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ON AN INEQUALITY OF GRONWALL

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ABSTRACT. In this paper, we obtain some new Gronwall-Bellman type integral inequalities, and we give an application of our results in the study of boundedness of the solutions of nonlinear integrodifferential equations.

Key words and phrases: Gronwall inequality, nonlinear integrodifferential equation, nondecreasing function, nonnegative continuous functions, partial derivatives, variational equation.

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1. INTRODUCTION

Integral inequalities play a significant role in the study of differential and integral equations. In particular, there has been a continuous interest in the following inequality.

Lemma 1.1. *Let $u(t)$ and $g(t)$ be nonnegative continuous functions on $I = [0, \infty)$ for which the inequality*

$$u(t) \leq c + \int_a^t g(s)u(s)ds, \ t \in I$$

holds, where c is a nonnegative constant. Then

$$u(t) \leq c \exp\left(\int_a^t g(s)ds\right), \ t \in I.$$

Due to various motivations, several generalizations and applications of this lemma have been obtained and used extensively, see the references under [1, 3].

Pachpatte [5] obtained a useful general version of this lemma. The aim of this work is to establish some useful generalizations of the inequalities obtained in [5]. Some consequences of our results are also given.

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2. STATEMENT OF RESULTS

Our main results are given in the following theorems:

Theorem 2.1. *Let $u(t)$, $f(t)$ be nonnegative continuous functions in a real interval $I = [a,b]$. Suppose that $k(t,s)$ and its partial derivatives $k_t(t,s)$ exist and are nonnegative continuous functions for almost every $t, s \in I$. If the inequality*

$$u(t) \le c + \int_a^t f(s)u(s)ds + \int_a^t f(s)\left(\int_a^s k(s,\tau)u(\tau)d\tau\right)ds,\ a \le \tau \le s \le t \le b, \tag{2.1}$$

holds, where c is a nonnegative constant, then

$$u(t) \le c\left[1 + \int_a^t f(s)\exp\left(\int_a^s (f(\tau) + k(\tau,\tau))d\tau\right)ds\right]. \tag{2.2}$$

Proof. Define a function $v(t)$ by the right hand side of (2.1). Then it follows that

$$u(t) \le v(t). \tag{2.3}$$

Therefore

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)\int_a^t k(t,\tau)u(\tau)d\tau, \quad v(a) = c \\ &\le f(t)\left(v(t) + \int_a^t k(t,\tau)v(\tau)d\tau\right). \text{ (by (2.3))} \end{aligned} \tag{2.4}$$

If we put

$$m(t) = v(t) + \int_a^t k(t,\tau)v(\tau)d\tau, \tag{2.5}$$

then it is clear that

$$v(t) \le m(t). \tag{2.6}$$

Therefore

$$\begin{aligned} m'(t) &= v'(t) + k(t,t)v(t) + \int_a^t k_t(t,\tau)v(\tau)d\tau, \quad m(a) = v(a) = c \\ &\le v'(t) + k(t,t)v(t), \\ &\le f(t)m(t) + k(t,t)v(t), \text{ (by (2.4))} \\ &\le (f(t) + k(t,t))\,m(t). \text{ (by (2.6))} \end{aligned} \tag{2.7}$$

Integrate (2.7) from a to t, we obtain

$$m(t) \le c\exp\left(\int_a^t \big(f(s) + k(s,s)\big)ds\right). \tag{2.8}$$

Substitute (2.8) into (2.4), we have

$$v'(t) \le cf(t)\exp\left(\int_a^t \big(f(s) + k(s,s)\big)ds\right). \tag{2.9}$$

Integrating both sides of (2.9) from a to t, we obtain

$$v(t) \le c\left[1 + \int_a^t f(s)\exp\left(\int_a^s \big(f(\tau) + k(\tau,\tau)\big)d\tau\right)ds\right].$$

By (2.3) we have the desired result. □

Remark 2.2. If in Theorem 2.1 we set $k(t,s) = g(s)$, our estimate reduces to Theorem 1 obtained in [5].

Theorem 2.3. *Let $u(t)$, $f(t)$, $h(t)$ and $g(t)$ be nonnegative continuous functions in a real interval $I = [a,b]$. Suppose that $h'(t)$ exists and is a nonnegative continuous function. If the following inequality*

$$u(t) \le c + \int_a^t f(s)u(s)ds + \int_a^t f(s)h(s)\left(\int_a^s g(\tau)u(\tau)d\tau\right) ds \; a \le \tau \le s \le t \le b,$$

holds, where c is a nonnegative constant, then

$$u(t) \le c\left[1 + \int_a^t f(s)\exp\left(\int_a^s (f(\tau) + g(\tau)h(\tau) + h'(\tau)\int_a^\tau g(\sigma)d\sigma)d\tau\right) ds\right].$$

Proof. This follows by similar argument as in the proof of Theorem 2.1. We omit the details. □

Remark 2.4. If in Theorem 2.3, we set $h(t) = 1$, then our result reduces to Theorem 1 obtained in [5].

Remark 2.5. If in Theorem 2.3, $h'(t) = 0$ then our estimate is more general than Theorem 1 obtained by Pachpatte in [5].

Lemma 2.6. *Let $v(t)$ be a positive differentiable function satisfying the inequality*

$$v'(t) \le f(t)v(t) + g(t)v^p(t),\; t \in I = [a,b], \tag{2.10}$$

where the functions $f(t)$ and $g(t)$ are continuous in I, and $p \ge 0,\; p \ne 1$, is a constant. Then

$$v(t) \le \exp\left(\int_a^t f(s)ds\right)\left[v^q(a) + q\int_a^t g(s)\exp\left(-q\int_a^s f(\tau)d\tau\right)ds\right]^{\frac{1}{q}}, \tag{2.11}$$

for $t, s \in [a,\beta)$, where $q = 1-p$ and β is chosen so that the expression

$$\left[v^q(a) + q\int_a^t g(s)\exp\left(-q\int_a^s f(\tau)d\tau\right)ds\right]^{\frac{1}{q}}$$

is positive in the subinterval $[a,\beta)$.

Proof. We reduce (2.10) to a simpler differential inequality by the following substitution. Let

$$z(t) = \frac{v^q(t)}{q}.$$

Then

$$\begin{aligned} z'(t) &= v^{q-1}(t) \times v'(t) \\ &\le v^{q-1}(t)\left(f(t)v(t) + g(t)v^p(t)\right), \quad \text{(by (2.10))} \\ &= qf(t)z(t) + g(t) \text{ (since } q = 1-p\text{)}. \end{aligned} \tag{2.12}$$

By Lemma 1.1 [1], (2.12) gives

$$z(t) \le \frac{v^q(a)}{q}\exp\left(\int_a^t qf(s)ds\right) + \int_a^t g(s)\exp\left(\int_s^t qf(\tau)d\tau\right)ds.$$

That is

$$v^q(t) \le \exp\left(\int_a^t qf(s)ds\right)\left[v^q(a) + \int_a^t g(s)\exp\left(-\int_a^s qf(\tau)d\tau\right)ds\right].$$

From this, it follows that

$$v(t) \leq \exp\left(\int_a^t f(s)ds\right)\left[c^q + q\int_a^t g(s)\exp\left(-q\int_a^s f(\tau)d\tau\right)ds\right]^{\frac{1}{q}}.$$

□

Theorem 2.7. *Let $u(t)$, $f(t)$ be nonnegative continuous functions in a real interval $I = [a, b]$. Suppose that the partial derivatives $k_t(t, s)$ exist and are nonnegative continuous functions for almost every $t, s \in I$. If the the inequality*

$$u(t) \leq c + \int_a^t f(s)u(s)ds + \int_a^t f(s)\left(\int_a^s k(s,\tau)u^p(\tau)d\tau\right)ds, \quad a \leq \tau \leq s \leq t \leq b \tag{2.13}$$

holds, where $0 \leq p < 1, q = 1 - p$ and $c > 0$ are constants.
Then

$$u(t) \leq c + \int_a^t f(s)\exp\left(\int_a^s f(\tau)d\tau\right) \times \left[c^{1-p} + (1-p)\int_a^s k(\tau,\tau)\exp\left(-(1-p)\int_a^\tau f(\sigma)d\sigma\right)d\tau\right]^{\frac{1}{1-p}} ds. \tag{2.14}$$

Proof. Define a function $v(t)$ by the right hand side of (2.13) from which it follows that

$$u(t) \leq v(t). \tag{2.15}$$

Then

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)\int_a^t k(t,\tau)u^p(\tau)d\tau, \quad v(a) = c \\ &\leq f(t)\left(v(t) + \int_a^t k(t,\tau)v^p(\tau)d\tau\right). \text{ (by (2.15))} \end{aligned} \tag{2.16}$$

If we put

$$m(t) = v(t) + \int_a^t k(t,\tau)v^p(\tau)d\tau, \tag{2.17}$$

then it is clear that

$$v(t) \leq m(t). \tag{2.18}$$

Hence

$$\begin{aligned} m'(t) &= v'(t) + k(t,t)v^p(t) + \int_a^t k_t(t,\tau)v^p(\tau)d\tau, \quad m(a) = v(a) = c \\ &\leq v'(t) + k(t,t)v^p(t), \\ &\leq f(t)m(t) + k(t,t)v^p(t), \text{ (by (2.16))} \\ &\leq f(t)m(t) + k(t,t)m^p(t). \text{ (by (2.18))} \end{aligned} \tag{2.19}$$

By Lemma 2.6 we have

$$m(t) \leq \exp\left(\int_a^t (f(s)ds\right)\left[m^q + q\int_a^s k(s,s)\exp\left(-q\int_a^s f(\tau)d\tau\right)ds\right]^{\frac{1}{q}}. \tag{2.20}$$

Substituting (2.20) into (2.16), we have

$$\text{(2.21)} \qquad v'(t) \le f(t)\exp\left(\int_a^t (f(s)ds\right)\left[m^q + q\int_a^s k(s,s)\exp\left(-q\int_a^s f(\tau)d\tau\right)ds\right]^{\frac{1}{q}}.$$

Integrate both sides of (2.21) from a to t and using (2.15), we obtain

$$u(t) \le c + \int_a^t f(s)\exp\left(\int_a^s f(\tau)d\tau\right)\Big[c^{1-p} + (1-p)\int_a^s k(\tau,\tau)\exp\left(-(1-p)\int_a^\tau f(\sigma)d\sigma\right)d\tau\Big]^{\frac{1}{1-p}} ds.$$

This completes the proof of the theorem □

Remark 2.8. If in Theorem 2.7, we put $k(t,s) = g(s)$, then our result reduces to Theorem 2 obtained in [5].

Theorem 2.9. *Let $u(t)$, $f(t)$, $h(t)$ and $g(t)$ be nonnegative continuous functions in a real interval $I = [a,b]$. Suppose that $h'(t)$ exists and is a nonnegative continuous function. If the following inequality*

$$\text{(2.22)} \quad u(t) \le c + \int_a^t f(s)u(s)ds + \int_a^t f(s)h(s)\left(\int_a^s g(\tau)u^p(\tau)d\tau\right)ds \; a \le \tau \le s \le t \le b,$$

holds, where $0 \le p < 1, q = 1-p$ and $c > 0$ are nonnegative constant. Then

$$\text{(2.23)} \quad u(t) \le c + \int_a^t f(s)\exp\left(\int_a^s f(\tau)d\tau\right)\Big[c^{1-p} + (1-p)\int_a^s (h(\tau)f(\tau) + h'(\tau)\int_a^\tau f(\sigma)d\sigma\Big)\exp\left(-(1-p)\int_a^\tau f(\sigma)d\sigma\right)d\tau\Big]^{\frac{1}{1-p}} ds.$$

Proof. This follows by similar argument as in the proof of Theorem 2.7. We also omit the details. □

Remark 2.10. If in Theorem 2.9, we set $h(t) = 1$ then our result reduces to the estimate in Theorem 2 obtained by Pachpatte in [5].

Remark 2.11. If in Theorem 2.9, $h'(t) = 0$ then our result is more general than Theorem 2 obtained in [5].

3. APPLICATIONS

There are many applications of the inequalities obtained in Section 2. Here we shall give an application which is just sufficient to convey the importance of our results. We shall consider the nonlinear integrodifferential equation

$$\text{(3.1)} \qquad x'(t) = f(t,u(t)) + \int_{t_0}^t g\left(t,s,x(s)\right)ds,$$

and the corresponding perturbed equation

$$\text{(3.2)} \qquad u'(t) = f(t,u(t)) + \int_{t_0}^t g\left(t,s,u(s)\right)ds + h\left(t,u(t),\int_{t_0}^t k(t,s,u(s))ds\right)$$

for all $t_0, t \in \mathbb{R}^+$ and $x,u,f,g,h \in \mathbb{R}^n$.

If we let $x(t) = x(t;t_0,x_0)$ and $u(t) = u(t;t_0,x_0)$ be the solutions of (3.1) and (3.2) respectively with $x(t_0) = u(t_0) = x_0$ and $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $f_x : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$,

$g, k : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g_x : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions in their respective domains. Then we have by [2] that $\frac{\partial x}{\partial x_0}(t, t_0, x_0) = \Phi(t, t_0, x_0)$ exists and satisfies the variational equation

$$x'(t) = f_x(t, x(t; t_0, x_0))z(t) + \int_{t_0}^{t} g_x\left(t, s, x(s; t_0, x_0)\right) z(s)ds, \quad z(t_0) = I \tag{3.3}$$

and

$$\frac{\partial x}{\partial t_0}(t; t_0, x_0) + \Phi(t, t_0, x_0) f(t_0, x_0) \int_{t_0}^{t} \Phi(t, s, x_0) g(s, t_0, x_0) ds = 0. \tag{3.4}$$

Thus the solutions $x(t)$ and $u(t)$ are related by

$$u(t) = x(t) \int_{t_0}^{t} \Phi(t, s, u(s)) h\left(s, u(s), \int_{t_0}^{t} k(s, \tau, u(\tau)) d\tau\right) ds. \tag{3.5}$$

Theorem 3.1. *Let f, f_x, g, g_x, k, h, as earlier defined, be nonnegative continuous functions. Suppose that the following inequalities hold:*

$$|\Phi(t, s, u)| \leq Me^{-\alpha(t-s)}, \tag{3.6}$$

$$|\Phi(t, s, u)h(s, u, z)| \leq p(s)\left(|u| + |z|\right), \tag{3.7}$$

$$|k(t, s, u)| \leq q(s, s)\,|y| \tag{3.8}$$

for $0 \leq s \leq t,\ u, z \in \mathbb{R}^n,\ M \geq 1$ and $\alpha > 0$ are constants. If $p(t)$ and $q(t, t)$ are continuous and nonnegative and

$$\int^{\infty} p(s)ds < \infty, \ \int^{\infty} q(s, s)ds < \infty. \tag{3.9}$$

Then for any bounded solution $x(t; t_0, x_0)$ of (3.1) in $\mathbb{R}^+$, then the corresponding solutions of (3.2) is bounded in $\mathbb{R}^+$.

Proof. We have from (3.6)– (3.8) that equation (3.2) gives

$$|u(t)| \leq M\,|x_0| + \int_{t_0}^{t} p(s)\,|u(s)|\,ds + \int_{t_0}^{t} p(s)\left(\int_{t_0}^{t} q(\tau, \tau)\,|u(\tau)|\,d\tau\right) ds.$$

Hence by Theorem 2.1, we have

$$|u(t)| \leq M\,|x_0| \left[1 + \int_{t_0}^{t} p(s) \exp\left(\int_{s_0}^{s} (p(\tau) + q(\tau, \tau)) d\tau\right) ds\right].$$

Hence by (3.9), we easily see that $|u(t)|$ is bounded and the proof is complete. □

REFERENCES

[1] D. BAINOV AND P. SIMEONOV, *Integral inequalities and Applications,* Academic Publishers, Dordrecht, 1992.

[2] F. BRAUER, A nonlinear variation of constants formula for Volterra equations, *Mat. Systems Th.*, **6** (1972), 226 – 234.

[3] J. CHANDRA AND B.A. FLEISHMAN, On a generalization of Gronwall-Bellman lemma in partially ordered Banach spaces, *J. Math. Anal. Appl.*, **31** (1970), 668 – 681.

[4] J.A. OGUNTUASE, Remarks on Gronwall type inequalities, *An. Stiint. Univ. "Al. I. Cuza"*, t.**45** (1999), in press

[5] B.G. PACHPATTE, A note on Gronwall-Bellman inequality, *J. Math. Anal. Appl.*, **44** (1973), 758–762.