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ON INTEGRAL INEQUALITIES OF GRONWALL-BELLMAN-BIHARI TYPE IN SEVERAL VARIABLES

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ABSTRACT. We present some new results on the linear and non-linear integral inequalities of Gronwall-Bellman-Bihari type to n-dimensional integrals with a kernel of the form $k(x,t)$ where x and t are in $S \subset \mathbb{R}^n$.

These inequalities extend and compliment some existing results in the literature on Gronwall-Bellman-Bihari type inequalities.

Key words and phrases: Gronwall-Bellman-Bihari inequality, good kernel, nonincreasing function, nondecreasing function, nonnegative function.

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1. INTRODUCTION

The results obtained in this paper originated from the celebrated Gronwall-Bellman-Bihari inequality which has been of vital importance in the study of existence, uniqueness, continuous dependence, comparison, perturbation, boundedness and stability of solutions of differential and integral equations (see for example [1, 2, 3, 4, 5, 6] and the references cited therein).

In the last three decades, more than one variable generalizations of these inequalities have been obtained and these results have generated a lot of research interests due to its usefulness in the theory of differential and integral equations (see for example [1, 3, 6, 7, 8, 9, 10] and the references cited therein).

The purpose of this paper is to establish some new integral inequalities in n independent variables which will compliment the existing results in the literature on Gronwall- Bellman-Bihari type inequalities in several variables.

Throughout this paper, we shall assume that S is any bounded open set in the n dimensional Euclidean space $\mathbb{R}^n$ and that our integrals are on $\mathbb{R}^n (n \geq 1)$, unless otherwise specified.

For $x = (x_1, x_2, \ldots, x_n),\ x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in S$, we shall denote the integral

$$\int_{x_1^0}^{x_1} \int_{x_1^0}^{x_2} \cdots \int_{x_1^0}^{x_n} \ldots dt_n \ldots dt_1 \qquad \text{by} \qquad \int_{x^0}^{x} \ldots dt$$

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and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, \dots, n$.

Furthermore, for $x, t \in \mathbb{R}^n$, we shall write $t \le x$ whenever $t_i \le x_i, i = 1, 2, \dots, n$. Unless otherwise specified, all functions considered are functions of n-variables which are nonnegative and continuous on $[x^0, x], x \ge x^0 \ge 0$ and $x \in S$.

2. LINEAR INEQUALITIES

In this section, we shall obtain bounds to the linear Gronwall-Bellman-Bihari type integral inequalities for a more general kernel $k(x,t)$ and a product kernel $k(x,t) = h(x)f(t)$.

Definition 2.1. *A function $k(x,t)$ of the $2n$ variables $x_1, \dots, t_n$ is called a good kernel if*

(1) $k(\cdot, \cdot) \ge 0$.

(2) $k(\cdot, \cdot)$ *is a continuous function of its $2n$ variables.*

(3) $k(\cdot, \cdot)$ *is monotone non-decreasing in its first n variables, i.e. $k(x,t) \ge k(y,t)$ whenever $x \ge y$.*

Theorem 2.1. *Let $k(x,t)$ be a good kernel, $u(x)$ is a real valued nonnegative continuous function on S and $g(x)$ be a positive, nondecreasing continuous function on S. Suppose that the following inequality*

$$u(x) \le g(x) + \int_{x^0}^{x} k(x,t)u(t)dt \tag{2.1}$$

holds for all $x \in S$ with $x \ge x^0$, then

$$u(x) \le g(x) \left\{ 1 + \int_{x^0}^{x} k(s,s) \exp\left(\int_{x^0}^{s} k(t,t)dt \right) ds \right\}. \tag{2.2}$$

Proof. Since $g(x)$ is positive and nondecreasing, we can write (2.1) as

$$\frac{u(x)}{g(x)} \le 1 + \int_{x^0}^{x} k(x,t)\frac{u(t)}{g(t)}dt.$$

Setting $\frac{u(x)}{g(x)} = r(x)$, then we have

$$r(x) \le 1 + \int_{x^0}^{x} k(x,t)r(t)dt.$$

Let

$$v(x) = 1 + \int_{x^0}^{x} k(x,t)r(t)dt.$$

Then

$$r(x) \le v(x)$$

and $v(x^0) = 1$ or $x_i = x_i^0,\ i = 1, 2, \dots, n$. Hence

$$D_1 \dots D_n v(x) = k(x,x)r(x) \le k(x,x)v(x). \tag{2.3}$$

From (2.3) we obtain

$$\frac{v(x)D_1 \dots D_n v(x)}{v^2(x)} \le k(x,x).$$

That is

$$\frac{v(x)D_1 \dots D_n v(x)}{v^2(x)} \le k(x,x) + \frac{(D_n v(x))(D_1 \dots D_{n-1} v(x))}{v^2(x)}.$$

Hence

$$D_n \left(\frac{D_1 \dots D_{n-1} v(x)}{v(x)} \right) \le k(x,x).$$

Integrating with respect to x_n from x_n^0 to x_n, we have

$$\frac{D_1 \dots D_{n-1} v(x)}{v(x)} \leq \int_{x_n^0}^{x_n} k(x_1, x_2, \dots, x_{n-1}, t_n, x_1, x_2, \dots, x_{n-1}, t_n) dt_n.$$

Thus

$$\frac{v(x) D_1 \dots D_{n-1} v(x)}{v^2(x)} \leq \int_{x_n^0}^{x_n} k(x_1, x_2, \dots, x_{n-1}, t_n, x_1, x_2, \dots, x_{n-1}, t_n) dt_n + \frac{(D_{n-1} v(x))(D_1 \dots D_{n-2} v(x))}{v^2(x)}.$$

That is

$$D_{n-1} \left(\frac{D_1 \dots D_{n-2} v(x)}{v(x)} \right) \leq \int_{x_n^0}^{x_n} k(x_1, x_2, \dots, x_{n-1}, t_n, x_1, x_2, \dots, x_{n-1}, t_n) dt_n.$$

Integrating with respect to x_{n-1} from x_{n-1}^0 to x_{n-1}, we have

$$\frac{D_1 \dots D_{n-2} v(x)}{v(x)} \leq \int_{x_{n-1}^0}^{x_{n-1}} \int_{x_n^0}^{x_n} k(x_1, x_2, \dots, x_{n-2}, t_{n-1}, t_n, x_1, x_2, \dots, x_{n-2}, t_{n-1}, t_n) dt_n dt_{n-1}.$$

Continuing this process, we obtain

$$\frac{D_1 D_2 v(x)}{v(x)} \leq \int_{x_3^0}^{x_3} \dots \int_{x_n^0}^{x_n} k(x_1, x_2, t_3, \dots, t_n, x_1, x_2, t_3, \dots, t_n) dt_n \dots dt_3.$$

From this we obtain

$$D_2 \left(\frac{D_1 v(x)}{v(x)} \right) \leq \int_{x_3^0}^{x_3} \dots \int_{x_n^0}^{x_n} k(x_1, x_2, t_3, \dots, t_n, x_1, x_2, t_3, \dots, t_n) dt_n \dots dt_3.$$

Integrating with respect to the x_2 component from x_2^0 to x_2, we have

$$\frac{D_1 v(x)}{v(x)} \leq \int_{x_2^0}^{x_2} \dots \int_{x_n^0}^{x_n} k(x_1, t_2, t_3, \dots, t_n, x_1, t_2, t_3, \dots, t_n) dt_n \dots dt_2.$$

Integrating with respect to the x_1 component from x_1^0 to x_1, we obtain

$$\log \frac{v(x)}{v(x_1^0, x_2, \dots, x_n)} \leq \int_{x^0}^{x} k(t,t) dt.$$

That is

$$v(x) \leq \exp\left(\int_{x^0}^{x} k(t,t) dt \right). \tag{2.4}$$

Substituting (2.4) into (2.3) we have

$$D_1 \dots D_n r(x) \leq k(x,x) v(x) \leq k(x,x) \exp\left(\int_{x^0}^{x} k(t,t) dt \right).$$

Integrating this inequality with respect to the x_n component from x_n^0 to x_n, then with respect to the x_{n-1}^0 to x_{n-1}, and continuing until finally x_1^0 to x_1, and noting that $r(x) = 1$ at $x_i = x_i^0$, we have

$$r(x) \leq 1 + \int_{x^0}^{x} k(s,s) \exp\left(\int_{x^0}^{s} k(t,t) dt \right) ds.$$

Since $\frac{u(x)}{g(x)} = r(x)$, then we obtain

$$u(x) \leq g(x) \left\{ 1 + \int_{x^0}^{x} k(s,s) \exp\left(\int_{x^0}^{s} k(t,t) dt \right) ds \right\}.$$

This completes the proof of our result. □

Next, we shall consider the case in which $k(x,t) = h(x)f(t)$. Then we have the following result.

Theorem 2.2. *Let $h(x)$, $f(t)$, $u(x)$ be real valued nonnegative continuous functions on S and $g(x)$ be a positive, nondecreasing continuous function on S. If $h'(x) = 0$, where the prime denote $\frac{\partial^n}{\partial_{x_1}\ldots\partial_{x_n}}$ and the following inequality*

$$u(x) \leq g(x) + h(x)\int_{x^0}^{x} f(t)u(t)dt \tag{2.5}$$

holds for all $x \in S$ with $x \geq x^0$, then

$$u(x) \leq g(x)\left\{1 + \int_{x^0}^{x} h(s)f(s)\exp\left(\int_{x^0}^{s} h(t)f(t)dt\right)ds\right\}. \tag{2.6}$$

Proof. Similar to the proof of Theorem 2.1 and so the details are omitted. □

Remark 2.3. *If we set $k(x,t) = f(t)$ in Theorem 2.2, then our estimate reduces to*

$$u(x) \leq g(x)\left\{1 + \int_{x^0}^{x} f(s)\exp\left(\int_{x^0}^{s} f(t)dt\right)ds\right\}.$$

3. NON-LINEAR INEQUALITIES

Definition 3.1. *A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to belong to the class $\mathcal{F}$ if it satisfies the following conditions:*

(1) *ϕ is nondecreasing and continuous in $\mathbb{R}^+$ and $\phi(u) > 0$ for $u > 0$;*
(2) $\frac{1}{\alpha}\phi(u) \leq \phi\left(\frac{u}{\alpha}\right),\ u \geq 0,\ \alpha \geq 1.$

We observe from the above definition that $\mathcal{F}$ has the following properties:

(1) $\phi \in \mathcal{F}$ if and only if $\frac{\phi(u)}{u}$ is nonincreasing for $u > 0$;
(2) $\phi \in \mathcal{F}$ implies that ϕ is subadditive;
(3) If ϕ satisfies (1) of Definition 3.1 and is concave in $\mathbb{R}^+$, then $\phi \in \mathcal{F}$.

Theorem 3.1. *Let $k(x,t)$ be a good kernel and $u(x)$ be a real valued nonnegative continuous function on S. If $g(x)$ be a positive, nondecreasing continuous function on S and ϕ belong to class $\mathcal{F}$ for which the following inequality*

$$u(x) \leq g(x) + \int_{x^0}^{x} k(x,t)\phi(u(t))dt \tag{3.1}$$

holds for all $x \in S$ with $x \geq x^0$, then for $x^0 \leq x \leq x^*$,

$$u(x) \leq g(x)G^{-1}\left(G(1) + \int_{x^0}^{x} k(t,t)dt\right), \tag{3.2}$$

where

$$G(z) = \int_{z^0}^{z} \frac{ds}{\phi(s)},\ z \geq z^0 > 0,$$

G^{-1} is the inverse of G and x^* is chosen so that

$$G(1) + \int_{x^0}^{x} k(t,t)dt \in \mathrm{Dom}(G^{-1}).$$

Proof. Since $g(x)$ is positive and nondecreasing, we can write (3.1) as

$$\frac{u(x)}{g(x)} \leq 1 + \int_{x^0}^{x} k(x,t)\frac{\phi(u(t))}{g(t)}dt \leq 1 + \int_{x^0}^{x} k(x,t)\phi\left(\frac{u(t)}{g(t)}\right)dt.$$

Setting $\frac{u(x)}{g(x)} = v(x)$, then we have

$$v(x) \leq 1 + \int_{x^0}^{x} k(x,t)\phi(v(t))dt.$$

Let

$$r(x) = 1 + \int_{x^0}^{x} k(x,t)\phi(v(t))dt.$$

Then

$$v(x) \leq r(x)$$

and $v(x^0) = 1$ or $x_i = x_i^0,\ i = 1, 2, \dots, n$ and

$$D_1 \dots D_n r(x) = k(x,x)\phi(r(x)).$$

That is

$$\frac{D_1 \dots D_n r(x)}{\phi(r(x))} \leq k(x,x).$$

Since

$$D_n\left(\frac{D_1 \dots D_{n-1} r(x)}{\phi(v(x))}\right) = \frac{D_1 \dots D_n r(x)}{\phi(r(x))} - \frac{D_n\phi(r(x))D_1 \dots D_{n-1}r(x)}{\phi^2(r(x))}$$

and

$$D_n\phi(r(x)) = \phi'(r(x))D_n r(x) \geq 0,\ D_1 \dots D_{n-1}r(x) \geq 0.$$

The above inequality implies

$$D_n\left(\frac{D_1 \dots D_{n-1} r(x)}{\phi(r(x))}\right) \leq k(x,x)$$

provided $\phi'(r(x)) \geq 0$ for $r(x) \geq 0$.

Integrating with respect to x_n from x_n^0 to x_n and taking into account the fact that $D_1 \dots D_{n-1}r(x) = 0$ for $x_n = x_n^0$, we have

$$\frac{D_1 \dots D_{n-1} r(x)}{\phi(v(x))} \leq \int_{x_n^0}^{x_n} k(x_1, x_2, \dots, x_{n-1}, t_n, x_1, x_2, \dots, x_{n-1}, t_n)dt_n.$$

Repeating this, we find (after $n-1$ steps) that

$$\frac{D_1 r(x)}{\phi(r(x))} \leq \int_{x_1^0}^{x_1} \dots \left(\int_{x_n^0}^{x_n} k(x_1, \dots, x_{n-1}, t_n, x_1, \dots, x_{n-1}, t_n)dt_n\right) \dots dt_2.$$

We note that for

$$G(s) = \int_{s^0}^{s} \frac{dz}{\phi(z)},\ s \geq s^0 > 0.$$

It thus follows that

$$D_1 G(r(x)) = \frac{D_1 r(x)}{\phi(r(x))},$$

so that

$$D_1 G(r(x)) \leq \int_{x_2^0}^{x_2} k(x_1, t_2, \dots, t_n, x_1, t_2, \dots, t_n)dt_n \dots dt_2.$$

Integrating both sides of the above inequality with respect to the component

$$G(r(x_1, \dots, x_n)) - G(r(t_1, x_2, \dots, x_n)) \leq \int_{x^0}^{x} k(t,t)dt.$$

Since $r(t_1, x_2, \dots, x_n) = 1$ we have

$$r(x)) \leq G^{-1}\left(G(1) + \int_{x^0}^{x} k(t,t)dt\right).$$

From this we obtain

$$v(x) \leq r(x)) \leq G^{-1}\left(G(1) + \int_{x^0}^{x} k(t,t)dt\right).$$

Using the fact that $\frac{u(x)}{g(x)} = v(x)$, we have

$$u(x) \leq g(x)G^{-1}\left(G(1) + \int_{x^0}^{x} k(t,t)dt\right)$$

which is required and the proof is complete. □

If we set $k(x,t) = h(x)f(t)$, then we shall obtain the following result

Theorem 3.2. *Let $h(x)$, $f(t)$, $u(x)$ be real valued nonnegative continuous functions on S and $g(x)$ be a positive, nondecreasing continuous function on S, and ϕ belong to class $\mathcal{F}$ .If $h'(x) = 0$ and the following inequality*

$$u(x) \leq g(x) + h(x)\int_{x^0}^{x} f(t)\phi(u(t))dt \tag{3.3}$$

holds for all $x \in S$ with $x \geq x^0$, then for $x^0 \leq x \leq x^*$, then

$$u(x) \leq g(x)G^{-1}\left(G(1) + h(x)\int_{x^0}^{x} f(t)dt\right), \tag{3.4}$$

where

$$G(z) = \int_{z^0}^{z} \frac{ds}{\phi(s)},\ z \geq z^0 > 0,$$

G^{-1} is the inverse of G and x^* is chosen so that

$$G(1) + h(x)\int_{x^0}^{x} f(t)dt \in \mathrm{Dom}(G^{-1}).$$

Proof. Similar to the proof of Theorem 3.1 and so the details are omitted. □

Remark 3.3. *If we set $k(x,t) = f(t)$ in Theorem 3.2, then our estimate reduces to*

$$u(x) \leq g(x)G^{-1}\left(G(1) + \int_{x^0}^{x} f(t)dt\right).$$

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