_2\cup W_3.\end{cases} \tag{3.19}$$

By Corollary 2.2, Lemma 2.1, (3.17), (3.18) and (3.19) we have

$$\begin{aligned}\mathrm{E}H\left(X(r,n,m,k)\right) &\geq \int_0^{z_a} H\left(F^{-1}(t)\right)d\underline{\Phi}^l_{r,n}(t)+\int_{z_a}^1 H\left(F^{-1}(t)\right)d\overline{\Phi}^l_{r,n}(t)\\ &=\underline{\varphi}^l_{r,n}(z_a)\int_{z_a-\chi}^{z_a} H\left(F^{-1}(t)\right)dt+\overline{\varphi}^l_{r,n}(z_a)\int_{z_a}^{z_a+\psi} H\left(F^{-1}(t)\right)dt,\end{aligned}$$

which gives (ii). This completes the proof of Proposition 3.1. ∎

Remark 3.2. Observe that the bounds of Proposition 3.1 work under quite weak assumptions. In the case of the lower bounds we even do not need $\mathrm{E}H(X)$ to be finite – see Example 3.1 below.

Example 3.1. Let

$$F(t)=\begin{cases} (2+t^2)^{-1}, & \text{for } t<0,\\ (2-t^2)^{-1}, & \text{for } t\in[0,1),\\ 1, & \text{else .}\end{cases}$$

It is easy to check that $\mathrm{E}X^2_{2,3}=3.5$, $\mathrm{E}X^2=+\infty$ and the lower bound for $\mathrm{E}X^2_{2,3}$ in Proposition 3.1 (i) is meaningful (and equals 0.88).

Remark 3.3. If $\mathrm{E}X^2(r,n,m,k)<+\infty$ and $H(t)=(t-\mathrm{E}X(r,n,m,k))^2$, $t\in\mathbb{R}$, then Proposition 3.1 provides lower and upper bounds for variation of g OS's $X(r,n,m,k)$.

Remark 3.4. Note that right-hand sides of the inequalities (i) and (ii) of Proposition 3.1 depend on the parent distribution not only through a simple functional of the quantile function as the bounds of Proposition 1 of Gajek and Okolewski [6], but also through a value of distribution function at a single point determined by H. The reason of this drawback lays on difficulties which occur while quasiconvex function H is not monotone.

Remark 3.5. Equality in Proposition 3.1 (i) holds if $\overline{w}\in W_1$ or one of the following conditions is satisfied:

(a) F has exactly one atom;
(b) for $z_a \in (0, c_n^r)$, F has at most three atoms with the probability masses $(z_a, c_n^r - z_a, 1 - c_n^r)$ or $(z_a, 1 - z_a)$ or $(c_n^r, 1 - c_n^r)$, respectively;
(c) for $z_a \in [c_n^r, d_n^r]$, F has exactly two atoms with the probability masses $(z_a, 1 - z_a)$, respectively;
(d) for $z_a \in (d_n^r, 1)$, F has at most three atoms with the probability masses $(d_n^r, z_a - d_n^r, 1 - z_a)$ or $(z_a, 1 - z_a)$ or $(d_n^r, 1 - d_n^r)$, respectively.

Remark 3.6. Equality in Proposition 3.1 (ii) holds if $\overline{w} \in W_1$ or one of the following conditions is satisfied:

(a') F has exactly one atom;
(b') for $z_a \in (0, b_n^r)$, F has at most three atoms with the probability masses $(z_a, \overline{d}_n^r - z_a, 1 - \overline{d}_n^r)$ or $(z_a, 1 - z_a)$ or $(\overline{d}_n^r, 1 - \overline{d}_n^r)$, respectively;
(c') for $z_a = b_n^r$, F has exactly two atoms with the probability masses $(z_a, 1 - z_a)$, respectively;
(d') for $z_a \in (b_n^r, 1)$, F has at most three atoms with the probability masses $(\overline{c}_n^r, z_a - \overline{c}_n^r, 1 - z_a)$ or $(z_a, 1 - z_a)$ or $(\overline{c}_n^r, 1 - \overline{c}_n^r)$, respectively.

Remark 3.7. Under the additional assumptions that $H|_{I_F}$ is left-hand continuous and is not constant on any nonempty open interval, the conditions given in Remarks 3.5 and 3.6 are also sufficient. Indeed, denoting $\overline{S} = \{t \in (0, z_a);\ \overline{\Phi}_{r,n}^u(t) > \Phi_{r,n}(t)\}$, $\underline{S} = \{t \in (z_a, 1);\ \underline{\Phi}_{r,n}^u(t) < \Phi_{r,n}(t)\}$ observe that $\overline{S} = (0, z_a)$ and $\underline{S} = (z_a, 1)$ for $\overline{w} \in W_2 \cup W_4$ and these $\overline{w} \in W_3$ for which $z_a \in [c_n^r, d_n^r]$; $\overline{S} = (0, z_a)$ and $\underline{S} = (z_a, c_n^r) \cup (c_n^r, 1)$ for $\overline{w} \in W_3$ such that $z_a \in (0, c_n^r)$; $\overline{S} = (0, d_n^r) \cup (d_n^r, z_a)$ and $\underline{S} = (z_a, 1)$ for $\overline{w} \in W_3$ such that $z_a \in (d_n^r, 1)$. Combining this with the fact that $H \circ F^{-1}$ is left-hand continuous and that, by Lemma 2.1 and Corollary 2.2, the equality in the inequality (i) of Proposition 3.1 is attained iff $H \circ F^{-1}$ (or equivalently F^{-1}) is constant on each connected interval from the set $\overline{S} \cup \underline{S}$, proves Remark 3.5. A similar reasoning applies to Remark 3.6.

Remark 3.8. The proof of Proposition 3.1 (i) relies on applying Lemma 2.1 and Corollary 2.2 to the integrals $\int_{z_a}^1 H\left(F^{-1}(t)\right) d\Phi_{r,n}(t)$ and $\int_0^{z_a} H\left(F^{-1}(t)\right) d\Phi_{r,n}(t)$. The question arises whether one can use in Lemma 2.1 (Corollary 2.2) a minorant (a majorant) different than $\underline{\Phi}_{r,n}^u$ ($\overline{\Phi}_{r,n}^u$, respectively) in order to alter the parameter corresponding to κ (λ) and further improve the resulting bound. In the class of absolutely continuous nondecreasing minorants (majorants) of $\Phi_{r,n}$ which have the same values as $\Phi_{r,n}$ at the both ends of the interval $[z_a, 1]$ ($[0, z_a]$) and which Radon-Nikodym derivatives are essentially finite, the answer to the question is negative. Indeed, the form of the bound (i) implies that it is most precise when the minorant and the majorant provide the Radon-Nikodym derivatives with the least possible essential supremums. Since $\underline{\varphi}_{r,n}^u$ as well as $\overline{\varphi}_{r,n}^u$ satisfy this condition, Proposition 3.1 (i) provides in some sense optimal bounds. A similar remark refers to the case of the bound (ii) of Proposition 3.1.

Remark 3.9. Obviously, $\Phi_{r,n}$ is its own minorant (majorant, respectively) on any subinterval of $(0, 1)$ and $\varphi_{r,n}|_{(z_a,1)}$ ($\varphi_{r,n}|_{(0,z_a)}$) has a greater essential supremum than $\underline{\varphi}_{r,n}^u$ ($\overline{\varphi}_{r,n}$) whenever $\underline{\Phi}_{r,n}^u$ ($\overline{\Phi}_{r,n}^u$) is not identical with $\Phi_{r,n}|_{(z_a,1)}$ ($\Phi_{r,n}|_{(0,z_a)}$). According to Remark 3.8, the bounds of Proposition 3.1 for order and record statistics from a continuous parent distribution are more precise than (are the same as) their analogues from Proposition 1 of Gajek and Okolewski [5] except for (in the case of) the lower bounds if $z_a \neq b_n^r$ (if $z_a = b_n^r$).

Now, assuming that some additional conditions are satisfied we shall compare in Corollary 3.12 the upper bounds following from Proposition 3.1 (Corollary 3.11) with their counterparts following from easy to obtain modification of Proposition 1 of Gajek and Okolewski [6] (Corollary 3.10).

Corollary 3.10. *Let* $\overline{w} \in W \setminus W_5$, $H : \mathbb{R} \to \mathbb{R}$ *be quasiconvex and* $\beta_{r,n}$, a, z_a *be defined by (3.2), (3.4), (3.5), respectively. Suppose that* $P(X \geq a) = 1$, $z_a \in (0,1)$, $H(a) = 0$ *and* H *is not constant on any nonempty open interval.*

Then

$$\mathrm{E}H\left(X(r,n,m,k)\right) \leq \beta_{r,n} \int_{\max\left\{z_a, 1-1/\beta_{r,n}\right\}}^{1} H\left(F^{-1}(t)\right) dt.$$

Proof. On account of Proposition 3.1 (i) of Gajek and Okolewski [6] it suffices to show that, under the assumptions of Corollary 3.10, $H \circ F^{-1}$ is nondecreasing and $H \circ F^{-1}(t) = 0$ for $t \in (0, z_a)$. To this end observe that $H \circ F^{-1}(t) = H(a) = 0$ for $t \in (0, z_a)$, $H \circ F^{-1}(z_a) = H\left(F^{-1}(F(a))\right) \geq H(a) = 0$ and that, by definition, the function $H \circ F^{-1}$ is nondecreasing on $(z_a, 1)$. ∎

Corollary 3.11. *Let the assumptions of Corollary 3.10 be satisfied. Then*

$$\mathrm{E}H\left(X(r,n,m,k)\right) \leq \kappa^{-1}(1 - \Phi_{r,n}(z_a)) \int_{1-\kappa}^{1} H\left(F^{-1}(t)\right) dt,$$

where κ *is defined by (3.10).*

Proof. Combination of Proposition 3.1 and the fact that $(H \circ F^{-1})(t) = H(a) = 0$ for each $t \in (0, z_a)$ gives the result. ∎

Corollary 3.12. *Let* c_n^r *be such as in (3.2). Suppose that the assumptions of Corollary 3.10 are satisfied.*

(i) *If* $z_a \in (0,1) \setminus \{c_n^r\}$, *then the bounds of Corollary 3.11 are better than the bounds of Corollary 3.10,*
(ii) *If* $z_a = c_n^r$, *then Corollary 3.10 and Corollary 3.11 provide the identical bounds.*

Proof. Let us denote by A_u and B_u the right-hand sides of the inequalities in Corollary 3.11 and Corollary 3.10, respectively.

If $z_a \in (0, 1 - 1/\beta_{r,n}]$, then

$$A_u = \beta_{r,n} \int_{1-1/\beta_{r,n}}^{1} H\left(F^{-1}(t)\right) dt > \beta_{r,n} \int_{1-1/\beta_{r,n}+\Phi_{r,n}(z_a)/\beta_{r,n}}^{1} H\left(F^{-1}(t)\right) dt = B_u,$$

as $H\left(F^{-1}(t)\right) > H(a) = 0$ for $t > z_a$.

If $z_a \in (1 - 1/\beta_{r,n}, c_n^r]$, then

$$A_u = \beta_{r,n} \int_{z_a}^{1} H\left(F^{-1}(t)\right) dt \geq \beta_{r,n} \int_{1-(1-\Phi_{r,n}(z_a))/\beta_{r,n}}^{1} H\left(F^{-1}(t)\right) dt = B_u.$$

Indeed, since the function $f : (0,1) \to \mathbb{R}$ defined by $(1 - \Phi_{r,n}(t))/(1-t)$ obtains its maximum equal to $\beta_{r,n}$ at the unique point $t = c_n^r$, $z_a \leq 1 - (1 - \Phi_{r,n}(z_a))/\beta_{r,n}$ for $z_a \in (0,1)$ and the equality is attained only for $z_a = c_n^r$.

If $z_a \in (c_n^r, 1)$, then

$$A_u = \beta_{r,n} \int_{z_a}^{1} H\left(F^{-1}(t)\right) dt > \frac{1 - \Phi_{r,n}(z_a)}{1 - z_a} \int_{z_a}^{1} H\left(F^{-1}(t)\right) dt = B_u$$

and the proof is complete. ∎

Now, we present some H-moment bounds on $\mathrm{E}H\left(X(r,n,m,k)\right)$ provided that H is quasiconvex and nonnegative. The special cases $z_a=0$ and $z_a=1$ follow from Proposition 3 of Gajek and Okolewski [6], so, we shall formulate the result for H quasiconvex such that $z_a\in(0,1)$.

Proposition 3.13. *Suppose that $\overline{w}\in W\setminus W_5$. Then for an arbitrary quasiconvex function $H:\mathbb{R}\to\mathbb{R}_+\cup\{0\}$ such that $z_a\in(0,1)$, it holds that*

$$\mathrm{E}H\left(X(r,n,m,k)\right)\le M_{r,n}(z_a)\mathrm{E}H(X)\le\max\{\beta_{r,n},\gamma_{r,n}\}\mathrm{E}H(X),$$

where $M_{r,n}(z_a)=\max\{\lambda^{-1}\Phi_{r,n}(z_a),\kappa^{-1}[1-\Phi_{r,n}(z_a)]\}$ and z_a, λ,κ are given by (3.5), (3.9), (3.10), respectively.

Proof. For $\overline{w}\in W_1$ we have the obvious identity. So, let us consider the other cases. Estimating the right-hand side of Proposition 3.1 (i) we get

$$\mathrm{E}H\left(X(r,n,m,k)\right)\le\max\{\lambda^{-1}\Phi_{r,n}(z_a),\kappa^{-1}[1-\Phi_{r,n}(z_a)]\}\{\int_0^\lambda H\left(F^{-1}(t)\right)dt + \int_{1-\kappa}^1 H\left(F^{-1}(t)\right)dt\}.$$

Putting z_a instead of λ and $1-\kappa$ gives the first inequality. The second inequality follows from the first one as a consequence of the following facts:

(i) if $z_a\in(0,c_n^r)$, then
$$M^1_{r,n}(z_a)\equiv\lambda^{-1}\Phi_{r,n}(z_a)=z_a^{-1}\Phi_{r,n}(z_a)<\varphi_{r,n}(z_a)<\varphi_{r,n}(c_n^r)=\beta_{r,n},$$
$$M^2_{r,n}(z_a)\equiv\kappa^{-1}[1-\Phi_{r,n}(z_a)]=\beta_{r,n},$$
so, $M_{r,n}(z_a)=\max\{M^1_{r,n}(z_a),M^2_{r,n}(z_a)\}=\beta_{r,n}$;

(ii) if $z_a\in(d_n^r,1)$, then
$$M^1_{r,n}(z_a)=\gamma_{r,n},$$
$$M^2_{r,n}(z_a)=(1-z_a)^{-1}[1-\Phi_{r,n}(z_a)]<\varphi_{r,n}(z_a)<\varphi_{r,n}(d_n^r)=\gamma_{r,n},$$
so, $M_{r,n}(z_a)=\gamma_{r,n}$;

(iii) if $z_a\in[c_n^r,d_n^r]$, then
$$M^1_{r,n}(z_a)=z_a^{-1}\Phi_{r,n}(z_a)\le\gamma_{r,n},$$
$$M^2_{r,n}(z_a)=(1-z_a)^{-1}[1-\Phi_{r,n}(z_a)]\le\beta_{r,n},$$
so, $M_{r,n}(z_a)\le\max\{\beta_{r,n},\gamma_{r,n}\}$.

The proof is complete. ∎

Remark 3.14. Equality in the first inequality of Proposition 3.13 holds if $\overline{w}\in W_1$ or F has only one atom at $H^{-1}(0)$ (provided that there exists a point t_0 from the image of $(0,1)$ under F^{-1} such that $H(t_0)=0$) or $z_a=\Phi_{r,n}(z_a)$ and one of the following conditions is satisfied:

(a) F has exactly one atom;
(b) F has exactly two atoms with the probability masses $(z_a,1-z_a)$, respectively.

Under the additional assumptions that H is left-hand continuous and it is not constant on any nonempty open interval, the above conditions are also sufficient. Indeed, for $\overline{w}\in W_1$ we have the obvious identity. If $\overline{w}\in W_3$ and $z_a\in(0,c_n^r)\cup(d_n^r,1)$, or $\overline{w}\in W_2\cup W_4$, then $\lambda<1-\kappa$ and the equality is attained iff $H\circ F^{-1}(t)=0$ for $t\in(0,1)$. If $\overline{w}\in W_3$ and $z_a\in[c_n^r,d_n^r]$, then $\lambda=z_a$, $\kappa=1-z_a$, so, the equality is attained iff $\lambda^{-1}\Phi_{r,n}(z_a)=\kappa^{-1}[1-\Phi_{r,n}(z_a)]$ (i.e. iff $z_a=\Phi_{r,n}(z_a)$) and one of the conditions (a) or (c) of Remark 3.5 is satisfied.

Remark 3.15. Equality in the second inequality of Proposition 3.13 holds iff $\overline{w} \in W_1$ or F has only one atom at $H^{-1}(0)$ (provided that there exists a point t_0 from the image of $(0,1)$ under F^{-1} such that $H(t_0) = 0$).

Under some additional restrictions on the function $H \circ F^{-1}$ we can formulate another consequence of Proposition 3.1.

Proposition 3.16. *Let a, z_a, λ, κ, χ and ψ be defined by (3.4), (3.5), (3.9), (3.10), (3.11) and (3.12), respectively. Suppose that $H : \mathbb{R} \to \mathbb{R}$ is a given quasiconvex function such that $z_a \in (0,1)$.*

(i) *If $\overline{w} \in W$ and the function $H \circ F^{-1}$ is convex on the interval $[z_a - \chi, z_a + \psi]$, then*

$$\mathrm{E}H\left(X(r,n,m,k)\right) \geq \Phi_{r,n}(z_a)(H \circ F^{-1})(z_a - \chi/2) + (1 - \Phi_{r,n}(z_a))(H \circ F^{-1})(z_a + \psi/2).$$

(ii) *If $\overline{w} \in W \setminus W_5$ and the function $H \circ F^{-1}$ is concave on the intervals $[0,\lambda]$ and $[1-\kappa, 1]$, then*

$$\mathrm{E}H\left(X(r,n,m,k)\right) \leq \Phi_{r,n}(z_a)(H \circ F^{-1})(\lambda/2) + (1 - \Phi_{r,n}(z_a))(H \circ F^{-1})(1 - \kappa/2).$$

Proof. Applying Jensen's inequality to the bound (ii) of Proposition 3.1 we have

$$\begin{aligned}
\mathrm{E}H\left(X(r,n,m,k)\right) &\geq \chi^{-1}\Phi_{r,n}(z_a)\int_{z_a-\chi}^{z_a} H\left(F^{-1}(t)\right)dt \\
&\qquad + \psi^{-1}(1-\Phi_{r,n}(z_a))\int_{z_a}^{z_a+\psi} H\left(F^{-1}(t)\right)dt \\
&\geq \Phi_{r,n}(z_a)\left(H \circ F^{-1}\right)\left(\chi^{-1}\int_{z_a-\chi}^{z_a} t\,dt\right) \\
&\qquad + (1-\Phi_{r,n}(z_a))\left(H \circ F^{-1}\right)\left(\psi^{-1}\int_{z_a}^{z_a+\psi} t\,dt\right) \\
&= \Phi_{r,n}(z_a)(H \circ F^{-1})(z_a - \chi/2) \\
&\qquad + (1-\Phi_{r,n}(z_a))(H \circ F^{-1})(z_a + \psi/2).
\end{aligned}$$

The proof of (i) is complete. The case (ii) can be proven in a similar way. ∎

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