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WEIGHTED MODULAR INEQUALITIES FOR HARDY-TYPE OPERATORS ON MONOTONE FUNCTIONS

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ABSTRACT. If $(Kf)(x) = \int_0^x k(x,y)f(y)\,dy$, $x > 0$, is a Hardy-type operator defined on the cone of monotone functions, then weight characterizations for which the modular inequality

$$Q^{-1}\left(\int_0^\infty Q[\theta(Kf)]w\right) \le P^{-1}\left(\int_0^\infty P[Cf]v\right)$$

holds, are given for a large class of modular functions P, Q. Specifically, these functions need not both be N-functions, and the class includes the case where $Q \circ P^{-1}$ is concave. Our results generalize those in [7, 24], where the case $Q \circ P^{-1}$ convex, with P, Q, N-function was studied. Applications involving the Hardy averaging operator, its dual, the Hardy-Littlewood maximal function, and the Hilbert transform are also given.

Key words and phrases: Hardy-type operators, modular inequalities, weights, N-functions, characterizations, Orlicz-Lorentz spaces.

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1. INTRODUCTION

An integral operator K defined by

$$(Kf)(x) = \int_0^x k(x,y)f(y)\,dy, \qquad x > 0,\ f \ge 0$$

is called a *Hardy type operator*, if the kernel k satisfies

(1.1) (i) $k(x,y) > 0,\ x > y > 0,\ k$ is increasing in x and decreasing in y.
(ii) $k(x,y) \le D[k(x,z) + k(z,y)],\ 0 < y < z < x,$
for some constant $D > 0$.

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$k(x,y) = 1$; $k(x,y) = \phi(x-y)$, ϕ increasing, $\phi(a+b) \leq D[\phi(a)+\phi(b)]$ $0 < a,b < \infty$; and $k(x,y) = \psi(y/x)$, ψ decreasing, $\psi(ab) \leq D[\psi(a)+\psi(b)]$ $0 < a,b < 1$; are examples of kernels satisfying (1.1) and hence define Hardy-type operators.

If $k(x,y)$ has no monotonicity properties, satisfies (ii) and its reverse, then k is said to satisfy the Oinarov condition ([22]) and we write $k(x,y) \approx k(x,z) + k(z,y)$, $0 < y < z < x$.

In this paper we study Hardy-type operators (and its duals) defined on the cone of monotone functions. Specifically, weight functions θ, w, v are characterized for which the modular inequality

$$Q^{-1}\left(\int_0^\infty Q[\theta(x)(Kf)(x)]w(x)\,dx\right) \leq P^{-1}\left(\int_0^\infty P[Cf(x)]v(x)\,dx\right) \tag{1.2}$$

is satisfied for a large class of modular functions P, Q, and $f \geq 0$, monotone.

For example, if $K = I$, the identity operator and $0 \leq f\downarrow$, then the weights are characterized for which (1.2) holds with P, Q increasing and P weakly convex (cf. Theorem 3.1). For general K, defined on $0 \leq f\downarrow$, weight characterizations are given for which (1.2) holds with P an N-function, P, $\tilde{P} \in \Delta_2$ and Q weakly convex (cf. Theorem 3.4). Specifically, $Q \circ P^{-1}$ may be concave. These results together with the corresponding results where K is defined on the cone of increasing functions are new. The case $0 < q < 1 < p$ for the general K, defined on $0 \leq f\uparrow$, was unknown until this paper.

If $P(x) = x^p$, $Q(x) = x^q$, $0 < p,q < \infty$, $\theta(x) = 1$, then our results reduce to weighted Lebesgue space inequalities and in particular if $k(x,y) = 1$, to the weight characterizations of Ariño-Muckenhoupt [1] ($p = q > 1$ $w = v$), Sawyer [21] ($1 < p,q < \infty$) and Stepanov [23] ($0 < q < p,\ p > 1$). The general case where P and Q are N-functions, such that P and its complementary function $\tilde{P}$ satisfy Δ_2 with $Q \circ P^{-1}$ convex (more precisely $P \ll Q$) was studied by Sun [24] with $k(x,y)$ the convolution kernel.

To explain the scope of our results we require some definitions and known facts.

A non-negative function P on $\mathbb{R}^+$ is called an N-function if it has the form

$$P(x) = \int_0^x p(t)\,dt, \qquad x > 0, \tag{1.3}$$

where p is non-decreasing, right continuous on $(0,\infty)$, $p(0+) = 0$, $p(\infty) = \infty$ and $p(t) > 0$ if $t > 0$. Clearly

$$\lim_{x\to 0+} \frac{P(x)}{x} = \lim_{x\to\infty} \frac{x}{P(x)} = 0.$$

Given an N-function P, then its complementary function $\tilde{P}$ is defined by $\tilde{P}(y) = \sup_{x>0}\{xy - P(x)\}$ and

$$t \leq P^{-1}(t)\tilde{P}^{-1}(t) \leq 2t, \quad p(t/2)/2 \leq P(t)/t \leq p(t), \quad t > 0 \tag{1.4}$$

holds. It is easily seen that if P is an N-function so is $\tilde{P}$, and the complement relation is symmetric.

If (X,μ) is a σ-finite measure space, then a μ-measurable function f belongs to the *Orlicz-space* $L_{P(\mu)}$ if the *Luxemburg norm*

$$\|f\|_{P(\mu)} = \inf\left\{\lambda > 0 : \int_X P\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}$$

is finite. The *Orlicz norm* in $L_{P(\mu)}$ is defined by

$$\|f\|'_{P(\mu)} = \sup\left\{\left|\int_X fg\,d\mu\right| : \int_X \tilde{P}(g)d\mu \leq 1\right\}.$$

We note that the Luxemburg and Orlicz norm are equivalent and

$$\|f\|_{P(\mu)} \leq 1 \quad \text{if and only if} \quad \int_X P(f)d\mu \leq 1. \tag{1.5}$$

Given an N-function P, we always use the Luxemburg norm in $L_{P(\mu)}$ and define that *associate space* $L_{\tilde{P}(\mu)}$ of $L_{P(\mu)}$ consists of those μ-measurable g, for which the Orlicz norm

$$\|g\|_{\tilde{P}(\mu)} = \sup\left\{ \left| \int_X fg\,d\mu \right| : \|f\|_{P(\mu)} \leq 1 \right\}$$

is finite.

A *weight function* u ($u \not\equiv 0$, $u \not\equiv \infty$) is a non-negative measurable and locally integrable function on $\mathbb{R}^+$, and if $d\mu(x) = u(x)dx$, then we write $P(\mu) = P(u)$. The standard duality principle in Orlicz spaces may be written as

$$\sup_{0 \leq f} \frac{\int_0^\infty fg}{\|f\|_{P(u)}} = \left\| \frac{g}{u} \right\|_{\tilde{P}(u)}, \qquad g \geq 0.$$

For these and other facts see [13, 14, 20].

Definition 1.1.

a) An increasing function $P : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy Δ_2, ($P \in \Delta_2$), if there is a constant $C > 1$, such that $P(2t) \leq CP(t)$, $t \geq 0$.
b) A strictly increasing function $Q : \mathbb{R}^+ \to \mathbb{R}^+$ is weakly convex, ($Q \in \Delta^2$), if $Q(0) = 0$, $Q(\infty) = \infty$ and $2Q(t) \leq Q(Mt)$, $t \geq 0$, for some constant $M > 1$.
c) ([16]) If P and Q are increasing, then we write $P \ll Q$, if there is a constant $A > 0$, such that
$$\sum_j Q \circ P^{-1}(a_j) \leq Q \circ P^{-1}\left(A \sum_j a_j \right)$$
is satisfied for all non-negative sequences $\{a_j\}_{j\in\mathbb{Z}}$.

A convex function Q satisfying $Q(0) = 0$, $Q(\infty) = \infty$ is weakly convex (with $M = 2$). However, the weakly convex function $Q(t) = t^\alpha$, $t \geq 0$, $0 < \alpha < 1$, is not convex, and $Q(t) = \ln(1 + t)$, $t \geq 0$ is not weakly convex. Observe also that if $Q \circ P^{-1}$ is convex, then $P \ll Q$.

The main result of this paper (Theorem 3.4) characterizes the weights θ, w, v for which (1.2) is satisfied for decreasing $f \geq 0$ with P an N-function, P, $\tilde{P} \in \Delta_2$ and Q weakly convex. This characterization is expressed in terms of estimates involving covering sequences.

Definition 1.2. A strictly increasing positive sequence $\{x_j\}_{j\in\mathbb{Z}}$ is called a covering sequence if the sequence is of the form $\{x_j\}_{j=-\infty}^{\infty}$ or of the form $\{x_j\}_{j=N}^{M}$, where M and/or N is finite. In the latter case we define $x_{N-1} = 0$ and/or $x_{M+1} = \infty$.

In some instances covering sequences satisfy $\int_0^{x_j} v = 2^k$, $k \in \mathbb{Z}$, where v is a weight function. If $2^N < \int_0^\infty v < 2^{N+1}$ then in the case $2^N < \int_0^\infty v < 3 \cdot 2^{N-1}$ we set $x_N = \infty$ and the covering sequence is $\{x_j\}_{j=-\infty}^{N-1}$. In the remaining case we set $x_{N+1} = \infty$ and the covering sequence is $\{x_j\}_{j=-\infty}^{N}$. Under these conventions $2^{k-1} \leq \int_{x_j}^{x_{j+1}} v \leq 3 \cdot 2^{k-1}$ for $0 < x_j < \infty$.

The manuscript is divided into four sections. The next section contains the weight characterization of a modular Hardy-type inequality for Young's and weakly convex functions by Qinsheng Lai [19]. As a consequence a corresponding result for the dual operator follows. In addition, modular Hardy and conjugate Hardy inequalities (Lemma 2.3) are given. Section 3, the main results, contain the weighted modular inequalities for the identity operator (Theorem 3.1) and Hardy-type operator (Theorem 3.4) defined on decreasing functions. Some special cases given there are needed in Section 4 and seem to be new even in the Lebesgue space case. In the last section results for the Hardy operator on increasing functions are given. Moreover, the boundedness of the Hardy-Littlewood maximal function and the Hilbert transform in weighted Orlicz-Lorentz spaces are characterized.

The notation is standard, $\mathbb{R}^+$ and $\mathbb{R}$ denote the non-negative real and real numbers respectively, while $\mathbb{Z}$ denotes the set of integers. The symbol χ_E stands for the characteristic function of a set E. All functions are assumed measurable and u, v, w, θ denote weight functions. If u is a weight function $u(E) = \int_E u(x)\,dx$, $U(x) = \int_0^x u$ and $U^*(x) = \int_x^\infty u$, $(x > 0)$. Instead of non-increasing, non-decreasing we shall say decreasing and increasing respectively, otherwise we shall prefix it by "strictly". If $f \geq 0$ is increasing (decreasing) we shall write $0 \leq f\!\uparrow$ $(0 \leq f\!\downarrow)$ and similarly for sequences. Expressions of the form $A \approx B$ are interpreted to mean that A/B are bounded above and below by positive constants. Constants are (with the exception of those of Definition 1.1) denoted by B and C and they may have different values at different places. Inequalities, such as (1.2), are interpreted to mean that if the right side is finite, so is the left side and the inequality holds.

Other notations and concepts are introduced when needed.

2. PRELIMINARY RESULTS

In order to prove weighted modular inequalities for Hardy type operators defined on the cone of monotone functions, a number of results are required. The first result (Theorem 2.1) by Q. Lai [19] is a weight characterization of the Hardy-type operator for which a weighted modular inequality is satisfied. This theorem extends corresponding work of [3, 4, 18, 22, 24] to Young's functions P and weakly convex functions Q without the assumption that $Q \circ P^{-1}$ (or more precisely $P \ll Q$) is convex.

Theorem 2.1. *(*[19, Thm. 1]*) Suppose K is a Hardy-type operator, P a Young's function and Q weakly convex. Let θ, w, ρ and v be weight functions, then the modular inequality*

$$Q^{-1}\left(\int_0^\infty Q[\theta(x)Kf(x)]w(x)\,dx\right) \leq P^{-1}\left(\int_0^\infty P[C\rho(x)f(x)]v(x)\,dx\right)$$

is satisfied for all $f \geq 0$, if and only if there are constants $B > 0$, such that,

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{B}\left\|\frac{k(x_j,\cdot)\chi_{(x_{j-1},x_j)}}{\varepsilon_j v\rho}\right\|_{\tilde{P}(\varepsilon_j v)}\right]w(x)\,dx\right) \leq P^{-1}\left(\sum_j 1/\varepsilon_j\right)$$

and

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)k(x,x_j)}{B}\left\|\frac{\chi_{(x_{j-1},x_j)}}{\varepsilon_j v\rho}\right\|_{\tilde{P}(\varepsilon_j v)}\right]w(x)\,dx\right) \leq P^{-1}\left(\sum_j 1/\varepsilon_j\right)$$

hold for all positive sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and all covering sequences $\{x_j\}_{j\in\mathbb{Z}}$.

A corresponding result for the conjugate Hardy-type operator

$$(K^*h)(x) = \int_x^\infty k(y,x)h(y)\,dy, \qquad x > 0,\ h \geq 0,$$

where k satisfies (1.1), also holds. In fact, writing $\tilde{k}(x,y) = k(\frac{1}{y},\frac{1}{x})$ and $\bar{\bar{h}}(y) = h(1/y)/y^2$, then a change of variables shows that

$$(K^*h)(1/x) = \int_0^x \tilde{k}(x,y)\bar{\bar{h}}(y)\,dy \equiv (K\bar{\bar{h}})(x)$$

is a Hardy-type operator since $\tilde{k}(x,y)$ satisfies the same conditions as $k(x,y)$. Writing $\bar{g}(x) = g(1/x)$ and $\bar{\bar{g}}(x) = g(1/x)/x^2$ it follows that

$$Q^{-1}\left(\int_0^\infty Q[\theta(x)K^*h(x)]w(x)\,dx\right) = Q^{-1}\left(\int_0^\infty Q[\bar{\theta}(x)K\bar{\bar{h}}(x)]\bar{\bar{w}}(x)\,dx\right)$$

and

$$P^{-1}\left(\int_0^\infty P[Cx^2\bar{\rho}(x)\bar{\bar{h}}(x)]\bar{\bar{v}}(x)\,dx\right) = P^{-1}\left(\int_0^\infty P[C\rho(x)h(x)]v(x)\,dx\right).$$

Also

$$\left\|\frac{\tilde{k}(x_j,\cdot)\chi_{(x_{j-1},x_j)}}{\varepsilon_j\bar{\rho}\bar{v}}\right\|_{\tilde{P}(\varepsilon_j\bar{\bar{v}})} = \left\|\frac{k(\cdot,\frac{1}{x_j})\chi_{(1/x_j,1/x_{j-1})}}{\varepsilon_j\rho v}\right\|_{\tilde{P}(\varepsilon_j v)}$$

and

$$\left\|\frac{\chi_{(x_{j-1},x_j)}}{\varepsilon_j\bar{\rho}\bar{v}}\right\|_{\tilde{P}(\varepsilon_j\bar{\bar{v}})} = \left\|\frac{\chi_{(1/x_j,1/x_{j-1})}}{\varepsilon_j\rho v}\right\|_{\tilde{P}(\varepsilon_j v)}.$$

Therefore, if $1/x_j = y_{-k}$, $k \in \mathbb{Z}$, then $\{y_j\}_{j\in\mathbb{Z}}$ is also a covering sequence, whenever $\{x_j\}_{j\in\mathbb{Z}}$ is. Thus, the following characterization follows from Theorem 2.1.

Proposition 2.2. *If K^* is the conjugate Hardy-type operator, P a Young's function and Q weakly convex, then*

$$Q^{-1}\left(\int_0^\infty Q[\theta(x)(K^*h)(x)]w(x)\,dx\right) \le P^{-1}\left(\int_0^\infty P[C\rho(x)h(x)]v(x)\,dx\right)$$

is satisfied for all $h \ge 0$, if and only if there is a constant $B > 0$, such that

$$Q^{-1}\left(\sum_j \int_{y_{j-1}}^{y_j} Q\left[\frac{\theta(x)}{B}\left\|\frac{k(\cdot,y_j)\chi_{(y_j,y_{j+1})}}{\varepsilon_j\rho v}\right\|_{\tilde{P}(\varepsilon_j v)}\right]w(x)\,dx\right) \le P^{-1}\left(\sum_j 1/\varepsilon_j\right)$$

and

$$Q^{-1}\left(\sum_j \int_{y_{j-1}}^{y_j} Q\left[\frac{\theta(x)k(y_j,x)}{B}\left\|\frac{\chi_{(y_j,y_{j+1})}}{\varepsilon_j\rho v}\right\|_{\tilde{P}(\varepsilon_j v)}\right]w(x)\,dx\right) \le P^{-1}\left(\sum_j 1/\varepsilon_j\right)$$

holds for all positive sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and all covering sequences $\{y_j\}_{j\in\mathbb{Z}}$.

Note that if Q is an N-function, then Q is convex and in particular, weakly convex. Hence Theorem 2.1 and Proposition 2.2 hold in this case.

The following result is required in the next section.

Lemma 2.3. *Suppose P and $\tilde{P}$ are N-functions, $V(x) = \int_0^x v$, $V^*(x) = \int_x^\infty v$ and v is a weight function.*

(i) *If $V(\infty) = \infty$, then there exists a constant $C > 0$, such that*

$$\int_0^\infty P\left[\frac{1}{V(x)}\int_0^x fv\right]v(x)\,dx \le \int_0^\infty P[Cf(x)]v(x)\,dx, \tag{2.1}$$

is satisfied for all $f \ge 0$ if and only if $\tilde{P} \in \Delta_2$, and

$$\int_0^\infty P\left[\int_x^\infty \frac{fv}{V}\right]v(x)\,dx \le \int_0^\infty P[Cf(x)]v(x)\,dx, \tag{2.2}$$

is satisfied for all $f \ge 0$ if and only if $P \in \Delta_2$.

(ii) *If* $V^*(0)=\infty$, *then*

$$\int_0^\infty P\left[\frac{1}{V^*(x)}\int_x^\infty fv\right]v(x)\,dx \le \int_0^\infty P[Cf(x)]v(x)\,dx, \tag{2.3}$$

is satisfied for all $f\ge 0$ *if and only if* $\tilde{P}\in\Delta_2$, *and*

$$\int_0^\infty P\left[\int_0^x \frac{fv}{V^*}\right]v(x)\,dx \le \int_0^\infty P[Cf(x)]v(x)\,dx, \tag{2.4}$$

is satisfied for all $f\ge 0$ *if and only if* $P\in\Delta_2$.

The conditions $V(\infty)=\infty$ and $V^*(0)=\infty$ are only required in the necessity part of the proof.

Proof. First observe that if $\bar{f}(x)=f(1/x)$, $\bar{\bar{v}}(x)=v(1/x)/x^2$ then via obvious changes of variables, (2.3) reduces to (2.1) with f replaced by $\bar{f}$ and v by $\bar{\bar{v}}$. A similar change of variable shows that (2.4) reduces to (2.2). Note that $V^*(1/t)=\int_0^t \bar{\bar{v}}$. Therefore it suffices to prove only part (i) of Lemma 2.3.

Next we observe that (2.1) is equivalent to

$$\int_0^\infty \tilde{P}\left[\int_x^\infty \frac{fv}{V}\right]v(x)\,dx \le \int_0^\infty \tilde{P}[Cf(x)]v(x)\,dx. \tag{2.5}$$

To see this, recall that by [4, Prop. 2.5] (see also [11]) that (2.1) holds if and only if for every $\varepsilon>0$,

$$\|Tf\|_{P(\varepsilon v)} \le C\|f\|_{P(\varepsilon v)} \quad \text{where} \quad Tf(x)=\frac{1}{V(x)}\int_0^x fv.$$

But by the standard duality principle in Orlicz spaces this is equivalent to

$$\left\|\frac{T^*g}{\varepsilon v}\right\|_{\tilde{P}(\varepsilon v)} \le C\left\|\frac{g}{\varepsilon v}\right\|_{\tilde{P}(\varepsilon v)}, \quad \text{where} \quad T^*g(x)=v(x)\int_x^\infty \frac{g(t)}{V(t)}\,dt$$

is the conjugate operator of T. By homogeneity of the norm and again applying [4, Prop. 2.5] it follows that this inequality is equivalent to

$$\int_0^\infty \tilde{P}\left[\frac{1}{v(x)}\,(T^*g)(x)\right]v(x)\,dx \le \int_0^\infty \tilde{P}\left[\frac{Cg(x)}{v(x)}\right]v(x)\,dx,$$

which is (2.5) with $g=fv$. Hence we only need to show that (2.1) is satisfied, if and only if $\tilde{P}\in\Delta_2$.

Let $\tilde{P}\in\Delta_2$ and define $f^+(x)=f(x)$ if $x\ge 0$ and zero otherwise, $v(|x|)dx=d\mu(x)$, then

$$\frac{1}{V(x)}\int_0^x fv \le (M_\mu f^+)(x) := \sup_{x\in I}\frac{1}{\mu(I)}\int_I f^+ d\mu, \quad I\in\mathbb{R}.$$

Clearly M_μ is sublinear and of type (∞,∞), and weak type $(1,1)$, with respect to $d\mu$. Now the argument of [5, p. 149-150] shows that $\tilde{P}\in\Delta_2$ is sufficient for

$$\int_0^\infty P\left(M_\mu f^+(x)\right)d\mu(x) \le \int_0^\infty P\left(Cf(x)\right)d\mu(x),$$

from which (2.1) follows.

To prove that (2.1) implies $\tilde{P}\in\Delta_2$, it suffices (see [5, Prop. 3]) to prove that there exists a $\delta>0$, such that $p(\delta x)\le 1/2\,p(x)$, where $p(x)=P'(x)$ with $p(0)=0$.

By Theorem 2.1, with $Q = P$, $k(x,y) = 1$, $\theta = 1/V$, $\rho = 1/v$, f replaced by fv, $x_j = r > 0$, $x_{j-1} = 0$ and $x_{j+1} = \infty$, (2.1) implies

$$\int_r^\infty P\left[\frac{1}{B\,V(t)}\left\|\frac{\chi_{(0,r)}}{\varepsilon}\right\|_{\tilde{P}(\varepsilon v)}\right] v(t)\,dt \le 1/\varepsilon$$

for all $\varepsilon > 0$ and $r > 0$. But by the definition of the Luxemburg norm and (1.4) with $t = 1/(\varepsilon\,V(r))$

$$\begin{aligned}\left\|\frac{\chi_{(0,r)}}{\varepsilon}\right\|_{\tilde{P}(\varepsilon v)} &= \frac{1}{\varepsilon}\inf\left\{\lambda > 0 : \int_0^r \tilde{P}\left(\frac{1}{\lambda}\right)\varepsilon v(t)\,dt \le 1\right\} \\ &= \frac{1}{\varepsilon \tilde{P}^{-1}\left(\frac{1}{\varepsilon V(r)}\right)} \\ &\ge \frac{V(r)}{2}\,P^{-1}\left(\frac{1}{\varepsilon V(r)}\right).\end{aligned}$$

Hence (2.1) implies

$$\int_r^\infty P\left[\frac{V(r)}{2BV(t)}\,P^{-1}\left(\frac{1}{\varepsilon V(r)}\right)\right] v(t)\,dt \le 1/\varepsilon.$$

If $x = \frac{V(r)}{2BV(t)}\,P^{-1}\left(\frac{1}{\varepsilon V(r)}\right)$, this inequality is

$$\int_0^{P^{-1}(\frac{1}{\varepsilon V(r)})/(2B)} \frac{P(x)}{x^2}\,dx \le \frac{2B}{\varepsilon V(r) P^{-1}\left(\frac{1}{\varepsilon V(r)}\right)}.$$

Writing

$$y = P^{-1}\left(\frac{1}{\varepsilon V(r)}\right)$$

one obtains

$$\int_0^{y/(2B)} \frac{P(x)}{x^2}\,dx \le 2B\,P(y)/y, \qquad y > 0. \tag{2.6}$$

Then it follows from (1.4) that $\int_0^{y/(4B)} \frac{p(x)}{x}\,dx \le 4Bp(y)$. Now let $0 < \eta < 1$, then on integrating by parts

$$\begin{aligned}4Bp(y) &\ge \int_0^{y/(4B)} \frac{p(x)}{x}\,dx \\ &\ge \int_0^{y/(4B)} \log\left(\frac{y}{4Bt}\right) dp(t) \\ &\ge \int_0^{\eta y/(4B)} \log\left(\frac{y}{4Bt}\right) dp(t) \\ &\ge \log(1/\eta) p(\eta y/(4B)).\end{aligned}$$

Choose η so that $\log(1/\eta) \ge 8B$ and $\delta = \eta/(4B)$, then $p(\delta y) \le \frac{1}{2}\,p(y)$. This, as was noted, implies by [5, Prop. 3] that $\tilde{P} \in \Delta_2$. □

3. MAIN RESULTS

Our first result concerns the identity operator defined on monotone functions.

Theorem 3.1. *Suppose P and Q are increasing and P is weakly convex. Then*

$$Q^{-1}\left(\int_0^\infty Q[\theta(x)f(x)]w(x)\,dx\right) \le P^{-1}\left(\int_0^\infty P[Cf(x)]v(x)\,dx\right) \tag{3.1}$$

holds for all $0 \le f\downarrow$, if and only if there is a constant $B > 0$, such that,

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\varepsilon_j}{B}\,\theta(x)\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j P(\varepsilon_j)\int_{x_j}^{x_{j+1}} v(x)\,dx\right) \tag{3.2}$$

is satisfied for all non-negative decreasing sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and the covering sequence $\{x_j\}_{j\in\mathbb{Z}}$ such that $\int_0^{x_j} v = 2^k$, $k \in \mathbb{Z}$.

Similarly, (3.1) holds for all $0 \le f\uparrow$, if and only if (3.2) is satisfied for all non-negative increasing sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and the covering sequence $\{x_j\}_{j\in\mathbb{Z}}$ satisfying $\int_{x_j}^\infty = 2^{-k}$.

Proof. We only prove the first part of the theorem since the argument for the second part is similar.

Let $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ be any decreasing sequence, then $f(x) = \sum_j \varepsilon_j \chi_{(x_j,x_{j+1})}(x)$ is decreasing and substituting f into (3.1), (3.2) follows with $B = C$.

Conversely if (3.2) holds then, since $\int_0^{x_j} v = 2^k$ and $2P(x) \le P(Mx)$, $M > 1$

$$\begin{aligned}
Q^{-1}\left(\int_0^\infty Q[\theta(x)f(x)]w(x)\,dx\right) &\le Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q[\theta(x)f(x_j)]w(x)\,dx\right)\\
&\le P^{-1}\left(\sum_j P(Bf(x_j))\int_{x_j}^{x_{j+1}} v\right)\\
&= P^{-1}\left(\sum_j 2P(Bf(x_j))\int_{x_{j-1}}^{x_j} v\right)\\
&\le P^{-1}\left(\sum_j \int_{x_{j-1}}^{x_j} P(MBf(x))v(x)\,dx\right)\\
&= P^{-1}\left(\int_0^\infty P[MBf(x)]v(x)\,dx\right).
\end{aligned}$$

This proves Theorem 3.1. □

If $Q \circ P^{-1}$ is convex, Theorem 3.1 has the following form:

Corollary 3.2. *Suppose P and Q are increasing, P is weakly convex and $P \ll Q$. Then (3.1) holds for all $0 \le f\downarrow$, if and only if for all $\varepsilon > 0$ and $r > 0$, there is a constant $B > 0$, such that,*

$$Q^{-1}\left(\int_0^r Q\left[\frac{\theta(x)}{B}\,P^{-1}\left(\frac{\varepsilon}{\int_0^r v}\right)\right] w(x)\,dx\right) \le P^{-1}(\varepsilon). \tag{3.3}$$

Similarly, (3.1) is satisfied for all $0 \le f\uparrow$, if and only if

$$Q^{-1}\left(\int_r^\infty Q\left[\frac{\theta(x)}{B}\,P^{-1}\left(\frac{\varepsilon}{\int_r^\infty v}\right)\right] w(x)\,dx\right) \le P^{-1}(\varepsilon) \tag{3.4}$$

is satisfied.

Proof. By Theorem 3.1 is suffices to show that (3.2) with increasing (decreasing) sequence $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ is equivalent to (3.3) (respectively (3.4)).

First fix $j = k_0 \in \mathbb{Z}$ and let $x_{j_0} = r > 0$. Then for fixed $\varepsilon > 0$ define $\varepsilon_m = P^{-1}(\varepsilon / \int_0^r v)$, if $m < k_0$ and zero otherwise. Clearly $\{\varepsilon_m\}_{m\in\mathbb{Z}}$ is decreasing and by (3.2)

$$
\begin{aligned}
Q^{-1}&\left(\int_0^r Q\left[\frac{\theta(x)}{B}\,P^{-1}\left(\frac{\varepsilon}{\int_0^r v}\right)\right] w(x)\,dx\right) \\
&= Q^{-1}\left(\sum_{j<k_0}\int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{B}\,P^{-1}\left(\frac{\varepsilon}{\int_0^r v}\right)\right] w(x)\,dx\right) \\
&= Q^{-1}\left(\sum_{j<k_0}\int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)\varepsilon_j}{B}\right] w(x)\,dx\right) \\
&\le\; P^{-1}\left(\sum_{j<k_0} P\left(P^{-1}\left(\frac{\varepsilon}{\int_0^r v}\right)\right)\int_{x_j}^{x_{j+1}} v(x)\,dx\right) = P^{-1}(\varepsilon).
\end{aligned}
$$

To prove the converse, recall that since P is weakly convex, there is an $M \ge 1$, such that $2P(x) \le P(Mx)$. Hence with $y = P(Mx)$

$$
P^{-1}(y) \le MP^{-1}(y/2), \qquad y > 0. \tag{3.5}
$$

If $\{x_j\}_{j\in\mathbb{Z}}$ is a covering sequence satisfying $\int_0^{x_j} v = 2^k$ and $\eta_j > 0$, to be determined later, then by (3.5) and (3.3) with $\varepsilon = \eta_j$ and $r = x_{j+1}$,

$$
\begin{aligned}
\int_{x_j}^{x_{j+1}} & Q\left[\frac{\theta(x)}{BM}\,P^{-1}\left(\frac{\eta_j}{\int_{x_j}^{x_{j+1}} v}\right)\right] w(x)\,dx \\
&\le \int_0^{x_{j+1}} Q\left[\frac{\theta(x)}{B}\,P^{-1}\left(\frac{\eta_j}{\int_0^{x_{j+1}} v}\right)\right] w(x)\,dx \le Q\circ P^{-1}(\eta_j).
\end{aligned}
$$

Since $P \ll Q$, summing over $k \in \mathbb{Z}$ yields

$$
\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{BM}\,P^{-1}\left(\frac{\eta_j}{\int_{x_j}^{x_{j+1}} v}\right)\right] w(x)\,dx \le \sum_j Q\circ P^{-1}(\eta_j) \le Q\circ P^{-1}\left(\sum_j A\eta_j\right),
$$

where A is the constant arising from condition $P \ll Q$ (cf. Defn. 1.1c)). Now choose η_j so that $\{\eta_j/2^k\}$ is decreasing, hence $\varepsilon_j = P^{-1}(A\eta_j / \int_{x_j}^{x_{j+1}} v)$ defines a decreasing sequence. Therefore

$$
Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{MB}\,P^{-1}\left(\frac{P(\varepsilon_j)}{A}\right)\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j P(\varepsilon_j)\int_{x_j}^{x_{j+1}} v\right),
$$

and applying (3.5) α-times so that $2^\alpha/A \ge 1$, then

$$
P^{-1}\left(\frac{2P(\varepsilon_j)}{2A}\right) \ge \frac{1}{M}\,P^{-1}\left(\frac{2}{A}\,P(\varepsilon_j)\right) \ge \cdots \ge \frac{1}{M^\alpha}\,\varepsilon_j
$$

and the result follows.

If $0 \le f\uparrow$, fix $k_0 \in \mathbb{Z}$ such that $x_{j_0} = r > 0$ and define

$$
\varepsilon_m = P^{-1}(\varepsilon / \int_r^\infty v) \text{ if } m \ge k_0 \text{ and zero otherwise.}
$$

Then $\{\varepsilon_m\}_{m\in\mathbb{Z}}$ is increasing and the previous argument shows that (3.4) follows from (3.2). Also if $\{x_j\}_{j\in\mathbb{Z}}$ is a covering sequence such that $\int_{x_j}^{\infty} v = 2^{-k}$ and $\eta_j > 0$, then by (3.5) and (3.4), since $2\int_{x_j}^{x_{j+1}} v = 2^{-k}$,

$$\int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{MB}\, P^{-1}\left(\frac{\eta_j}{\int_{x_j}^{x_{j+1}} v}\right)\right] w(x)\,dx \leq \int_{x_j}^{\infty} Q\left[\frac{\theta(x)}{B}\, P^{-1}\left(\frac{\eta_j}{\int_{x_j}^{\infty} v}\right)\right] w(x)\,dx \leq Q\circ P^{-1}(\eta_j).$$

Summing over k, and choosing η_j so that $\{\frac{\eta_j}{2^{-k}}\}$ is an increasing sequence, then with

$$\varepsilon_j = P^{-1}\left(\frac{A\eta_j}{\int_{x_j}^{x_{j+1}} v}\right), \qquad k\in\mathbb{Z},$$

defines an increasing sequence, where A is the constant arising from the condition $P \ll Q$. The inequality (3.2) now follows as before. □

Corollary 3.2 was proved by J. Q. Sun [24, Lemma 3.1] in the case when P and Q are N-functions (hence convex). If $P(x) = x^p$, $Q(x) = x^q$, $0 < p \leq q < \infty$, one obtains (with $\theta(x) = 1$) the well known weight conditions ([21, 23]) which characterize (3.1). If $0 < q < p < \infty$ we have:

Corollary 3.3. *Let* $0 < q < p < \infty$ *and* $1/r = 1/q - 1/p$, *then the following are equivalent:*

$$\left(\int_0^\infty f^q w\right)^{1/q} \leq C\left(\int_0^\infty f^p v\right)^{1/p} \tag{3.6}$$

is satisfied for all $0 \leq f\downarrow$.

$$\int_0^\infty [W^{1/p}V^{-1/p}]^r w \equiv B_0^r < \infty, \tag{3.7}$$

$$\sum_j \left[w(E_j)^{1/q}(E_j)^{-1/p}\right]^r \equiv B_1^r < \infty, \tag{3.8}$$

where $w(E_j) = \int_{x_j}^{x_{j+1}} w$, $v(E_j) = \int_{x_j}^{x_{j+1}} v$ *and the covering sequence* $\{x_j\}$ *satisfies* $V(x_j) = 2^k$.

$$\left[\sum_j \varepsilon_j^q w(E_j)\right]^{1/q} \leq B\left[\sum_j \varepsilon_j^p v(E_j)\right]^{1/p} \tag{3.9}$$

holds for all decreasing sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ *and covering sequences* $\{x_j\}$ *with* $V(x_j) = 2^k$. *(Recall:* $W(t) = \int_0^t w$, $V(t) = \int_0^t v$.*)*

If $0 \leq f\uparrow$ the above statement holds with W and V replaced by W^* and V^*, respectively, the covering sequence $\{x_j\}$ satisfies $V^*(x_j) = 2^{-k}$ and $\{\varepsilon_j\}$ is taken to be increasing.

Proof. We only prove the corollary in the case $0 \leq f\downarrow$ since the case $0 \leq f\uparrow$ is proved, with obvious modifications, in the same way.

The equivalence of (3.6) and (3.9) follows at once from Theorem 3.1 with $Q(x) = x^q$, $P(x) = x^p$, $\theta(x) = 1$. Since the equivalence of (3.6) and (3.7) was proved in [21, 23], it remains to prove (3.7) ⇒ (3.8) ⇒ (3.9).

Since $r/q = r/p + 1$ and $w(E_j) = \int_{x_j}^{x_{j+1}} w$, it follows that

$$w(E_j)^{r/q} = \frac{r}{q} \int_{x_j}^{x_{j+1}} \left(\int_{x_j}^{t} w \right) w(t)\, dt \leq \frac{r}{q} \int_{x_j}^{x_{j+1}} W(t)^{r/p} w(t)\, dt$$

on integrating. Since $v(E_j) = 2^k = V(x_j)$ it follows therefore that

$$\begin{aligned} \sum_j [w(E_j)^{1/q} v(E_j)^{-1/p}]^r &\leq \frac{r}{q} \sum_j \int_{x_j}^{x_{j+1}} 2^{-rk/p} W(t)^{r/p} v(t)\, dt \\ &\leq \frac{r 2^{r/p}}{q} \sum_j \int_{x_j}^{x_{j+1}} V(t)^{-r/p} W(t)^{r/p} w(t)\, dt \\ &= \frac{r 2^{r/p}}{q} B_0^r. \end{aligned}$$

Hence (3.7) ⇒ (3.8).

Since the dual of the sequence space $\ell^{p/q}$ is $\ell^{r/q}$, it follows that $\sum_j [w(E_j) v(E_j)^{-q/p}]^{r/q} = B_1^r < \infty$ implies

$$\sum_j \eta_j w(E_j) v(E_j)^{-q/p} \leq B_1^q \left(\sum_j \eta_j^{p/q} \right)^{q/p}$$

for any positive sequence $\{\eta_j\}$ in $\ell^{p/q}$. Now choose $\{\eta_j\}$ so that $\eta_j^{1/q} = \varepsilon_j v(E_j)^{1/p} = \varepsilon_j 2^{k/p}$ with $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ decreasing. Thus (3.8) ⇒ (3.9), which completes the proof. □

Note that if $0 < q < p = 1$, then with $q = \frac{\alpha}{\alpha+1}$, $\alpha > 0$, $r = \alpha$ and one shows that

$$\frac{B_1}{2(1+\alpha)^{1/\alpha}} \leq B_0 \leq 2(1+\alpha)^{1/\alpha} B_1.$$

Here of course

$$B_0 = \left(\int_0^\infty W^\alpha V^{-\alpha} w \right)^{1/\alpha} \quad \text{and} \quad B_1 = \left(\sum_j w(E_j)^{\alpha+1} v(E_j)^{-\alpha} \right)^{1/\alpha}.$$

We now give the main result of this section.

Theorem 3.4. *Suppose P is an N-function, P and $\tilde{P}$ satisfy the Δ_2 condition and Q weakly convex. If K is a Hardy-type operator defined on the cone of decreasing functions, then*

$$Q^{-1}\left(\int_0^\infty Q[\theta(x) Kf(x)] w(x)\, dx \right) \leq P^{-1}\left(\int_0^\infty P[Cf(x)] v(x)\, dx \right) \tag{3.10}$$

is satisfied, if and only if there is a constant $B > 0$, such that for all positive sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and all covering sequences $\{x_j\}_{j\in\mathbb{Z}}$ with $i(x) = x$

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{B} \left\| \frac{k(x_j,\cdot)\chi_{(x_{j-1},x_j)} i}{\varepsilon_j V} \right\|_{\tilde{P}(\varepsilon_j v)} \right] w(x)\, dx \right) \leq P^{-1}\left(\sum_j \frac{1}{\varepsilon_j} \right) \tag{3.11}$$

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x) k(x,x_j)}{B} \left\| \frac{\chi_{(x_{j-1},x_j)} i}{\varepsilon_j V} \right\|_{\tilde{P}(\varepsilon_j v)} \right] w(x)\, dx \right) \leq P^{-1}\left(\sum_j \frac{1}{\varepsilon_j} \right) \tag{3.12}$$

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{B}\left\|\frac{(K1)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \leq P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \tag{3.13}$$

are satisfied, and for all positive decreasing sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ *and the covering sequence* $\{x_j\}_{j\in\mathbb{Z}}$ *satisfying* $\int_0^{x_j} v = 2^k$

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\varepsilon_j\theta(x)(K1)(x)}{B}\right] w(x)\,dx\right) \leq P^{-1}\left(\sum_j P(\varepsilon_j)\int_{x_j}^{x_{j+1}} v\right) \tag{3.14}$$

is satisfied.

Proof. (Sufficiency.) The idea comes from [23]. We may assume that f has the form $f(x) = \int_x^\infty h$, $h > 0$, for once the result has been proved for such f, a limiting argument (see e.g. [24]) gives the general case. Clearly since $(K1)(x) = \int_0^x k(x,y)\,dy$

$$\begin{aligned}(Kf)(x) &= \int_0^x k(x,y)\int_y^\infty h(t)\,dtdy \\ &= (K1)(x)f(x) + \int_0^x h(t)\left(\int_0^t k(x,y)\,dy\right)dt.\end{aligned}$$

But since

$$\frac{1}{V(t)} - \frac{1}{V(x)} = \int_t^x V(y)^{-2}v(y)\,dy \quad \text{and} \quad \int_0^x h(s)V(s)\,ds \leq \int_0^x f(t)v(t)\,dt,$$

it follows again on interchanging the order of integration that

$$\begin{aligned}&\int_0^x h(t)\int_0^t k(x,y)\,dydt \\ &\quad= \int_0^x\int_0^t k(x,y)h(t)V(t)\left[\frac{1}{V(x)} + \int_t^x V(s)^{-2}v(s)\,ds\right]dydt \\ &\quad= \frac{1}{V(x)}\int_0^x k(x,y)\int_y^x h(t)V(t)\,dtdy \\ &\qquad+ \int_0^x V(s)^{-2}v(s)\int_0^s h(t)V(t)\left(\int_0^t k(x,y)\,dy\right)dtds \\ &\quad\leq \frac{1}{V(x)}\int_0^x k(x,y)\int_0^x f(t)v(t)\,dtdy \\ &\qquad+ \int_0^x V(s)^{-2}v(s)\left(\int_0^s k(x,y)\,dy\right)\int_0^s f(t)v(t)\,dtds \\ &\quad\leq (K1)(x)\,\frac{1}{V(x)}\int_0^x f(t)v(t)\,dt + I(x) \quad \text{(by definition of } I(x)\text{) respectively.}\end{aligned}$$

Now since $k(x,y) \leq D[k(x,s) + k(s,y)]$, $y < s < x$,

$$\begin{aligned}I(x) \leq D\Bigg[&\int_0^x k(x,s)V(s)^{-2}v(s)s\int_0^s f(t)v(t)\,dtds \\ &+ \int_0^x V(s)^{-2}v(s)\left(\int_0^s f(t)v(t)\,dt\right)(K1)(s)\,ds\Bigg]\end{aligned}$$

and writing

$$F(s) = V(s)^{-2} v(s) s \int_0^s fv,$$

one obtains

$$\begin{aligned}(Kf)(x) &\le (K1)(x) f(x) + (K1)(x)\, \frac{1}{V(x)} \int_0^x f(t)v(t)\,dt \\ &\quad + D \int_0^x k(x,s) F(s)\,ds + D \int_0^x \frac{(K1)(s)}{s}\, F(s)\,ds \\ &\equiv (I_1 + I_2 + I_3 + I_4)(x),\end{aligned}$$

respectively. Now

$$\theta(x)(Kf)(x) \le \theta(x) \sum_{i=1}^4 I_i(x) \le 4\theta(x) \max_{s=1,2,3,4} I_s(x) = 4\theta(x) I_{s(x)}(x),$$

where $s(x) \in \{1,\, 2,\, 3,\, 4\}$, and since Q is increasing and satisfies $2Q(x) \le Q(Mx)$, $M > 1$, we have

$$\begin{aligned}Q[\theta(x)(Kf)(x)] &\le Q[4\theta(x) I_{s(x)}(x)] \\ &\le \sum_{s=1}^4 Q[4\theta(x) I_s(x)] \\ &\le \sum_{i=1}^4 \frac{1}{4} Q[4M^2 \theta(x) I_s(x)].\end{aligned}$$

Integration yields

$$\int_0^\infty Q[\theta(x)(Kf)(x)] w(x)\,dx \le \sum_{s=1}^4 \frac{1}{4} \int_0^\infty Q[4M^2\theta(x) I_s(x)] w(x)\,dx$$

and therefore it suffices to prove that

$$\int_0^\infty Q[4M^2\theta(x) I_s(x)] w(x)\,dx \le Q \circ P^{-1} \left(\int_0^\infty P[Cf(x)] v(x)\,dx \right) \tag{3.15}$$

$s = 1, 2, 3, 4$ is satisfied.

Since $I_1(x) = (K1)(x) f(x)$, then by Theorem 3.1 with θ replaced by $\theta(x)(K1)(x)$, (3.15) holds if and only if (3.14) is satisfied.

Next, since $0 \le f\downarrow$ so is $\frac{1}{V(x)} \int_0^x fv$, and since $I_2(x) = (K1)(x)\, \frac{1}{V(x)} \int_0^x fv$, Theorem 3.1 shows (with θ replaced by $\theta(x)(K1)(x)$) that (3.14) is equivalent to

$$\begin{aligned}\int_0^\infty Q[4M^2\theta(x) I_2(x)] w(x)\,dx &\le Q \circ P^{-1} \left(\int_0^\infty P\left[C\, \frac{1}{V(x)} \int_0^x fv \right] v(x)\,dx \right) \\ &\le Q \circ P^{-1} \left(\int_0^\infty P[Cf(x)] v(x)\,dx \right).\end{aligned}$$

Here the last inequality follows from (2.1) of Lemma 2.3.

Next, $I_3(x) = D(KF)(x)$, so that by Theorem 2.1 with $\rho(x) = V(x)/(xv(x))$

$$\int_0^\infty Q[4M^2\theta(x)I_3(x)]w(x) \le Q\circ P^{-1}\left(\int_0^\infty P\left[C\,\frac{1}{V(x)}\int_0^x fv\right]v(x)\,dx\right)$$
$$\le Q\circ P^{-1}\left(\int_0^\infty P[Cf(x)]v(x)\,dx\right).$$

Here the first inequality holds if (3.11) and (3.12) are satisfied and the second follows from (2.1) of Lemma 2.3.

Since $I_4(x) = D\int_0^x (K1)(s)\ \frac{F(s)}{s}\,ds$ we apply Theorem 2.1 with $k(x,y) = 1$ and $\rho(x) = V(x)/(v(x)(K1)(x))$, so that

$$\int_0^\infty Q[4M^2\theta(x)I_4(x)]w(x)\,dx \le Q\circ P^{-1}\left(\int_0^\infty P\left[C\,\frac{1}{V(x)}\int_0^x fv\right]v(x)\,dx\right)$$
$$\le Q\circ P^{-1}\left(\int_0^\infty P[Cf(x)]v(x)\,dx\right)$$

is satisfied if and only if (3.13) holds. The last inequality follows of course again from (2.1) of Lemma 2.3.

(Necessity.) Since $0 \le f\downarrow$, $(Kf)(x) \ge (K1)(x)f(x)$ so that (3.10) implies (3.1) with θ replaced by $\theta(x)(K1)(x)$. Now Theorem 3.1 applies if P is an N-function and Q weakly convex and so (3.14) follows.

To prove that (3.10) implies (3.11) observe first that for fixed k,

$$\left\|\frac{ik(x_j,\cdot)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)} \tag{3.16}$$

is bounded. If this is not the case, then there is a sequence $\{f_n\}$ of non-negative functions satisfying $\|Cf_n\|_{P(\varepsilon_j v)} \le 1$, with C the constant of (3.10), and a sequence $\{\alpha_n\}$ with $\alpha_n \to \infty$, $n \to \infty$, such that (by definition of Orlicz norm) for each n

$$\begin{aligned}\alpha_n &< C\int_{x_{j-1}}^{x_j} \frac{xk(x_j,x)f_n(x)v(x)}{V(x)}\,dx\\ &\le C\int_0^{x_j} \frac{k(x_j,x)f_n(x)v(x)}{V(x)}\left(\int_0^x\,dy\right)dx\\ &= C\int_0^{x_j}\int_y^{x_j} \frac{k(x_j,x)f_n(x)v(x)}{V(x)}\,dxdy\\ &\le C\int_0^{x_j} k(x_j,y)F_n(y)\,dy,\end{aligned}$$

since $k(x_j,\cdot)$ is decreasing. Here $F_n(y) = \int_y^\infty \frac{f_n(x)v(x)}{V(x)}\,dx$. But since

$$\int_0^\infty P[Cf_n(x)]\varepsilon_j v(x)\,dx \le 1,$$

(2.2) of Lemma 2.3 and (3.10) show that

$$\begin{aligned} P^{-1}\left(\frac{1}{\varepsilon_j}\right) &\geq P^{-1}\left(\int_0^\infty P[Cf_n(x)]v(x)\,dx\right) \\ &\geq P^{-1}\left(\int_0^\infty P\left[\frac{C}{C_1}\,F_n(x)\right]v(x)\,dx\right) \\ &\geq Q^{-1}\left(\int_0^\infty Q\left[\frac{\theta(x)}{C_1}(KF_n)(x)\right]w(x)\,dx\right) \\ &\geq Q^{-1}\left(\int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{C_1}\int_0^{x_j} k(x_j,y)F_n(y)\,dy\right]w(x)\,dx\right) \\ &> Q^{-1}\left(\int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)\alpha_n}{C\,C_1}\right]w(x)\,dx\right), \end{aligned}$$

where C_1 is the constant of (2.2). But this is a contradiction since $\alpha_n \to \infty$. Hence (3.16) is bounded.

Now suppose (3.11) fails to be satisfied. Then for any $B > 0$ there exists a covering sequence $\{x_j\}_{j\in\mathbb{Z}}$ and a positive sequence $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ such that

$$P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) < Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{2BC_1}\left\|\frac{k(x_j,\cdot)\chi_{(x_{j-1},x_j)}i}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right]w(x)\,dx\right)$$

where C_1 is taken to be the constant of (2.2). Now for $k \in \mathbb{Z}$, choose $f_j \geq 0$, such that $\operatorname{supp} f_j \subset (x_{j-1}, x_j)$ with

$$\int_0^\infty P[BC_1 f_j(x)]\varepsilon_j v(x)\,dx \leq 1 \tag{3.17}$$

and

$$\begin{aligned} \frac{1}{2BC_1}\left\|\frac{k(x_j,\cdot)\chi_{(x_{j-1},x_j)}i}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)} &\leq \int_{x_{j-1}}^{x_j} \frac{xk(x_j,x)f_j(x)v(x)}{V(x)}\,dx \\ &\leq \int_0^{x_j} k(x_j,y)F_j(y)\,dy, \end{aligned}$$

where

$$F_j(y) = \int_y^\infty \frac{f_j(x)v(x)}{V(x)}\,dx.$$

Let $f(x) = \sum_j f_j(x)$ and $F(x) = \int_x^\infty \frac{f(t)v(t)}{V(t)}\,dt$, then by (2.2) of Lemma 2.3, (3.17) and our assumption

$$\begin{aligned}
&P^{-1}\left(\int_0^\infty P[BF(x)]v(x)\,dx\right) \\
&\qquad \le P^{-1}\left(\int_0^\infty P[BC_1 f(x)]v(x)\,dx\right) \\
&\qquad = P^{-1}\left(\sum_j \int_{x_{j-1}}^{x_j} P[BC_1 f_j(x)]v(x)\,dx\right) \\
&\qquad \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \\
&\qquad < Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{2BC_1}\left\|\frac{k(x_j,\cdot)\chi_{(x_{j-1},x_j)}i}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \\
&\qquad \le Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\theta(x)\int_0^{x_j} k(x_j,y)F_j(y)\,dy\right] w(x)\,dx\right) \\
&\qquad \le Q^{-1}\left(\int_0^\infty Q\left[\theta(x)(KF)(x)\right] w(x)\,dx\right) \\
&\qquad \le P^{-1}\left(\int_0^\infty P[CF(x)]v(x)\,dx\right),
\end{aligned}$$

where the last inequality is (3.10). But this is impossible for $B > C$ and hence (3.11) must be satisfied.

To show that (3.12) and (3.13) are satisfied one proceeds as before. First one shows that both

$$\left\|\frac{\chi_{(x_{j-1},x_j)}i}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)} \quad \text{and} \quad \left\|\frac{(K1)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}$$

are bounded for fixed k. Then with $f(x)$ and $F(x)$ defined as above one has

$$\int_{x_{j-1}}^{x_j} \frac{x\, f_j(x)v(x)}{V(x)}\,dx \le \int_0^{x_j}\int_y^\infty \frac{f_j(x)v(x)}{V(x)}\,dxdy$$

and for (3.13)

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \frac{(K1)(x)f_j(x)v(x)}{V(x)}\,dx &\le \int_0^{x_j}\left(\int_0^x k(x,y)\,dy\right)\frac{f_j(x)v(x)}{V(x)}\,dx \\
&\le \int_0^{x_j} k(x_j,y)\int_y^\infty \frac{f_j(x)v(x)}{V(x)}\,dxdy.
\end{aligned}$$

Inequality (3.12) is then obtained as (3.11) was shown to hold. To prove (3.13) assume to the contrary that (3.13) fails. Then for any $B > 0$, we have

$$\begin{aligned}
&P^{-1}\left(\int_0^\infty P[BF(x)]v(x)\,dx\right)\\
&\quad\le P^{-1}\left(\int_0^\infty P[BC_1 f(x)]v(x)\,dx\right)\\
&\quad\le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right)\\
&\quad< Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{2BC_1}\left\|\frac{(K1)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right)\\
&\quad\le Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\theta(x)\int_0^{x_j} k(x,y)\int_y^\infty \frac{f_j(t)v(t)}{V(t)}\,dtdy\right] w(x)\,dx\right)\\
&\quad\le Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\theta(x)\int_0^{x_j} k(x,y)F(y)\,dy\right] w(x)\,dx\right)\\
&\quad\le Q^{-1}\left(\int_0^\infty Q\left[\theta(x)(KF)(x)\right] w(x)\,dx\right)\\
&\quad\le P^{-1}\left(\int_0^\infty P[CF(x)]v(x)\,dx\right)
\end{aligned}$$

from which the contradiction follows for $B > C$. This proves Theorem 3.4. □

If $k(x,y) = 1$, $\theta(x) = x^a$, $-1 \le a < \infty$, the conditions (3.11), (3.12), (3.13) coincide and since $(K1)(x) = x$ we get the following corollary.

Corollary 3.5. *Let P and Q be as in Theorem 3.4 and $a \ge -1$. Then*

$$Q^{-1}\left(\int_0^\infty Q\left[x^a\int_0^x f\right] w(x)\,dx\right) \le P^{-1}\left(\int_0^\infty P[Cf(x)]v(x)\,dx\right)$$

is satisfied for all $0 \le f\downarrow$, if and only if for all decreasing sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and the covering sequence $\{x_j\}$ satisfying $\int_0^{x_j} v = 2^k$,

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\varepsilon_j}{B}x^{a+1}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j P(\varepsilon_j)\int_{x_j}^{x_{j+1}} v\right) \tag{3.18}$$

holds, and

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{x^a}{B}\left\|\frac{i\chi_{(x_{j-1},x_j)}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \tag{3.19}$$

is satisfied for all positive sequences $\{\varepsilon_j\}$ and all covering sequences $\{x_j\}$.

If Q is also an N-function, then a result corresponding to Corollary 3.5 holds also for the dual operator.

Corollary 3.6. *Let P and Q be N-functions and P, $\tilde{P} \in \Delta_2$. If $a \ge -1$ then*

$$Q^{-1}\left(\int_0^\infty Q\left[\int_x^\infty t^a f(t)\,dt\right] w(x)\,dx\right) \le P^{-1}\left(\int_0^\infty P[Cf(x)]v(x)\,dx\right) \tag{3.20}$$

is satisfied for all $0 \le f\downarrow$*, if and only if*

$$Q^{-1}\left(\sum_j \int_{y_{j-1}}^{y_j} Q\left[\frac{1}{B}\left\|\frac{k(\cdot,y_j)\chi_{(y_j,y_{j+1})}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \tag{3.21}$$

and

$$Q^{-1}\left(\sum_j \int_{y_{j-1}}^{y_j} Q\left[\frac{k(y_j,x)}{B}\left\|\frac{\chi_{(y_j,y_{j+1})}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \tag{3.22}$$

holds for all positive sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ *and all covering sequences* $\{y_j\}_{j\in\mathbb{Z}}$*. Here*

$$k(y,x) = \begin{cases} \ln(y/x) & \text{if } a=-1, \\ y^{a+1}-x^{a+1} & \text{if } a>-1. \end{cases}$$

Proof. By [7, Thm. 2.2], (3.20) is equivalent to

$$Q^{-1}\left(\int_0^\infty Q\left[\int_x^\infty t^a \int_t^\infty h(s)\,dsdt\right] w(x)\,dx\right) \le P^{-1}\left(\int_0^\infty P\left[C\,\frac{V(x)h(x)}{v(x)}\right] v(x)\,dx\right),$$

$h \ge 0$. However, since

$$\int_x^\infty t^a \int_t^\infty h(s)\,dsdt = \int_x^\infty k(y,x)h(y)\,dy,$$

the result follows from Proposition 2.2 with $\theta(x)=1$, $\rho(x)=V(x)/v(x)$. □

Remark 3.1.

(i) Let $P(x)=x^p$, $Q(x)=x^q$, $0<q<p<\infty$, $p>1$ and $a=-1$ then (3.18) is

$$\left(\sum_j \varepsilon_j^q w(E_j)\right)^{1/q} \le C\left(\sum_j \varepsilon_j^p v(E_j)\right)^{1/p},$$

where $w(E_j)=\int_{x_j}^{x_{j+1}} w$ and $v(E_j)=\int_{x_j}^{x_{j+1}} v$. But by Corollary 3.3 this is equivalent to

$$\int_0^\infty [W^{1/p}V^{-1/p}]^r w < \infty, \qquad \frac{1}{r}=\frac{1}{q}-\frac{1}{p}.$$

Also, if $\eta_j=\varepsilon_j^{-q/p}$ then (3.19) takes the form

$$\sum_j \eta_j\left(\int_{x_j}^{x_{j+1}} x^{-q}w(x)\,dx\right)\left(\int_{x_{j-1}}^{x_j} t^{p'}V(t)^{-p'}v(t)\,dt\right)^{q/p'} \le C\left(\sum_j \eta_j^{p/q}\right)^{q/p}.$$

But the dual space of $\ell^{p/q}$ is $\ell^{r/q}$, where $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$ and hence (3.19) is in this case

$$\left\{\sum_j\left(\int_{x_j}^{x_{j+1}} x^{-q}w(x)\,dx\right)^{r/q}\left(\int_{x_{j-1}}^{x_j} t^{p'}V(t)^{-p'}v(t)\,dt\right)^{r/p'}\right\}^{1/r} \le C.$$

(Cf. [21, Thm. 2], where this was proved in case $1<q<p<\infty$ and [23] in the remaining case.)

(ii) Considering Corollary 3.6 in the case $P(x)=x^p$, $Q(x)=x^q$, $1<q<p<\infty$, $a=-1$ we see that (3.21) takes the form

$$\sum_j \eta_j\left(\int_{y_{j-1}}^{y_j} w\right)\left(\int_{y_j}^{y_{j+1}} \ln^{p'}(t/y_j)V(t)^{-p'}v(t)\,dt\right)^{q/p'} \le C\left(\sum_j \eta_j^{p/q}\right)^{q/p},$$

where again $\eta_j = \varepsilon_j^{-q/p}$. But again since $\ell^{r/q}$ is the dual of $\ell^{p/q}$ it follows that in this case (3.21) is equivalent to

$$\left\{\sum_j \left(\int_{y_{j-1}}^{y_j} w\right)^{r/q} \left(\int_{y_j}^{y_{j+1}} \ln^{p'}(t/y_j) V(t)^{-p'} v(t)\,dt\right)^{r/p'}\right\}^{1/r} \le C,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Similarly, (3.22) takes the form

$$\left\{\sum_j \left(\int_{y_{j-1}}^{y_j} \ln^q(y_j/x) w(x)\,dx\right)^{r/q} \left(\int_{y_j}^{y_{j+1}} V(t)^{-p'} v(t)\,dt\right)^{r/p'}\right\}^{1/r} \le C,$$

$\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, for all covering sequences $\{y_j\}_{j\in\mathbb{Z}}$.

Hence these two conditions are necessary and sufficient for the inequality

$$\left(\int_0^\infty w(x) \left(\int_x^\infty \frac{f(t)}{t}\,dt\right)^q dx\right)^{1/q} \le C \left(\int_0^\infty f(x)^p v(x)\,dx\right)^{1/p}$$

to be satisfied for all $0 \le f\downarrow$.

4. HARDY-TYPE OPERATORS ON INCREASING FUNCTIONS

In order to obtain weight characterizations for which modular inequalities for the Hardy-type operator

$$(Kf)(x) = \int_0^x k(x,y) f(y)\,dy, \qquad 0 \le f\uparrow$$

are satisfied, we require also that the kernel $\bar{k}$ defined by

$$\bar{k}(x,y) = \int_y^x k(x,t)\,dt \tag{4.1}$$

satisfies also conditions (i) and (ii) of (1.1). That is,

$$\bar{k}(x,y) \le D[\bar{k}(x,z) + \bar{k}(z,y)], \qquad 0 < y < z < x. \tag{4.2}$$

Note that if $k(x,t) = (x-t)^\alpha$, $\alpha \ge 0$ then $\bar{k}$ satisfies (4.2). On the other hand if $k(x,t) = \ln(x/t)$ then $\bar{k}$ does not satisfy (4.2) for any $D \ge 1$.

The principal result for Hardy-type operators defined on the cone of increasing functions is the following:

Theorem 4.1. *Suppose K is a Hardy-type operator and $\bar{k}$ defined by (4.1) satisfies (4.2). Let P be an N-function with P, $\tilde{P} \in \Delta_2$ and Q weakly convex. Then the modular inequality*

$$Q^{-1}\left(\int_0^\infty Q[\theta(x)(Kf)(x)] w(x)\,dx\right) \le P^{-1}\left(\int_0^\infty P[Cf(x)] v(x)\,dx\right) \tag{4.3}$$

is satisfied for all $0 \le f\uparrow$, if and only if there is a constant $B > 0$, such that,

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{B} \left\| \frac{\bar{k}(x_j,\cdot)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V^*} \right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \tag{4.4}$$

and

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\bar{k}(x,x_j)\theta(x)}{B} \left\| \frac{\chi_{(x_{j-1},x_j)}}{\varepsilon_j V^*} \right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \tag{4.5}$$

holds for all positive sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ *and all covering sequences* $\{x_j\}_{j\in\mathbb{Z}}$. *Here again* $V^*(x) = \int_x^\infty v$ *with* $V^*(0) = \infty$.

Proof. Without loss of generality we may assume that f has the form $f(x) = \int_0^x h$, $h \geq 0$ (cf. [24, Lemma 3.2]). Since $V^*(x)^{-1} = \int_0^x V^*(t)^{-2}v(t)\,dt$, changing the order of integration we show that

$$\begin{aligned}(Kf)(x) &= \int_0^x k(x,y)\int_0^y h(s)\,dsdy \\ &= \int_0^x h(s)\bar{k}(x,s)\,\frac{V^*(s)}{V^*(s)}\,ds \\ &= \int_0^x h(s)\bar{k}(x,s)V^*(s)\int_0^s V^*(t)^{-2}v(t)\,dtds \\ &\leq \int_0^x V^*(t)^{-2}v(t)\bar{k}(x,t)\int_t^x h(s)V^*(s)\,dsdt \\ &\leq \int_0^x V^*(t)^{-2}v(t)\bar{k}(x,t)\int_t^\infty v(y)f(y)\,dydt.\end{aligned}$$

Hence if $F(t) = V^*(t)^{-2}v(t)\int_t^\infty fv$, then $Kf(x) \leq \int_0^x \bar{k}(x,t)F(t)\,dt$ and by Theorem 2.1 with $\rho(x) = V^*(x)/v(x)$

$$\begin{aligned}\int_0^\infty Q[\theta(x)(Kf)(x)]w(x)\,dx &\leq \int_0^\infty Q\left[\theta(x)\int_0^x \bar{k}(x,t)F(t)\,dt\right]w(x)\,dx \\ &\leq Q\circ P^{-1}\left(\int_0^\infty P\left[\frac{C}{V^*(x)}\int_x^\infty vf\right]v(x)\,dx\right)\end{aligned}$$

if and only if (4.4) and (4.5) are satisfied. Now (4.3) follows from (2.3) of Lemma 2.3.

To prove necessity one proves first that for fixed k

$$\left\|\frac{\bar{k}(x_j,\cdot)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V^*}\right\|_{\tilde{P}(\varepsilon_j v)}$$

is bounded. But this is proved (via contradiction) in the same way as the boundedness of (3.16) in the proof of Theorem 3.4, only now k and V are replaced by $\bar{k}$ and V^*, respectively. To prove that (4.4) is satisfied assume to the contrary that for every $B > 0$ there exist $\{x_j\}$ and $\{\varepsilon_j\}$ such that

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{2BC_1}\left\|\frac{\bar{k}(x_j,\cdot)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V^*}\right\|_{\tilde{P}(\varepsilon_j v)}\right]w(x)\,dx\right) > P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right).$$

By duality of Orlicz spaces there exists $f_j \geq 0$ such that supp$f_j \subset (x_{j-1},x_j)$, $\int_0^\infty P[BC_1 f_j]\varepsilon_j v$ ≤ 1 and

$$\frac{1}{2BC_1}\left\|\frac{\bar{k}(x_j,\cdot)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V^*}\right\|_{\tilde{P}(\varepsilon_j v)} < \int_{x_{j-1}}^{x_j}\frac{\bar{k}(x_j,x)f_j(x)v(x)}{V^*(x)}\,dx.$$

Now let $f = \sum f_j$ and $F(x) = \int_0^x \frac{fv}{V^*}$, so $F\uparrow$. Also

$$\begin{aligned}\int_{x_{j-1}}^{x_j} \frac{\bar{k}(x_j,x) f_j(x) v(x)}{V^*(x)}\,dx &\le \int_0^{x_j} \frac{f_j(x)v(x)}{V^*(x)} \int_x^{x_j} k(x_j,s)\,dsdx \\ &= \int_0^{x_j} k(x_j,s) \int_0^s \frac{f_j(x)v(x)}{V^*(x)}\,dxds \\ &\le \int_0^{x_j} k(x_j,s)F(s)\,ds\end{aligned}$$

and therefore by (2.4) of Lemma 2.3

$$\begin{aligned}&P^{-1}\left(\int_0^\infty P[BF(x)]v(x)\,dx\right) \\ &\qquad \le P^{-1}\left(\int_0^\infty P[BCf(x)]v(x)\,dx\right) \\ &\qquad \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \\ &\qquad < Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{\theta(x)}{2BC_1}\left\|\frac{\bar{k}(x_j,\cdot)\chi_{(x_{j-1},x_j)}}{\varepsilon_j V^*}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \\ &\qquad \le Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q[\theta(x)\int_0^{x_j} k(x_j,s)F(s)\,ds] w(x)\,dx\right) \\ &\qquad \le Q^{-1}\left(\int_0^\infty Q[\theta(x)(KF)(x)]w(x)\,dx\right) \\ &\qquad \le P^{-1}\left(\int_0^\infty P[CF(x)]v(x)\,dx\right).\end{aligned}$$

Here the last inequality is (4.3). But this is a contradiction for $B > C$. Hence (4.4) is satisfied.

The proof of (4.5) is similar, only now f_j is chosen so that

$$\frac{1}{2C_1}\left\|\frac{\chi_{(x_{j-1},x_j)}}{\varepsilon_j V^*}\right\|_{\tilde{P}(\varepsilon_j v)} < \int_{x_{j-1}}^{x_j} \frac{f_j(y)v(y)}{V^*(y)}\,dy$$

and

$$\begin{aligned}\bar{k}(x,x_j)\int_0^{x_j} \frac{f_j(y)v(y)}{V^*(y)}\,dy &\le \int_0^{x_j} \bar{k}(x,y)\,\frac{f_j(y)v(y)}{V^*(y)}\,dy \\ &\le \int_0^x \bar{k}(x,y)\,\frac{f_j(y)v(y)}{V^*(y)}\,dy \\ &= \int_0^x k(x,s)\int_0^s \frac{f_j(y)v(y)}{V^*(y)}\,dyds,\end{aligned}$$

$x \in (x_j, x_{j+1})$. We omit the details. This proves the theorem. □

Remark 4.1.

(i) If $V^*(0) < \infty$, Theorem 4.1 still holds, provided that in addition to (4.4) and (4.5) the weight condition

$$Q^{-1}\left(\int_0^\infty Q\left[\frac{1}{B}\,P^{-1}\left(\frac{1}{\varepsilon V^*(0)}\right)\theta(x)\bar{k}(x,0)\right] w(x)\,dx\right) \le P^{-1}\left(\frac{1}{\varepsilon}\right) \tag{4.6}$$

is also satisfied for all $\varepsilon > 0$.

(ii) If Q is an N-function and hence convex, the result may also be proved via the duality principle given in [7, Thm. 2.2].

A consequence of Theorem 4.1 is the following:

Corollary 4.2.

(i) *(*[10, Thm. 2.1]*) If* $1 < p \leq q < \infty$*, then*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f\right)^q w(x)\,dx\right)^{1/q} \leq C\left(\int_0^\infty f^p v\right)^{1/p} \tag{4.7}$$

is satisfied for all $0 \leq f\uparrow$*, if and only if, for all* $t > 0$

$$\left(\int_t^\infty (x-t)^q x^{-q} w(x)\,dx\right)^{1/q} V^*(t)^{-1/p}$$

and

$$\left(\int_t^\infty x^{-q}w(x)\,dx\right)^{1/q}\left(\int_0^t (t-x)^{p'}V^*(x)^{-p'}v(x)\,dx\right)^{1/p'}$$

are bounded.

(ii) *If* $0 < q < p < \infty$*,* $p > 1$ *then (4.7) is satisfied for all* $0 \leq f\uparrow$ *if and only if for all covering sequences* $\{x_j\}$

$$\left[\sum_j \left(\int_{x_j}^{x_{j+1}} x^{-q}w(x)\,dx\right)^{r/q}\left(\int_{x_{j-1}}^{x_j}(x_j-x)^{p'}V^*(x)^{-p'}v(x)\,dx\right)^{r/p'}\right]^{1/r} \leq C \tag{4.8}$$

and

$$\left[\sum_j \left(\int_{x_j}^{x_{j+1}} (x-x_j)^q x^{-q}w(x)\,dx\right)^{r/q}\left(\int_{x_{j-1}}^{x_j} V^*(x)^{-p'}v(x)\,dx\right)^{r/p'}\right]^{1/r} \leq C$$

$\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$*, are satisfied.*

(If $V^*(0) < \infty$ *the condition (4.6) must also be taken into account.)*

Proof. Let $Q(x) = x^q$, $P(x) = x^p$, $1 < p \leq q < \infty$, $\theta(x) = \frac{1}{x}$, $k(x,y) = 1$ in Theorem 4.1. Since $P \ll Q$ we may take in Theorem 4.1 $x_j = t > 0$, $x_{j-1} = 0$, $x_{j+1} = \infty$ and the result (i) follows. If $0 < q < p < \infty$, $p > 1$, then Q is weakly convex and by Theorem 4.1, (4.7) is satisfied for all $0 \leq f\uparrow$, if and only if for all covering sequences $\{x_j\}$

$$\left[\sum_j \eta_j \left(\int_{x_j}^{x_{j+1}} x^{-q}w(x)\,dx\right)\left(\int_{x_{j-1}}^{x_j}(x_j-x)^{p'}V^*(x)^{-p'}v(x)\,dx\right)^{q/p'}\right] \leq C_1\left(\sum_j \eta_j^{p/q}\right)^{q/p}$$

and

$$\left[\sum_j \eta_j \left(\int_{x_j}^{x_{j+1}} (x-x_j)^q x^q w(x)\,dx\right)\left(\int_{x_{j-1}}^{x_j} V^*(x)^{-p'}v(x)\,dx\right)^{q/p'}\right] \leq C_2\left(\sum_j \eta_j^{p/q}\right)^{q/p}$$

where we have taken $\eta_j = \varepsilon_j^{-q/p}$ in (4.4) and (4.5). But since the dual of $\ell^{p/q}$ is $\ell^{r/q}$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, the previous two estimates are equivalent to (4.8) and (4.9) respectively. □

The result of Corollary 4.2 (ii) in the case $1 < q < p < \infty$ was also proved in [10, Thm. 2.2], but the case $0 < q < 1 < p$ seems to be new.

In the remaining portion of this section we apply the results of the previous section to show that the Hardy-Littlewood maximal function and the Hilbert transform are bounded in weighted Orlicz-Lorentz spaces. This, in particular, extends the Lorentz space results of Ariño-Muckenhoupt [1] and Sawyer [21] to this general setting.

If P is an increasing function of $\mathbb{R}^+$ with $P(0) = 0$, then the Orlicz-Lorentz spaces $\bigwedge_P(v)$, with weight v consist of all Lebesgue measurable f on $\mathbb{R}^n$ such that $P^{-1}\left(\int_0^\infty P(f^*(x))v(x)\,dx\right) < \infty$. Here $f^*(t) = \inf\{s > 0 : |\{x : |f(x)| > s\}| \le t\}$ denotes the equimeasurable decreasing rearrangement of $|f|$.

Recall that if $(Mf)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy$ is the Hardy-Littlewood maximal function, then it is well known (cf. [2]) that $(Mf)^*(x) \approx \frac{1}{x}\int_0^x f^*$. It follows therefore from Corollary 3.5 with $a = -1$ that the following proposition holds:

Proposition 4.3. *Suppose P is an N-function, P, $\tilde{P} \in \Delta_2$ and Q weakly convex. Then $M : \bigwedge_P(v) \to \bigwedge_Q(w)$ is bounded, that is $Q^{-1}\left(\int_0^\infty Q((Mf)^*)w\right) \le CP^{-1}\left(\int_0^\infty P(Cf^*)v\right)$, if and if there are constants $B > 0$, such that*

$$Q^{-1}\left(\sum_j Q\left(\frac{\varepsilon_j}{B}\right)\int_{x_j}^{x_{j+1}} w\right) \le P^{-1}\left(\sum_j P(\varepsilon_j)\int_{x_j}^{x_{j+1}} v\right) \tag{4.9}$$

is satisfied for all decreasing sequences $\{\varepsilon_j\}$ and the covering sequence $\{x_j\}$ satisfying $\int_0^{x_j} v = 2^k$, and

$$Q^{-1}\left(\sum_j \int_{x_j}^{x_{j+1}} Q\left[\frac{1}{xB}\left\|\frac{i\,\chi_{(x_{j-1},x_j)}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right) \tag{4.10}$$

is satisfied for all positive sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and all covering sequences $\{x_j\}$.

Here again $V(x) = \int_0^x v$ and $i(x) = x$.

Another illustration involves the Hilbert transform defined by the principle value integral

$$(Hf)(x) = P.V.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt.$$

Then (see [21, (1.15)]) the rearrangement inequality

$$(Hf)^*(x) \le C_1\left[\frac{1}{x}\int_0^x f^*(t)\,dt + \int_x^\infty \frac{f^*(t)}{t}\,dt\right] \le C_2(Hf^*)^*(x)$$

is satisfied. But this implies that the Hilbert transform is bounded from $\bigwedge_P(v)$ to $\bigwedge_Q(w)$ if and only if

$$Q^{-1}\left(\int_0^\infty Q[Tf(x)]w(x)\,dx\right) \le P^{-1}\left(\int_0^\infty P[Cf(x)]v(x)\,dx\right)$$

is satisfied, where

$$Tf(x) = x^{-1}\int_0^x f(t)\,dt + \int_x^\infty \frac{f(t)}{t}\,dt, \quad 0 \le f\downarrow . \tag{4.11}$$

However, (4.11) will be satisfied if and only if it is satisfied for the averaging operator and its conjugate defined on decreasing functions. Hence Corollaries 3.5 and 3.6 apply with $a = -1$ and we have:

Proposition 4.4. *Suppose P and Q are N-functions and P, $\tilde{P} \in \Delta_2$. Then $H : \bigwedge_P(v) \to \bigwedge_Q(w)$ is bounded if and only if (4.9) (with $\{\varepsilon_j\}\downarrow$, $\{x_j\}$ satisfying $\int_0^{x_j} v = 2^k$), (4.10) and (see (3.18), (3.19))*

$$Q^{-1}\left(\sum_j \int_{y_{j-1}}^{y_j} Q\left[\frac{1}{B}\left\|\frac{\ln(\cdot/y_j)\chi_{(y_j,y_{j+1})}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right),$$

$$Q^{-1}\left(\sum_j \int_{y_{j-1}}^{y_j} Q\left[\frac{\ln(y_j/x)}{B}\left\|\frac{\chi_{(y_j,y_{j+1})}}{\varepsilon_j V}\right\|_{\tilde{P}(\varepsilon_j v)}\right] w(x)\,dx\right) \le P^{-1}\left(\sum_j \frac{1}{\varepsilon_j}\right),$$

are satisfied for all positive sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and all covering sequences $\{x_j\}$.

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