Journal of Inequalities in Pure and Applied Mathematics

ON A STRENGTHENED HARDY-HILBERT INEQUALITY

BICHENG YANG

Department of Mathematics
Guangdong Education College
Guangzhou
Guangdong 510303
PEOPLE'S REPUBLIC OF CHINA
EMail: bcyang@163.net

volume 1, issue 2, article 22, 2000.

Received 8 May, 2000; accepted 10 June 2000.

Communicated by: L. Debnath

Abstract
Contents
Home Page
Go Back
Close
Quit

ISSN (electronic): 1443-5756
012-00

Abstract

In this paper, a new inequality for the weight coefficient $W(n,r)$ of the form

$$W(n,r) = \sum_{m=0}^{\infty} \frac{1}{m+n+1} \left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}} \right)^{\frac{1}{r}} < \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{1}{13(n+1)(2n+1)^{1-\frac{1}{r}}} \quad (r>1,\ n \in N_0 = N \cup \{0\})$$

is proved. This is followed by a strengthened version of the more accurate Hardy-Hilbert inequality.

2000 Mathematics Subject Classification: 26D15
Key words: Hardy-Hilbert inequality, Weight Coefficient, Hölder's inequality.

Contents

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, and $0 < \sum_{n=1-\lambda}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1-\lambda}^{\infty} b_n^q < \infty$ $(\lambda = 0, 1)$, then the Hardy-Hilbert inequality is

$$\sum_{m=1-\lambda}^{\infty} \sum_{n=1-\lambda}^{\infty} \frac{a_m b_n}{m+n+\lambda} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\sum_{n=1-\lambda}^{\infty} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1-\lambda}^{\infty} b_n^q \right)^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\pi / \sin(\pi/p)$ is best possible (see [3, Chapter 9]). Inequality (1.1) is important in analysis and it's applications (see [4, Chapter 5]). In recent years, Yang and Gao [2, 7], have given a strengthened version of (1.1) for $\lambda = 0$ as

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1-\gamma}{n^{\frac{1}{p}}} \right] a_n^p \right\}^{\frac{1}{p}} \times \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1-\gamma}{n^{\frac{1}{q}}} \right] b_n^q \right\}^{\frac{1}{q}}, \tag{1.2}$$

where $\gamma = 0.5772^{+}$, is Euler's constant. Later, Yang and Debnath [6] proved a distinctly strengthened version of (1.1) for $\lambda = 0$ as

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{p}} + n^{-\frac{1}{q}}}\right] a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{q}} + n^{-\frac{1}{p}}}\right] b_n^q\right\}^{\frac{1}{q}}, \tag{1.3}$$

which is not comparable with (1.2).

Inequality (1.1) for $\lambda = 1$ is called the more accurate Hardy-Hilbert's inequality, which has been strengthened as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{\ln 2 - \gamma}{(2n+1)^{1+\frac{1}{p}}}\right] a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{\ln 2 - \gamma}{(2n+1)^{1+\frac{1}{q}}}\right] b_n^q\right\}^{\frac{1}{q}}, \tag{1.4}$$

by introducing an inequality of the weight coefficient $W(n, r)$ in the form (see

[5, equation (2.9)]):

$$W(n,r) = \sum_{m=0}^{\infty} \frac{1}{m+n+1}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{r}} < \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\ln 2-\gamma}{(2n+1)^{2-\frac{1}{r}}} \quad (r>1,\ n\in N_0)\,. \tag{1.5}$$

In this paper we will give another strengthened version of (1.1) for $\lambda = 1$, which is not comparable with (1.4). We need some preparatory works.

2. Some Lemmas

Lemma 2.1. *If* $f \in C^6[0,\infty)$, $f^{(2q)}(x) > 0$ $(q = 0,1,2,3)$, $f^{(i-1)}(\infty) = 0$ $(i = 1,2,\ldots,6)$, *and* $\sum_{n=0}^{\infty} f(n) < \infty$, *then we have*

$$\sum_{k=2}^{\infty} (-1)^k f(k) > \frac{1}{2} f(2) \quad \text{(see [1, equation (4.4)])}, \tag{2.1}$$

$$\sum_{m=0}^{\infty} f(m) = \int_0^{\infty} f(t)\,dt + \frac{1}{2} f(0) + \int_0^{\infty} \bar{B}_1 f'(t)\,dt, \tag{2.2}$$

$$\int_0^{\infty} \bar{B}_1 f'(t)\,dt = -\frac{1}{12} f'(0) + \delta_2,\ \delta_2 = \frac{1}{6} \int_0^{\infty} \bar{B}_3 f'''(t)\,dt < 0, \tag{2.3}$$

where $\bar{B}_i(t)$ $(i = 1,3)$ *are Bernoulli functions (see [5, equations (1.7)-(1.9)]).*

Setting the weight coefficient $W(n,r)$ in the form:

$$\begin{aligned} W(n,r) &= \sum_{m=0}^{\infty} \frac{1}{m+n+1} \left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}} \right)^{\frac{1}{r}} \\ &= \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\theta(n,r)}{(2n+1)^{2-\frac{1}{r}}} \quad (r > 1,\ n \in N_0), \end{aligned} \tag{2.4}$$

then we find

$$\theta(n,r) = \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} (2n+1)^{2-\frac{1}{r}} - (2n+1)^2 \sum_{m=0}^{\infty} \frac{1}{m+n+1} \left(\frac{1}{2m+1} \right)^{\frac{1}{r}}. \tag{2.5}$$

If we define the function $f(x)$ as $f(x) = \frac{1}{(x+n+1)}\left(\frac{1}{2x+1}\right)^{\frac{1}{r}}$, $x \in [0,\infty)$, then we have $f(0) = \frac{1}{(n+1)}$,

$$f'(x) = -\frac{1}{(x+n+1)^2}\left(\frac{1}{2x+1}\right)^{\frac{1}{r}} - \frac{2}{r(x+n+1)}\left(\frac{1}{2x+1}\right)^{1+\frac{1}{r}},$$
$$f'(0) = -\frac{1}{(n+1)^2} - \frac{2}{r(n+1)},$$

and

$$\int_0^\infty f(x)\,dx = \frac{1}{(2n+1)^{\frac{1}{r}}}\int_{\frac{1}{2n+1}}^\infty \frac{1}{(y+1)}\left(\frac{1}{y}\right)^{\frac{1}{r}} dy$$
$$= \frac{1}{(2n+1)^{\frac{1}{r}}}\left[\frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\left(1-\frac{1}{r}+\nu\right)(2n+1)^{\nu+1-\frac{1}{r}}}\right].$$

By (2.2) and (2.5), we have

(2.6)
$$\theta(n,r) = -\frac{(2n+1)^2}{2(n+1)} + \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\left(1-\frac{1}{r}+\nu\right)(2n+1)^{\nu-1}}$$
$$+\int_0^\infty \bar{B}_1(t)\left[\frac{(2n+1)^2}{(t+n+1)^2(2t+1)^{\frac{1}{r}}} + \frac{2(2n+1)^2}{r(t+n+1)(2t+1)^{1+\frac{1}{r}}}\right]dt.$$

By Lemma 4 of [5, p. 1106] and (2.4), we have

Lemma 2.2. *For $r > 1$, $n \in N_0$, we have $\theta(n,r) > \theta(n,\infty)$, and*

$$W(n,r) < \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\theta(n,\infty)}{(2n+1)^{2-\frac{1}{r}}} \quad (r > 1,\ n \in N_0), \tag{2.7}$$

where

$$\theta(n,\infty) = -\frac{(2n+1)^2}{2(n+1)} + \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(1+\nu)(2n+1)^{\nu-1}} + \int_0^\infty \bar{B}_1(t)\left[\frac{(2n+1)^2}{(t+n+1)^2}\right] dt. \tag{2.8}$$

Since by (2.3) and (2.1),we have

$$\int_0^\infty \bar{B}_1(t)\frac{1}{(t+n+1)^2}dt = -\frac{1}{12(n+1)^2} - \frac{1}{3!}\int_0^\infty \bar{B}_3(t)\left[\frac{1}{(t+n+1)}\right]''' dt$$
$$> -\frac{1}{12(n+1)^2}$$

and

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(1+\nu)(2n+1)^{\nu-1}} = (2n+1) - \frac{1}{2} + \sum_{\nu=2}^{\infty} \frac{(-1)^\nu}{(1+\nu)(2n+1)^{\nu-1}}$$
$$> (2n+1) - \frac{1}{2} + \frac{1}{6(2n+1)}.$$

Then by (2.8), we find

$$\theta(n,\infty) > \frac{1}{6} - \frac{1}{6(n+1)} - \frac{1}{12(n+1)^2} + \frac{1}{6(2n+1)}. \tag{2.9}$$

Lemma 2.3. *For $r > 1$, $n \in N_0$, we have*

$$W(n,r) < \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{1}{13\,(n+1)\,(2n+1)^{1-\frac{1}{r}}}. \tag{2.10}$$

Proof. Define the function $g(x)$ as

$$g(x) = \frac{1}{12} - \frac{1}{6\,(2x+1)} + \frac{1}{12\,(x+1)} + \frac{1}{12\,(2x+1)^2}, \quad x \in [0,\infty).$$

Then by (2.8), we have $\theta\,(n,\infty) > \frac{2n+1}{n+1} g\,(n)$. Since $g(1) > 0.0787 > \frac{1}{13}$, and for $x \in [1,\infty)$,

$$g'\,(x) = \frac{1}{3\,(2x+1)^2} - \frac{1}{12\,(x+1)^2} - \frac{1}{3\,(2x+1)^3} = \frac{4x^2+2x-1}{12\,(x+1)^2\,(2x+1)^3} > 0,$$

then for $n \geq 1$, we have $\theta\,(n,\infty) > \frac{2n+1}{(n+1)} g\,(1) > \frac{2n+1}{13(n+1)}$. Hence by (2.7), inequality (2.10) is valid for $n \geq 1$. Since $\ln 2 - \gamma = 0.1159^{+} > \frac{1}{13}$, then by (1.5), we find

$$W(0,r) < \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\ln 2 - \gamma}{(2\times 0+1)^{2-\frac{1}{r}}} < \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{1}{13\,(0+1)\,(2\times 0+1)^{1-\frac{1}{r}}}. \tag{2.11}$$

It follows that (2.10) is valid for $r > 1$,and $n \in N_0$. This proves the lemma. □

3. Theorem and Remarks

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, *and* $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, *then*

(3.1)
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{13(n+1)(2n+1)^{\frac{1}{p}}}\right]a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{13(n+1)(2n+1)^{\frac{1}{q}}}\right]b_n^q\right\}^{\frac{1}{q}},$$

(3.2)
$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{a_n}{m+n+1}\right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{p-1}\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{13(n+1)(2n+1)^{\frac{1}{p}}}\right]a_n^p.$$

Proof. By Hölder's inequality, we have

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} \\
&\quad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{a_m}{(m+n+1)^{\frac{1}{p}}}\left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right)^{\frac{1}{pq}}\right]\left[\frac{b_n}{(m+n+1)^{\frac{1}{q}}}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{pq}}\right] \\
&\quad \le \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right)^{\frac{1}{q}}\right]a_m^p\right\}^{\frac{1}{p}} \\
&\qquad\qquad \times\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{p}}\right]b_n^q\right\}^{\frac{1}{q}} \\
&\quad = \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right)^{\frac{1}{q}}\right]a_m^p\right\}^{\frac{1}{p}} \\
&\qquad\qquad \times\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{p}}\right]b_n^q\right\}^{\frac{1}{q}} \\
&\quad = \left\{\sum_{m=0}^{\infty}W(m,q)\,a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}W(n,p)\,b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Since $\sin(\pi/q)=\sin(\pi/p)$, by (2.10) for $r=p,q$, we have (3.1).

By (2.10), we have $W(n,p)<\pi/\sin(\pi/p)$. Then by Hölder's inequality, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{a_n}{m+n+1}\\
&=\sum_{n=0}^{\infty}\left[\frac{a_n}{(m+n+1)^{\frac{1}{p}}}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{pq}}\right]\left[\frac{1}{(m+n+1)^{\frac{1}{q}}}\left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right)^{\frac{1}{pq}}\right]\\
&\le\left\{\sum_{n=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{q}}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\frac{1}{(m+n+1)}\left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right)^{\frac{1}{p}}\right\}^{\frac{1}{q}}\\
&=\left\{\sum_{n=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{q}}\right]a_n^p\right\}^{\frac{1}{p}}\{W(m,p)\}^{\frac{1}{q}}\\
&<\left\{\sum_{n=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{q}}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right\}^{\frac{1}{q}}.
\end{aligned}$$

Then we find

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{a_n}{m+n+1}\right)^p\\
&\qquad<\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{q}}\right]a_n^p\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{\frac{p}{q}}
\end{aligned}$$

$$= \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{p-1} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{1}{(m+n+1)}\left(\frac{n+\frac{1}{2}}{m+\frac{1}{2}}\right)^{\frac{1}{q}}\right] a_n^p$$

$$= \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{p-1} \sum_{n=0}^{\infty} W(n,q)\, a_n^p.$$

Hence by (2.10) for $r = q$, we have (3.2). The theorem is proved. □

Remark 3.1. *Inequality (3.1) is a definite improvement over (1.1) for* $\lambda = 1$.

Remark 3.2. *Since for* $n \geq 2$, $13\,(2-\gamma) < 2 - \frac{1}{n+1}$, *then*

(3.3)
$$\frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\ln 2 - \gamma}{(2n+1)^{2-\frac{1}{r}}} > \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{1}{13\,(n+1)\,(2n+1)^{1-\frac{1}{r}}} \quad (r > 1,\ n \geq 2)\,.$$

In view of (2.11) and (3.3), it follows that (3.1) and (1.4) represent two distinct versions of strengthened inequalities. However, they are not comparable.

Remark 3.3. *Inequality (3.2) reduces to*

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{a_n}{m+n+1}\right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \sum_{n=0}^{\infty} a_n^p. \tag{3.4}$$

This is an equivalent form of the more accurate Hardy-Hilbert's inequality (see [3, Chapter 9]).

References

[1] R.P. BOAS, Estimating remainders, *Math. Mag.*, **51**(2) (1978), 83–89.

[2] M. GAO AND B. YANG, On the extended Hilbert's inequality, *Proc. Amer. Math. Soc.*, **126**(3) (1998), 751–759.

[3] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, 1934.

[4] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, 1991.

[5] B. YANG, On a strengthened version of the more accurate Hardy-Hilbert's inequality, *Acta Math. Sinica*, **42**(6) (1999), 1103–1110.

[6] B. YANG AND L. DEBNATH, On new strengthened Hardy-Hilbert's inequality, *Int. J. Math. Math. Sci.*, **21**(2) (1998), 403–408.

[7] B. YANG AND M. GAO, On a best value of Hardy-Hilbert's inequality, *Adv. Math.*, **26**(2) (1997), 156–164.