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INEQUALITIES FOR POWER-EXPONENTIAL FUNCTIONS

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ABSTRACT. The following inequalities for power-exponential functions are proved

$$\frac{y^{x^y}}{x^{y^x}} > \frac{y}{x} > \frac{y^x}{x^y}, \quad \left(\frac{y}{x}\right)^{xy} > \frac{y^y}{x^x},$$

where $0 < x < y < 1$ or $1 < x < y$.

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1. INTRODUCTION

It is well-known that, if $0 < x < y < e$, then

$$x^y < y^x. \tag{1.1}$$

If $e < x < y$, then inequality (1.1) is reversed. If $0 < x < e$, then

$$(e+x)^{e-x} > (e-x)^{e+x}. \tag{1.2}$$

For details about these inequalities, please refer to [1, p. 82] and [3, p. 365].

In [3, p. 365 and p. 768], an open problem was proposed: How do we compare the value of a^b with that of b^a for $1 < a < e < b$? Although it looks like a simple problem, not much progress has been made on it. Recently, some discussion was given in [1, p. 82] by Professor P.S. Bullen. Moreover, more detailed discussion on this open problem was given in [4] by Mr. Z. Luo and J.-J. Wen.

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There is a rich literature on inequalities for power-exponential functions, see [1, 2, 3].

In this paper, based on the revised Cauchy's mean-value theorem in integral form [7, 8], we will give some new inequalities for power-exponential functions, and propose an open problem.

2. Main Results

Theorem 2.1 (Main Theorem). *For $0 < x < y < 1$ or $1 < x < y$, we have*

$$\frac{y^{x^y}}{x^{y^x}} > \frac{y}{x} > \frac{y^x}{x^y}, \tag{2.1}$$

$$\left(\frac{y}{x}\right)^{xy} > \frac{y^y}{x^x}. \tag{2.2}$$

For $0 < x < 1 < y$, the right hand side of (2.1) *and* (2.2) *are reversed.*

If $0 < x < 1 < y$ or $0 < x < y < e$, then

$$1 < \frac{y \ln x}{x \ln y} \cdot \frac{y^x - 1}{x^y - 1} < \frac{y^x}{x^y}. \tag{2.3}$$

If $e < x < y$, inequality (2.3) *is reversed.*

First Proof of Theorem 2.1. We first prove the right hand side of inequality (2.1)

$$\frac{y}{x} > \frac{y^x}{x^y}, \tag{2.4}$$

where $0 < x < y < 1$ or $1 < x < y$. This inequality is equivalent to

$$\ln y - \ln x > x \ln y - y \ln x,$$

which can be written as

$$\frac{\frac{\ln y}{y} - \frac{\ln x}{x}}{\ln y - \ln x} < \frac{1}{xy}. \tag{2.5}$$

Since $\frac{d}{dt}\left(\frac{\ln t}{t}\right) = \frac{1-\ln t}{t^2}$, an integral form of inequality (2.4) follows from (2.5), that is

$$\int_x^y \frac{1 - \ln t}{t^2}\, dt < \frac{1}{xy} \int_x^y \frac{1}{t}\, dt. \tag{2.6}$$

The reciprocal change of variables in (2.6) gives

$$\int_{1/y}^{1/x} (1 + \ln t)\, dt < \frac{1}{xy} \int_{1/y}^{1/x} \frac{1}{t}\, dt. \tag{2.7}$$

Substituting $u = \frac{1}{x}$ and $v = \frac{1}{y}$ in (2.7) yields the following result

$$\int_v^u (1 + \ln t)\, dt < uv \int_v^u \frac{1}{t}\, dt. \tag{2.8}$$

In order to prove (2.4), it is sufficient to show (2.6) for $1 < x < y$, and (2.8) for $1 < v < u$. Introduce the following

$$f(x,y) = \int_x^y \frac{1 - \ln t}{t^2}\, dt - \frac{1}{xy} \int_x^y \frac{1}{t}\, dt, \quad y > x > 1; \tag{2.9}$$

$$h(u,v) = \int_v^u (1 + \ln t)\, dt - uv \int_v^u \frac{1}{t}\, dt, \quad u > v > 1. \tag{2.10}$$

After some straightforward calculations, we obtain the following results

$$\begin{aligned}
\frac{\partial f(x,y)}{\partial y} &= \left(\int_x^y \frac{1}{t}\,dt + x(1-\ln y) - 1\right)\frac{1}{xy^2} \equiv \frac{g(x,y)}{xy^2},\\
\frac{\partial g(x,y)}{\partial y} &= \frac{1-x}{y} < 0,\\
f(x,x) &= 0,\\
g(x,x) &= x(1-\ln x) - 1,\\
\frac{dg(x,x)}{dx} &= -\ln x < 0,\\
g(1,1) &= 0,\\
\frac{\partial h(u,v)}{\partial u} &= 1 + \ln u - v - v\int_v^u \frac{1}{t}\,dt,\\
\frac{\partial^2 h(u,v)}{\partial u^2} &= \frac{1-v}{u} < 0.
\end{aligned}$$

Since $\frac{dg(x,x)}{dx} < 0$, then $g(x,x)$ decreases, thus $g(x,x) < g(1,1) = 0$. From $\frac{\partial g(x,y)}{\partial y} < 0$, we have that $g(x,y)$ decreases in y, so $g(x,y) < g(x,x) < 0$. Then $\frac{\partial f(x,y)}{\partial y} < 0$, $f(x,y)$ decreases in y, therefore $f(x,y) < f(x,x) = 0$. This completes the proof of inequality (2.6) for $1 < x < y$.

Since $\frac{\partial^2 h(u,v)}{\partial u^2} < 0$, then $\frac{\partial h(u,v)}{\partial u}$ decreases in u, hence

$$\frac{\partial h(u,v)}{\partial u} < \left.\frac{\partial h(u,v)}{\partial u}\right|_{u=v} = 1 - v + \ln v < 0,$$

so $h(u,v)$ decreases in u, then $h(u,v) < h(v,v) = 0$. This completes the proof of (2.8) for $u > v > 1$.

Next, we prove the left hand side of inequality (2.1)

$$\frac{y^{x^y}}{x^{y^x}} > \frac{y}{x}, \tag{2.11}$$

where $0 < x < y < 1$ or $1 < x < y$. We can rewrite (2.11) in the form

$$x^y \ln y - y^x \ln x > \ln y - \ln x. \tag{2.12}$$

This is equivalent to

$$\frac{x^y - 1}{y^x - 1} > \frac{\ln x}{\ln y}. \tag{2.13}$$

Since $x^y - 1 = (\ln x)\int_0^y x^t\,dt$, $y^x - 1 = (\ln y)\int_0^x y^t\,dt$, then inequality (2.13) can be rewritten in the integral form

$$\frac{\int_0^y x^t\,dt}{\int_0^x y^t\,dt} > 1. \tag{2.14}$$

Making the change of variables, $t = ys$, gives

$$\int_0^y x^t\,dt = y\int_0^1 (x^y)^s\,ds, \tag{2.15}$$

$$\int_0^x y^t\,dt = x\int_0^1 (y^x)^s\,ds. \tag{2.16}$$

Therefore, the equivalent form of (2.14) is

$$\frac{\int_0^1 (y^x)^s\,ds}{\int_0^1 (x^y)^s\,ds} < \frac{y}{x}. \tag{2.17}$$

Hence, it is sufficient to show that inequality (2.17) is valid for $0 < x < y < 1$ or $1 < x < y$.

From the revised Cauchy's mean value theoren in integral form in [7, 8], we get

$$\frac{\int_0^1 (y^x)^s\,ds}{\int_0^1 (x^y)^s\,ds} = \left(\frac{y^x}{x^y}\right)^\theta, \quad \theta \in (0,1). \tag{2.18}$$

Using inequality (2.4) leads to

$$\left(\frac{y^x}{x^y}\right)^\theta < \left(\frac{y}{x}\right)^\theta < \frac{y}{x}.$$

Thus the inequality (2.17) is proved and the proof of (2.11) is complete.

It follows from (2.4) and (2.11) that the inequality (2.1) holds.

It is clear that inequality (2.2) is equivalent to

$$\frac{y}{x} > \frac{(y-1)\ln x}{(x-1)\ln y}. \tag{2.19}$$

It is evident that

$$\frac{(y-1)\ln x}{(x-1)\ln y} = \frac{\ln x^{y-1}}{\ln y^{x-1}} = \frac{\ln x^y - \ln x}{\ln y^x - \ln y} = \frac{\int_0^1 y^s\,ds}{\int_0^1 x^s\,ds} = \left(\frac{y}{x}\right)^\theta < \frac{y}{x}, \quad \theta \in (0,1).$$

This leads to the inequality (2.2).

Finally, inequality (2.3) can easily be derived from (1.1) and (2.18). Making similar arguments as above enables us to establish the reversed inequalities. □

Second Proof of Inequalities (2.2) *and* (2.4). It is easy to see that

$$t > 1 + \ln t, \quad t > 0, t \neq 1.$$

Therefore

$$\left(\frac{t\ln t}{t-1}\right)' = \frac{t-1-\ln t}{(t-1)^2} > 0,$$

and the function $\frac{t\ln t}{t-1}$ is increasing. This gives

$$\frac{y\ln y}{y-1} > \frac{x\ln x}{x-1}, \quad 1 < x < y \text{ or } 0 < x < y < 1.$$

This can be written as

$$xy(\ln y - \ln x) > y\ln y - x\ln x.$$

Thus, the desired inequality (2.2) follows.

Since $t < 1 + \ln t$ for $t \neq 1$ and $t > 0$, we have

$$\left(\frac{\ln t}{t-1}\right)' = \frac{t-1-t\ln t}{t(t-1)^2} < 0,$$

that is, the function $\frac{\ln t}{t-1}$ is decreasing, thus

$$\frac{\ln y}{y-1} < \frac{\ln x}{x-1},$$
$$\ln y - \ln x > x\ln y - y\ln x$$

hold for $0 < x < y < 1$ or $1 < x < y$. This yields inequality (2.4). □

Remark 2.2. It has been pointed out by Professor P.S. Bullen that inequality (2.4), the right hand side of inequality (2.1), is equivalent to inequality (2.2), this can be seen only if we replace x, y by $\frac{1}{x}, \frac{1}{y}$ respectively.

3. OPEN PROBLEM

Adopting the following notations:

$$f_1(x,y) = x, \tag{3.1}$$

$$f_{k+1}(x,y) = x^{f_k(y,x)}, \tag{3.2}$$

$$F_k(x,y) = \frac{f_k(y,x)}{f_k(x,y)} \tag{3.3}$$

for $0 < x < y < 1$ or $1 < x < y$, and $k \geqslant 1$.

The following inequalities need to be proved or disproved

$$F_{2k-1}(x,y) > F_{2k}(x,y), \tag{3.4}$$

$$F_{2k+4}(x,y) > F_{2k+1}(x,y). \tag{3.5}$$

That is,

$$F_2(x,y) < F_1(x,y) < F_4(x,y) < F_3(x,y) < F_6(x,y) < \cdots. \tag{3.6}$$

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