

## A GRUSS TYPE INEQUALITY FOR SEQUENCES OF VECTORS IN INNER PRODUCT SPACES AND APPLICATIONS

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volume 1, issue 2, article 12,  
2000.

*Received 5 February, 2000;*  
*accepted 29 February, 2000.*

*Communicated by: B. Mond*

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Abstract

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## Abstract

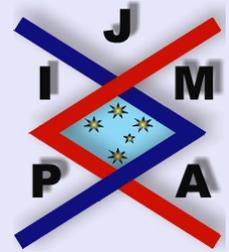
A Grüss type inequality for sequences of vectors in inner product spaces which complement a recent result from [6] and applications for differentiable convex functions defined on inner product spaces and applications for Fourier and Mellin transforms, are given.

*2000 Mathematics Subject Classification:* 26D15, 26D99, 46Cxx

*Key words:* Grüss' Inequality, Inner Product Spaces.

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# 1. Introduction

In 1935, G. Grüss proved the following integral inequality (see [11] or [12])

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma),$$

provided that  $f$  and  $g$  are two integrable functions on  $[a, b]$  and satisfy the condition

$$(1.2) \quad \phi \leq f(x) \leq \Phi \text{ and } \gamma \leq g(x) \leq \Gamma \text{ for all } x \in [a, b].$$

The constant  $\frac{1}{4}$  is the *best possible* and is achieved for

$$f(x) = g(x) = \operatorname{sgn} \left( x - \frac{a+b}{2} \right).$$

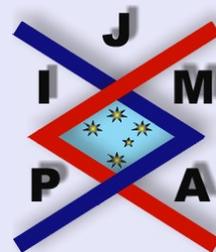
The discrete version of (1.1) states that:

If  $a \leq a_i \leq A, b \leq b_i \leq B$  ( $i = 1, \dots, n$ ) where  $a, A, a_i, b, B, b_i$  are real numbers, then

$$(1.3) \quad \left| \frac{1}{n} \sum_{i=1}^n a_i b_i - \frac{1}{n} \sum_{i=1}^n a_i \frac{1}{n} \sum_{i=1}^n b_i \right| \leq \frac{1}{4} (A - a) (B - b)$$

and the constant  $\frac{1}{4}$  is the best possible.

In the recent paper [2], the author proved the following generalisation in inner product spaces.



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**Theorem 1.1.** Let  $(X; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}, \mathbb{K} = \mathbb{C}, \mathbb{R}$ , and  $e \in X, \|e\| = 1$ . If  $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$  and  $x, y \in X$  such that

$$(1.4) \quad \operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0$$

holds, then we have the inequality

$$(1.5) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|.$$

The constant  $\frac{1}{4}$  is the best possible.

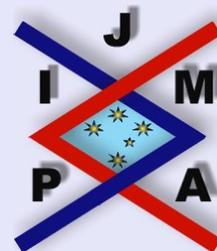
It has been shown in [1] that the above theorem, for the real case, contains the usual integral and discrete Grüss inequality and also some Grüss type inequalities for mappings defined on infinite intervals.

Namely, if  $\rho : (-\infty, +\infty) \rightarrow (-\infty, +\infty)$  is a probability density function, i.e.,  $\int_{-\infty}^{\infty} \rho(t) dt = 1$ , then  $\rho^{\frac{1}{2}} \in L^2(-\infty, \infty)$  and obviously  $\|\rho^{\frac{1}{2}}\|_2 = 1$ . Consequently, if we assume that  $f, g \in L^2(-\infty, \infty)$  and

$$(1.6) \quad \alpha \rho^{\frac{1}{2}} \leq f \leq \psi \rho^{\frac{1}{2}}, \quad \beta \rho^{\frac{1}{2}} \leq g \leq \theta \rho^{\frac{1}{2}} \quad \text{a.e. on } (0, \infty),$$

then we have the inequality

$$(1.7) \quad \left| \int_{-\infty}^{\infty} f(t) g(t) dt - \int_{-\infty}^{\infty} f(t) \rho^{\frac{1}{2}}(t) dt \int_{-\infty}^{\infty} g(t) \rho^{\frac{1}{2}}(t) dt \right| \leq \frac{1}{4} (\psi - \alpha) (\theta - \beta).$$



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In a similar way, if  $e = (e_i)_{i \in \mathbb{N}} \in l^2(\mathbb{R})$  with  $\sum_{i \in \mathbb{N}} |e_i|^2 = 1$  and  $x = (x_i)_{i \in \mathbb{N}}$ ,  $y = (y_i)_{i \in \mathbb{N}} \in l^2(\mathbb{R})$  are such that

$$(1.8) \quad \alpha e_i \leq x_i \leq \psi e_i, \quad \beta e_i \leq y_i \leq \theta e_i$$

for all  $i \in \mathbb{N}$ , then we have

$$(1.9) \quad \left| \sum_{i \in \mathbb{N}} x_i y_i - \sum_{i \in \mathbb{N}} x_i e_i \sum_{i \in \mathbb{N}} y_i e_i \right| \leq \frac{1}{4} (\psi - \alpha) (\theta - \beta).$$

In the recent paper [6], the author also proved the following discrete inequality in inner product spaces:

$$(1.10) \quad \left\| \sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i x_i \right\| \leq \frac{1}{4} |A - a| \|X - x\|$$

provided  $x_i \in H$ ,  $a_i \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) and  $a, A \in \mathbb{K}$ ,  $x, X \in H$  are such that

$$(1.11) \quad \operatorname{Re} [(A - a_i) (\bar{a}_i - \bar{a})] \geq 0 \quad \text{and} \quad \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{for all } i \in \{1, \dots, n\}.$$

The constant  $\frac{1}{4}$  is sharp.

For other recent developments of the Grüss inequality, see the papers [1]-[6], [10] and the website <http://rgmia.vu.edu.au/Gruss.html>

In this paper we point out some other Grüss type inequalities in inner product spaces which will complement the above result (1.10).



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## 2. Preliminary Results

The following lemma is of interest in itself (see also [6]).

**Lemma 2.1.** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$ ,  $x_i \in H$  and  $p_i \geq 0$  ( $i = 1, \dots, n$ ) such that  $\sum_{i=1}^n p_i = 1$  ( $n \geq 2$ ).*

*If  $x, X \in H$  are such that*

$$(2.1) \quad \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \text{ for all } i \in \{1, \dots, n\},$$

*then we have the inequality*

$$(2.2) \quad 0 \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq \frac{1}{4} \|X - x\|^2.$$

*The constant  $\frac{1}{4}$  is sharp.*

*Proof.* Define

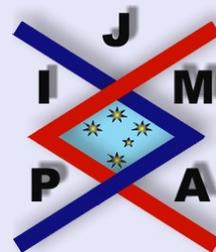
$$I_1 := \left\langle X - \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i - x \right\rangle$$

and

$$I_2 := \sum_{i=1}^n p_i \langle X - x_i, x_i - x \rangle.$$

Then

$$I_1 = \sum_{i=1}^n p_i \langle X, x_i \rangle - \langle X, x \rangle - \left\| \sum_{i=1}^n p_i x_i \right\|^2 + \sum_{i=1}^n p_i \langle x_i, x \rangle$$



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and

$$I_2 = \sum_{i=1}^n p_i \langle X, x_i \rangle - \langle X, x \rangle - \sum_{i=1}^n p_i \|x_i\|^2 + \sum_{i=1}^n p_i \langle x_i, x \rangle.$$

Consequently

$$(2.3) \quad I_1 - I_2 = \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2.$$

Taking the real value in (2.3) we can state

$$(2.4) \quad \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 = \operatorname{Re} \left\langle X - \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i - x \right\rangle - \sum_{i=1}^n p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle,$$

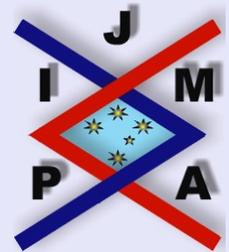
which is an identity of interest in itself.

Using the assumption (2.1), we can conclude, by (2.4), that

$$(2.5) \quad \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq \operatorname{Re} \left\langle X - \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i - x \right\rangle.$$

It is known that if  $y, z \in H$ , then

$$(2.6) \quad 4 \operatorname{Re} \langle z, y \rangle \leq \|z + y\|^2,$$



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with equality iff  $z = y$ .

Now, by (2.6), we can state that

$$\begin{aligned} \operatorname{Re} \left\langle X - \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i - x \right\rangle \\ \leq \frac{1}{4} \left\| X - \sum_{i=1}^n p_i x_i + \sum_{i=1}^n p_i x_i - x \right\|^2 = \frac{1}{4} \|X - x\|^2. \end{aligned}$$

Using (2.5), we can easily deduce (2.2).

To prove the sharpness of the constant  $\frac{1}{4}$ , let us assume that the inequality (2.2) holds with a constant  $c > 0$ , i.e.,

$$(2.7) \quad 0 \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq c \|X - x\|^2$$

for all  $p_i, x_i$  and  $x, X$  as in the hypothesis of Lemma 2.1.

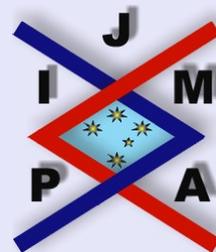
Assume that  $n = 2$ ,  $p_1 = p_2 = \frac{1}{2}$ ,  $x_1 = x$  and  $x_2 = X$  with  $x, X \in H$  and  $x \neq X$ . Then, obviously,

$$\langle X - x_1, x_1 - x \rangle = \langle X - x_2, x_2 - x \rangle = 0,$$

which shows that the condition (2.1) holds.

If we replace  $n, p_1, p_2, x_1, x_2$  in (2.7), we obtain

$$\begin{aligned} \sum_{i=1}^2 p_i \|x_i\|^2 - \left\| \sum_{i=1}^2 p_i x_i \right\|^2 &= \frac{1}{2} \left( \|x\|^2 + \|X\|^2 - \left\| \frac{x + X}{2} \right\|^2 \right) \\ &= \frac{\|X - x\|^2}{4} \leq c \|X - x\|^2, \end{aligned}$$



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from where we deduce  $c \geq \frac{1}{4}$ , which proves the sharpness of the constant factor  $\frac{1}{4}$ .  $\square$

**Remark 2.1.** The assumption (2.1) can be replaced by the more general condition

$$(2.8) \quad \sum_{i=1}^n p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0$$

and the conclusion (2.2) will still remain valid.

The following corollary is natural.

**Corollary 2.2.** Let  $a_i \in \mathbb{K}$ ,  $p_i \geq 0$  ( $i = 1, \dots, n$ ) ( $n \geq 2$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $a, A \in \mathbb{K}$  are such that

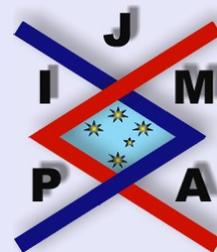
$$(2.9) \quad \operatorname{Re} [(A - a_i) (\bar{a}_i - \bar{a})] \geq 0 \text{ for all } i \in \{1, \dots, n\},$$

then we have the inequality

$$(2.10) \quad 0 \leq \sum_{i=1}^n p_i |a_i|^2 - \left| \sum_{i=1}^n p_i a_i \right|^2 \leq \frac{1}{4} |A - a|^2.$$

The constant  $\frac{1}{4}$  is sharp.

The proof follows by the above Lemma 2.1 by choosing  $H = \mathbb{K}$ ,  $\langle x, y \rangle := x\bar{y}$ ,  $x_i = a_i$ ,  $x = a$ ,  $X = A$ . We omit the details.



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**Remark 2.2.** The condition (2.9) can be replaced by the more general assumption

$$(2.11) \quad \sum_{i=1}^n p_i \operatorname{Re} [(A - a_i) (\bar{a}_i - \bar{a})] \geq 0.$$

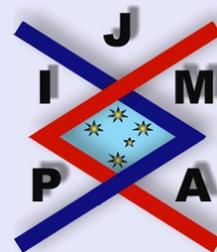
**Remark 2.3.** If we assume that  $\mathbb{K} = \mathbb{R}$ , then (2.8) is equivalent with

$$(2.12) \quad a \leq a_i \leq A \text{ for all } i \in \{1, \dots, n\}$$

and then, with the assumption (2.12), we get the discrete Grüss type inequality

$$(2.13) \quad 0 \leq \sum_{i=1}^n p_i a_i^2 - \left( \sum_{i=1}^n p_i a_i \right)^2 \leq \frac{1}{4} (A - a)^2$$

and the constant  $\frac{1}{4}$  is sharp.



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### 3. A Discrete Inequality of Grüss Type

The following Grüss type inequality holds.

**Theorem 3.1.** Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ ;  $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ,  $x_i, y_i \in H$ ,  $p_i \geq 0$  ( $i = 0, \dots, n$ ) ( $n \geq 2$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $x, X, y, Y \in H$  are such that

$$(3.1) \quad \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \text{ and } \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0 \text{ for all } i \in \{1, \dots, n\},$$

then we have the inequality

$$(3.2) \quad \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{4} \|X - x\| \|Y - y\|.$$

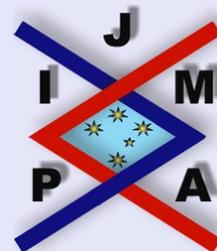
The constant  $\frac{1}{4}$  is sharp.

*Proof.* A simple calculation shows that

$$(3.3) \quad \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle = \frac{1}{2} \sum_{i,j=1}^n p_i p_j \langle x_i - x_j, y_i - y_j \rangle.$$

Taking the modulus in both parts of (3.3), and using the generalized triangle inequality, we obtain

$$(3.4) \quad \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^n p_i p_j |\langle x_i - x_j, y_i - y_j \rangle|.$$



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By Schwartz's inequality in inner product spaces we have

$$(3.5) \quad |\langle x_i - x_j, y_i - y_j \rangle| \leq \|x_i - x_j\| \|y_i - y_j\|$$

for all  $i, j \in \{1, \dots, n\}$ , and therefore

$$(3.6) \quad \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\| \|y_i - y_j\|.$$

Using the Cauchy-Buniakowsky-Schwartz inequality for double sums, we can state that

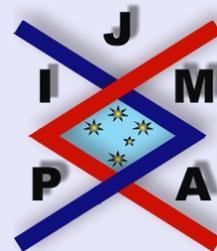
$$(3.7) \quad \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\| \|y_i - y_j\| \leq \left( \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^2 \right)^{\frac{1}{2}} \times \left( \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|y_i - y_j\|^2 \right)^{\frac{1}{2}}$$

and, a simple calculation shows that,

$$\frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^2 = \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2$$

and

$$\frac{1}{2} \sum_{i,j=1}^n p_i p_j \|y_i - y_j\|^2 = \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2.$$



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We obtain

$$(3.8) \quad \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ \leq \left( \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \right)^{\frac{1}{2}} \times \left( \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right)^{\frac{1}{2}} .$$

Using Lemma 2.1, we know that

$$\left( \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \|X - x\|$$

and

$$\left( \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \|Y - y\| .$$

Therefore, by (3.8) we may deduce the desired inequality (3.3).

To prove the sharpness of the constant  $\frac{1}{4}$ , let us assume that (3.2) holds with a constant  $c > 0$ , i.e.,

$$(3.9) \quad \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq c \|X - x\| \|Y - y\|$$

under the above assumptions for  $p_i$ ,  $x_i$ ,  $y_i$ ,  $x$ ,  $X$ ,  $y$ ,  $Y$  and  $n \geq 2$ .



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If we choose  $n = 2$ ,  $x_1 = x$ ,  $x_2 = X$ ,  $y_1 = y$ ,  $y_2 = Y$  ( $x \neq X$ ,  $y \neq Y$ ) and  $p_1 = p_2 = \frac{1}{2}$ , then

$$\begin{aligned} \sum_{i=1}^2 p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^2 p_i x_i, \sum_{i=1}^2 p_i y_i \right\rangle &= \frac{1}{2} \sum_{i,j=1}^2 p_i p_j \langle x_i - x_j, y_i - y_j \rangle \\ &= \sum_{1 \leq i < j \leq 2} p_i p_j \langle x_i - x_j, y_i - y_j \rangle \\ &= \frac{1}{4} \langle x - X, y - Y \rangle \end{aligned}$$

and then

$$\left| \sum_{i=1}^2 p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^2 p_i x_i, \sum_{i=1}^2 p_i y_i \right\rangle \right| = \frac{1}{4} |\langle x - X, y - Y \rangle|.$$

Choose  $X - x = z$ ,  $Y - y = z$ ,  $z \neq 0$ . Then using (3.9), we derive

$$\frac{1}{4} \|z\|^2 \leq c \|z\|^2, \quad z \neq 0$$

which implies that  $c \geq \frac{1}{4}$ , and the theorem is proved.  $\square$

**Remark 3.1.** The condition (3.1) can be replaced by the more general assumption

$$(3.10) \quad \sum_{i=1}^n p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0, \quad \sum_{i=1}^n p_i \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0$$

and the conclusion (3.2) still remains valid.



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The following corollary for real or complex numbers holds.

**Corollary 3.2.** Let  $a_i, b_i \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ),  $p_i \geq 0$  ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n p_i = 1$ .

If  $a, A, b, B \in \mathbb{K}$  are such that

$$(3.11) \quad \operatorname{Re} [(A - a_i) (\bar{a}_i - \bar{a})] \geq 0 \quad \text{and} \quad \operatorname{Re} [(B - b_i) (\bar{b}_i - \bar{b})] \geq 0,$$

then we have the inequality

$$(3.12) \quad \left| \sum_{i=1}^n p_i a_i \bar{b}_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i \bar{b}_i \right| \leq \frac{1}{4} |A - a| |B - b|$$

and the constant  $\frac{1}{4}$  is sharp.

The proof is obvious by Theorem 3.1 applied for the inner product space  $(\mathbb{C}, \langle \cdot, \cdot \rangle)$  where  $\langle x, y \rangle = x \cdot \bar{y}$ . We omit the details.

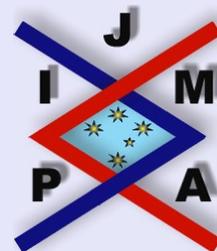
**Remark 3.2.** The condition (3.11) can be replaced by the more general condition

$$(3.13) \quad \sum_{i=1}^n p_i \operatorname{Re} [(A - a_i) (\bar{a}_i - \bar{a})] \geq 0, \quad \sum_{i=1}^n p_i \operatorname{Re} [(B - b_i) (\bar{b}_i - \bar{b})] \geq 0$$

and the conclusion of the above corollary will still remain valid.

**Remark 3.3.** If we assume that  $a_i, b_i, a, b, A, B$  are real numbers, then (3.11) is equivalent to

$$(3.14) \quad a \leq a_i \leq A, \quad b \leq b_i \leq B \quad \text{for all } i \in \{1, \dots, n\}$$



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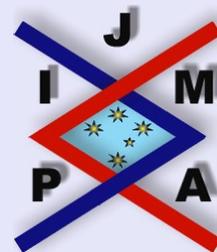
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and (3.12) becomes

$$(3.15) \quad 0 \leq \left| \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right| \leq \frac{1}{4} (A - a)(B - b),$$

which is the classical Grüss inequality for sequences of real numbers.



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## 4. Applications for Convex Functions

Let  $(H; \langle \cdot, \cdot \rangle)$  be a real inner product space and  $F : H \rightarrow \mathbb{R}$  a Fréchet differentiable convex mapping on  $H$ . Then we have the “gradient inequality”

$$(4.1) \quad F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle$$

for all  $x, y \in H$ , where  $\nabla F : H \rightarrow H$  is the gradient operator associated to the differentiable convex function  $F$ .

The following theorem holds.

**Theorem 4.1.** *Let  $F : H \rightarrow \mathbb{R}$  be as above and  $x_i \in H$  ( $i = 1, \dots, n$ ). Suppose that there exists the vectors  $\gamma, \phi \in H$  such that  $\langle x_i - \gamma, \phi - x_i \rangle \geq 0$  for all  $i \in \{1, \dots, m\}$  and  $m, M \in H$  such that  $\langle \nabla F(x_i) - m, M - \nabla F(x_i) \rangle \geq 0$  for all  $i \in \{1, \dots, m\}$ . Then for all  $p_i \geq 0$  ( $i = 1, \dots, m$ ) with  $P_m := \sum_{i=1}^m p_i > 0$ , we have the inequality*

$$(4.2) \quad 0 \leq \frac{1}{P_m} \sum_{i=1}^m p_i F(x_i) - F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \leq \frac{1}{4} \|\phi - \gamma\| \|M - m\|.$$

*Proof.* Choose in (4.1),  $x = \frac{1}{P_m} \sum_{i=1}^m p_i x_i$  and  $y = x_j$  to obtain

$$(4.3) \quad F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - F(x_j) \geq \left\langle \nabla F(x_j), \frac{1}{P_m} \sum_{i=1}^m p_i x_i - x_j \right\rangle$$

for all  $j \in \{1, \dots, n\}$ .



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If we multiply (4.3) by  $p_j \geq 0$  and sum over  $j$  from 1 to  $m$ , we have

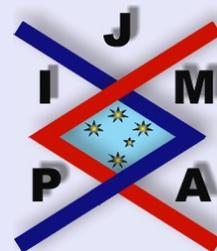
$$\begin{aligned}
 P_m F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) &= \sum_{j=1}^m p_j F(x_j) \\
 &\geq \frac{1}{P_m} \left\langle \sum_{j=1}^m p_j \nabla F(x_j), \sum_{i=1}^m p_i x_i \right\rangle - \sum_{i=1}^m \langle \nabla F(x_i), x_i \rangle.
 \end{aligned}$$

Dividing by  $P_m > 0$ , we obtain the inequality

$$\begin{aligned}
 (4.4) \quad 0 &\leq \frac{1}{P_m} \sum_{i=1}^m p_i F(x_i) - F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \\
 &\leq \frac{1}{P_m} \sum_{i=1}^m p_i \langle \nabla F(x_i), x_i \rangle \\
 &\quad - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i \nabla F(x_i), \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\rangle
 \end{aligned}$$

which is a generalisation for the case of inner product spaces of the result by Dragomir and Goh established in 1996 for the case of differentiable mappings defined on  $\mathbb{R}^n$  [9].

Applying Theorem 3.1 for real inner product spaces,  $X = \phi$ ,  $x = \gamma$ ,  $y_i =$



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$\nabla F(x_i)$ ,  $y = m$ ,  $Y = M$  and  $n = m$ , we easily deduce

$$(4.5) \quad \frac{1}{P_m} \sum_{i=1}^m p_i \langle x_i, \nabla F(x_i) \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i \nabla F(x_i) \right\rangle \leq \frac{1}{4} \|\Phi - \phi\| \|M - m\|$$

and then, by (4.4) and (4.5) we can conclude that the desired inequality (4.2) holds.  $\square$

**Remark 4.1.** *The conditions*

$$(4.6) \quad \langle x_i - \gamma, \phi - x_i \rangle \geq 0, \quad \langle \nabla F(x_i) - m, M - \nabla F(x_i) \rangle \geq 0,$$

for all  $i \in \{1, \dots, m\}$  can be replaced by the more general conditions

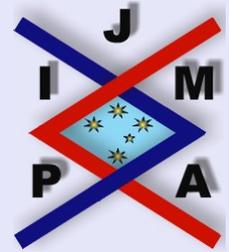
$$(4.7) \quad \sum_{i=1}^m p_i \langle x_i - \gamma, \phi - x_i \rangle \geq 0 \quad \text{and} \quad \sum_{i=1}^m p_i \langle \nabla F(x_i) - m, M - \nabla F(x_i) \rangle \geq 0$$

and the conclusion (4.2) will still be valid.

**Remark 4.2.** *Even if the inequality (4.2) is not as sharp as (4.4), it can be more useful in practice when only some bounds of the gradient operator  $\nabla F$  and of the vectors  $x_i$  ( $i = 1, \dots, n$ ) are known. In other words, it provides the opportunity to estimate the difference*

$$\Delta(F, x, p) := \frac{1}{P_m} \sum_{i=1}^m p_i F(x_i) - F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right),$$

where the differences  $\|\phi - \gamma\|$  and  $\|M - m\|$  are known.



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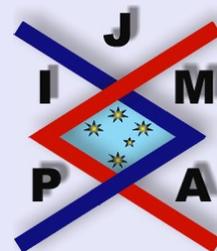
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**Remark 4.3.** For example, if we know that  $\langle \nabla F(x_i) - m, M - \nabla F(x_i) \rangle \geq 0$  for all  $i \in \{1, \dots, m\}$  and the vectors  $x_i$  ( $i = 1, \dots, n$ ) are not too far from each other in the sense that  $\langle x_i - \gamma, \phi - x_i \rangle \geq 0$  for all  $i \in \{1, \dots, m\}$  and  $\|\phi - \gamma\| \leq \frac{4\varepsilon}{\|M-m\|}$  ( $\varepsilon > 0$ ), then by (4.2), we can conclude that

$$0 \leq \Delta(F, x, p) \leq \varepsilon.$$




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## 5. Applications for Some Discrete Transforms

Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ ,  $\mathbb{K} = \mathbb{C}, \mathbb{R}$  and  $\bar{x} = (x_1, \dots, x_n)$  be a sequence of vectors in  $H$ .

For a given  $m \in \mathbb{K}$ , define the *discrete Fourier transform*

$$(5.1) \quad \mathcal{F}_w(\bar{x})(m) = \sum_{k=1}^n \exp(2wimk) \times x_k, \quad m = 1, \dots, n.$$

The complex number  $\sum_{k=1}^n \exp(2wimk) \langle x_k, y_k \rangle$  is actually the usual Fourier transform of the vector  $(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) \in \mathbb{K}^n$  and will be denoted by

$$(5.2) \quad \mathcal{F}_w(\bar{x} \cdot \bar{y})(m) = \sum_{k=1}^n \exp(2wimk) \langle x_k, y_k \rangle, \quad m = 1, \dots, n.$$

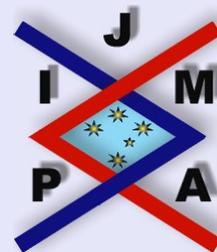
The following result holds.

**Theorem 5.1.** *Let  $\bar{x}, \bar{y} \in H^n$  be sequences of vectors such that there exists the vectors  $c, C, y, Y \in H$  with the properties*

$$(5.3) \quad \operatorname{Re} \langle C - \exp(2wimk) x_k, \exp(2wimk) x_k - c \rangle \geq 0, \quad k, m = 1, \dots, n$$

and

$$(5.4) \quad \operatorname{Re} \langle Y - y_k, y_k - y \rangle \geq 0, \quad k = 1, \dots, n.$$



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Then we have the inequality

$$(5.5) \quad \left| \mathcal{F}_w(\bar{x} \cdot \bar{y})(m) - \left\langle \mathcal{F}_w(\bar{x})(m), \frac{1}{n} \sum_{k=1}^n y_k \right\rangle \right| \leq \frac{n}{4} \|C - c\| \|Y - y\|,$$

for all  $m \in \{1, \dots, n\}$ .

The proof follows by Theorem 3.1 applied for  $p_k = \frac{1}{n}$  and for the sequences  $x_k \rightarrow c_k = \exp(2wimk) x_k$  and  $y_k$  ( $k = 1, \dots, n$ ). We omit the details.

We can also consider the Mellin transform

$$(5.6) \quad \mathcal{M}(\bar{x})(m) := \sum_{k=1}^n k^{m-1} x_k, \quad m = 1, \dots, n,$$

of the sequence  $\bar{x} = (x_1, \dots, x_n) \in H^n$ .

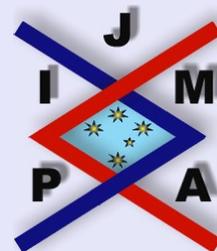
We remark that the complex number  $\sum_{k=1}^n k^{m-1} \langle x_k, y_k \rangle$  is actually the Mellin transform of the vector  $(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) \in \mathbb{K}^n$  and will be denoted by

$$(5.7) \quad \mathcal{M}(\bar{x} \cdot \bar{y})(m) := \sum_{k=1}^n k^{m-1} \langle x_k, y_k \rangle.$$

The following theorem holds.

**Theorem 5.2.** Let  $\bar{x}, \bar{y} \in H^n$  be sequences of vectors such that there exist the vectors  $d, D, y, Y \in H$  with the properties

$$(5.8) \quad \operatorname{Re} \langle D - k^{m-1} x_k, k^{m-1} x_k - d \rangle \geq 0$$



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for all  $k, m \in \{1, \dots, n\}$ , and (5.4) is fulfilled.

Then we have the inequality

$$(5.9) \quad \left| \mathcal{M}(\bar{x} \cdot \bar{y})(m) - \left\langle \mathcal{M}(\bar{x})(m), \frac{1}{n} \sum_{k=1}^n y_k \right\rangle \right| \leq \frac{n}{4} \|D - d\| \|Y - y\|$$

for all  $m \in \{1, \dots, n\}$ .

The proof follows by Theorem 3.1 applied for  $p_k = \frac{1}{n}$  and for the sequences  $x_k \rightarrow d_k = kx_k$  and  $y_k$  ( $k = 1, \dots, n$ ). We omit the details.

Another result which connects the Fourier transforms for different parameters  $w$  also holds.

**Theorem 5.3.** Let  $\bar{x}, \bar{y} \in H^n$  and  $w, z \in \mathbb{K}$ . If there exists the vectors  $e, E, f, F \in H$  such that

$$\operatorname{Re} \langle E - \exp(2wimk) x_k, \exp(2wimk) x_k - e \rangle \geq 0, \quad k, m = 1, \dots, n$$

and

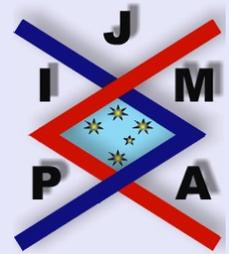
$$\operatorname{Re} \langle F - \exp(2zimbk) y_k, \exp(2zimbk) y_k - f \rangle \geq 0, \quad k, m = 1, \dots, n$$

then we have the inequality:

$$\left| \frac{1}{n} \mathcal{F}_{w+z}(\bar{x} \cdot \bar{y})(m) - \left\langle \frac{1}{n} \mathcal{F}_w(\bar{x})(m), \frac{1}{n} \mathcal{F}_z(\bar{y})(m) \right\rangle \right| \leq \frac{1}{4} \|E - e\| \|F - f\|,$$

for all  $m \in \{1, \dots, n\}$ .

The proof follows by Theorem 3.1 for the sequences  $\exp(2wimk) x_k$ ,  $\exp(2zimbk) y_k$  ( $k = 1, \dots, n$ ). We omit the details.



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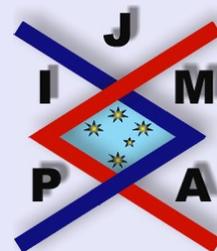
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## References

- [1] S.S. DRAGOMIR, Grüss inequality in inner product spaces, *Austral. Math. Soc. Gazette*, **26**(2) (1999), 66–70.
- [2] S.S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74–82.
- [3] S.S. DRAGOMIR, A Grüss type integral inequality for mappings of  $r$ -Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, **31**(1) (2000), 43–47.
- [4] S.S. DRAGOMIR, Some discrete inequalities of Grüss type and applications in guessing theory, *Honam Math. J.*, **21**(1) (1999), 115–126.
- [5] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.*, **31**(4) (2000), 397–415.
- [6] S.S. DRAGOMIR, A Grüss type discrete inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, (*in press*).
- [7] S.S. DRAGOMIR AND G.L. BOOTH, On a Grüss-Lupaş type inequality and its applications for the estimation of  $p$ -moments of guessing mappings, *Math. Comm.*, (*in press*).
- [8] S.S. DRAGOMIR AND I. FEDOTOV, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4) (1998), 286–292.



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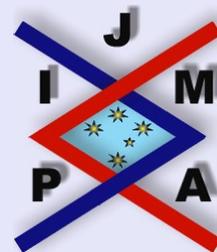
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- [9] S.S. DRAGOMIR AND C.J. GOH, A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory, *Mathl. Comput. Modelling*, **24**(2) (1996), 1–11.
- [10] A.M. FINK, A treatise on Grüss' inequality, *submitted*.
- [11] G. GRÜSS, Über das Maximum des absoluten Betrages von  $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx$ , *Math. Z.*, **39** (1935), 215–226.
- [12] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.




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