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A STEFFENSEN TYPE INEQUALITY

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Abstract

Steffensen's inequality deals with the comparison between integrals over a whole interval $[a, b]$ and integrals over a subset of $[a, b]$. In this paper we prove an inequality which is similar to Steffensen's inequality. The most general form of this inequality deals with integrals over a measure space. We also consider the discrete case.

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1. Introduction

The most basic inequality which deals with the comparison between integrals over a whole interval $[a, b]$ and integrals over a subset of $[a, b]$ is the following inequality, which was established by J.F. Steffensen in 1919, [3].

Theorem 1.1. (STEFFENSEN'S INEQUALITY) *Let a and b be real numbers such that $a < b$, f and g be integrable functions from $[a, b]$ into $\mathbb{R}$ such that f is nonincreasing and for every $x \in [a, b]$, $0 \leq g(x) \leq 1$. Then*

$$\int\limits_{b-\lambda}^{b} f(x)dx \leq \int\limits_{a}^{b} f(x)g(x)dx \leq \int\limits_{a}^{a+\lambda} f(x)dx,$$

where $\lambda = \int\limits_{a}^{b} g(x)dx$.

The following is a discrete analogue of Steffensen's inequality, [1]:

Theorem 1.2. (DISCRETE STEFFENSEN'S INEQUALITY). *Let $(x_i)_{i=1}^{n}$ be a nonincreasing finite sequence of nonnegative real numbers, and let $(y_i)_{i=1}^{n}$ be a finite sequence of real numbers such that for every i, $0 \leq y_i \leq 1$. Let k_1 $k_2 \in \{1, \ldots, n\}$ be such that $k_2 \leq \sum\limits_{i=1}^{n} y_i \leq k_1$. Then*

$$\sum_{i=n-k_2+1}^{n} x_i \leq \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{k_1} x_i.$$

In section 2 we consider the discrete case. Our first result is the following.

Theorem 1.3. *Let $\ell \geq 0$ be a real number, $(x_i)_{i=1}^n$ be a nonincreasing finite sequence of real numbers in $[\ell, \infty)$, and $(y_i)_{i=1}^n$ be a finite sequence of nonnegative real numbers. Let $\Phi : [\ell, \infty) \to [0, \infty)$ be strictly increasing, convex, and such that $\Phi(xy) \geq \Phi(x)\Phi(y)$ for all $x, y, xy \geq \ell$. Let $k \in \{1, \dots, n\}$ be such that $k \geq \ell$ and $\Phi(k) \geq \sum_{i=1}^n y_i$. Then either*

$$\sum_{i=1}^{n} \Phi(x_i) y_i \leq \Phi\left(\sum_{i=1}^{k} x_i\right) \quad \text{or} \quad \sum_{i=1}^{k} y_i \geq 1.$$

Theorem 1.3 takes an especially simple form if $\Phi(x) = x^\alpha$, where $\alpha \geq 1$.

Theorem 1.4. *Let $(x_i)_{i=1}^n$ be a nonincreasing finite sequence of nonnegative real numbers, and let $(y_i)_{i=1}^n$ be a finite sequence of nonnegative real numbers. Assume that $\alpha \geq 1$. Let $k \in \{1, \dots, n\}$ be such that*

$$k \geq \left(\sum_{i=1}^{n} y_i\right)^{\frac{1}{\alpha}}.$$

Then either

$$\sum_{i=1}^{n} x_i^\alpha y_i \leq \left(\sum_{i=1}^{k} x_i\right)^\alpha \quad \textit{or} \quad \sum_{i=1}^{k} y_i \geq 1.$$

As an example of an application of Theorem 1.4 we obtain the following result:

Theorem 1.5. *Let α and β be real numbers such that $\alpha \geq 1+\beta$, $0 \leq \beta \leq 1$. Let $(x_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative real numbers. Assume that*

$$\sum_{i=1}^{n} x_i \leq A, \qquad \sum_{i=1}^{n} x_i^{\alpha} \geq B^{\alpha},$$

where A and B are positive real numbers. Let $k \in \{1, 2 \ldots, n\}$ be such that

$$k \geq \left(\frac{A}{B}\right)^{\frac{\beta}{\alpha-1}}.$$

Then

$$\sum_{i=1}^{k} x_i^{\beta} \geq B^{\beta}.$$

For $\beta = 1$ this is a result from [1].

The main result of section 3 is Theorem 3.2. This theorem is similar to Theorem 1.3, but it involves integrals over a measure space instead of finite sums. The key tool that we use to state and to prove Theorem 3.2 is the concept of separating subsets introduced and studied in [1]. If we take a measure space to be just a closed interval of the real line $\mathbb{R}$, we obtain the following simplest case of Theorem 3.2:

Theorem 1.6. *Let $\ell \geq 0$ be a real number, a and b be real numbers such that $a < b$, f and g be integrable functions from $[a, b]$ into $[\ell, \infty)$ and $[0, \infty)$ respectively, such that f is nonincreasing. Let $\Phi : [\ell, \infty) \to [0, \infty)$ be strictly increasing,* *convex, and such that $\Phi(xy) \geq \Phi(x)\Phi(y)$ for all x, y, $xy \geq \ell$. Let λ be a real number such that $\Phi(\lambda) = \int_a^b g(x)dx$. Assume that $\lambda \leq b - a$ and*

$$f(a) - f(a-\lambda) \leq \int_a^{a+\lambda} [f(x) - f(a+\lambda)]\, dx.$$

Then either

$$\int_a^b (\Phi \circ f) g\, dx \leq \Phi\left(\int_a^{a+\lambda} f\, dx\right) \quad \text{or} \quad \int_a^{a+\lambda} g\, dx \geq 1.$$

Remark 1.1. In Theorems 1.3, 1.4, 1.6 and 3.2 the assumption that Φ is convex can be weakened: it is enough to assume that Φ is Wright-convex, where Wright-convexity means [4] that $\Phi(t_2) - \Phi(t_1) \leq \Phi(t_2 + \delta) - \Phi(t_1 + \delta)$ for all $t_1, t_2, \delta \in [0, \infty)$ such that $t_1 \leq t_2$. It is known that each convex function is Wright-convex, but the converse is not true.

2. The Discrete Case

Proof. of Theorem 1.3

$$\begin{aligned}
\sum_{i=1}^{n} \Phi(x_i) y_i &= \sum_{i=1}^{k} \Phi(x_i) y_i + \sum_{i=k+1}^{n} \Phi(x_i) y_i \\
&\le \sum_{i=1}^{k} \Phi(x_i) y_i + \Phi(x_k) \sum_{i=k+1}^{n} y_i \\
&= \sum_{i=1}^{k} \Phi(x_i) y_i + \Phi(x_k) \left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{k} y_i \right) \\
&= \sum_{i=1}^{k} y_i \left[\Phi(x_i) - \Phi(x_k) \right] + \Phi(x_k) \sum_{i=1}^{n} y_i .
\end{aligned}$$

Since $\Phi(k) \ge \sum_{i=1}^{n} y_i$ and $\Phi(kx_k) \ge \Phi(k)\Phi(x_k)$, we obtain

$$\sum_{i=1}^{n} \Phi(x_i) y_i \le \sum_{i=1}^{k} y_i \left[\Phi(x_i) - \Phi(x_k) \right] + \Phi(kx_k).$$

Since Φ is Wright-convex,

$$\begin{aligned}
\Phi(x_i) - \Phi(x_k) &\le \Phi(x_i + (k-1)x_k) - \Phi(x_k + (k-1)x_k) \\
&= \Phi(x_i + (k-1)x_k) - \Phi(kx_k) \\
&\le \Phi\left(\sum_{i=1}^{k} x_i \right) - \Phi(kx_k).
\end{aligned}$$

Therefore

$$\sum_{i=1}^{n} \Phi(x_i) y_i \leq \left[\Phi\left(\sum_{i=1}^{k} x_i \right) - \Phi(k x_k) \right] \sum_{i=1}^{k} y_i + \Phi(k x_k).$$

It follows that

$$(2.1) \quad \sum_{i=1}^{n} \Phi(x_i) y_i - \Phi\left(\sum_{i=1}^{k} x_i \right) \leq \left[\Phi\left(\sum_{i=1}^{k} x_i \right) - \Phi(k x_k) \right] \left(\sum_{i=1}^{k} y_i - 1 \right),$$

since

$$\sum_{i=1}^{k} x_i \geq k x_k, \qquad \Phi\left(\sum_{i=1}^{k} x_i \right) - \Phi(k x_k) \geq 0.$$

Assume first that

$$\Phi\left(\sum_{i=1}^{k} x_i \right) - \Phi(k x_k) = 0.$$

Since Φ is strictly increasing we obtain that

$$\sum_{i=1}^{k} x_i = k x_k \quad \text{and therefore} \quad x_1 = \cdots = x_k.$$

Then

$$\Phi\left(\sum_{i=1}^{k} x_i\right) - \sum_{i=1}^{n} \Phi(x_i) y_i \geq \Phi(k x_k) - \Phi(x_k) \sum_{i=1}^{n} y_i$$
$$\geq \Phi(k)\Phi(x_k) - \Phi(x_k) \sum_{i=1}^{n} y_i$$
$$= \Phi(x_k)\left(\Phi(k) - \sum_{i=1}^{n} y_i\right) \geq 0.$$

Thus, in the case $\Phi\left(\sum_{i=1}^{k} x_i\right) - \Phi(k x_k) = 0$ we obtain that

$$\sum_{i=1}^{n} \Phi(x_i) y_i \leq \Phi\left(\sum_{i=1}^{k} x_i\right),$$

and we are done.

Assume now that $\Phi\left(\sum_{i=1}^{k} x_i\right) - \Phi(k x_k) > 0$. Then equation (2.1) implies that either

$$\sum_{i=1}^{n} \Phi(x_i) y_i \leq \Phi\left(\sum_{i=1}^{k} x_i\right) \quad \text{or} \quad \sum_{i=1}^{k} y_i \geq 1.$$

□

Proof. of Theorem 1.5 Take x_i^β instead of x_i and $\frac{\alpha-1}{\beta}$ instead of α in Theorem 1.4. Then we get that

$$k \geq \left(\sum_{i=1}^{n} y_i\right)^{\frac{\beta}{\alpha-1}}$$

implies that either

$$\sum_{i=1}^{n} x_i^{\alpha-1} y_i \leq \left(\sum_{k=1}^{k} x_i^{\beta}\right)^{\frac{\alpha-1}{\beta}} \quad \text{or} \quad \sum_{i=1}^{k} y_i \geq 1.$$

Take $y_i = \frac{x_i}{B}$ for $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} y_i = \frac{1}{B} \sum_{i=1}^{n} x_i \leq \frac{A}{B}.$$

Since $k \geq \left(\frac{A}{B}\right)^{\frac{\beta}{\alpha-1}}$, we obtain that

$$k \geq \left(\sum_{i=1}^{n} y_i\right)^{\frac{\beta}{\alpha-1}}.$$

This implies that either

$$\sum_{i=1}^{k} x_i^{\beta} \geq \left(\sum_{i=1}^{n} x_i^{\alpha-1} y_i\right)^{\frac{\beta}{\alpha-1}} = \left(\frac{1}{B}\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{\beta}{\alpha-1}} \geq \left(\frac{B^{\alpha}}{B}\right)^{\frac{\beta}{\alpha-1}} = B^{\beta},$$

or

$$\sum_{i=1}^{k} x_i = B\sum_{i=1}^{k} y_i \geq B.$$

However, if

$$\sum_{i=1}^{k} x_i \geq B,$$

then, since $0 \leq \beta \leq 1$,

$$\sum_{i=1}^{k} x_i^{\beta} \geq \left(\sum_{i=1}^{k} x_i\right)^{\beta} \geq B^{\beta}.$$

Therefore in both cases we have that

$$\sum_{i=1}^{k} x_i^{\beta} \geq B^{\beta}.$$

□

Example 2.1. Let $(x_i)_{i=1}^n$ be a nonincreasing sequence in $[0, \infty)$ such that $\sum_{i=1}^{k} x_i \le 400$ and $\sum_{i=1}^{k} x_i^2 \ge 10,000$. Then $\sqrt{x_1} + \sqrt{x_2} \ge 10$. For a proof take $\alpha = 2$, $\beta = \frac{1}{2}$, $A = 400$, and $B = 100$ in Theorem 1.5. The result is the best possible since if $n \ge 16$ and $x_1 = \cdots = x_{16} = 25$, $x_{17} = \cdots = x_n = 0$, we have that $\sum_{i=1}^{n} x_i = 400$, $\sum_{i=1}^{n} x_i^2 = 10,000$, and $\sqrt{x_1} + \sqrt{x_2} = 10$.

3. The Case of Integrals over a Measure Space.

Let $X = (X, \mathcal{A}, \mu)$ be a measure space. From now on we will assume that $0 < \mu(X) < \infty$.

Definition 3.1. [1]. Let $f \in L^\circ(X)$, where $L^\circ(X)$ means the set of all measurable functions on X. Let $(U, c) \in \mathcal{A} \times \mathbb{R}$. We say that the pair (U, c) is *upper-separating* for f iff

$$\{x \in X : f(x) > c\} \overset{a}{\subseteq} U \overset{a}{\subseteq} \{x \in X : f(x) \geq c\}$$

where $A \overset{a}{\subseteq} B$ means that A is almost contained in B, i.e. $\mu(A \setminus B) = 0$. We say that a subset U of X is *upper-separating* for f if there exists $c \in \mathbb{R}$ such that (U, c) is an upper-separating pair for f.

It is possible to prove, [1], that if μ is continuous (for a definition of a continuous measure see, for example, [2]), then, given $f \in L^\circ(X)$, for any real number λ such that $0 \leq \lambda \leq \mu(X)$, there exists an upper-separating subset U for f such that $\mu(U) = \lambda$.

Lemma 3.1. *[1]. Let $\Phi : [0, \infty) \to \mathbb{R}$ be convex and increasing. Let $c \in [0, \infty)$ and let $f \in L^1(X)$ have nonnegative values and satisfy the condition*

$$0 \leq f - c \leq \int_X (f - c) d\mu \quad a.e. \tag{3.1}$$

Then

$$\Phi \circ f - \Phi(c) \leq \Phi\left(\int_X f d\mu\right) - \Phi\left(c\mu(X)\right) \quad a.e.$$

Proof. The conclusion is trivial if $f = c$ a.e. Suppose that $\mu(\{x \in X : f(x) > c\}) > 0$. Then the left inequality (3.1) implies that

$$\int_X (f - c)d\mu > 0.$$

On the other hand, by integrating the right inequality (3.1), we obtain

$$\int_X (f - c)d\mu \leq \left(\int_X (f - c)d\mu \right) \mu(X),$$

which implies $\mu(X) \geq 1$. Since Φ is Wright-convex, we obtain that

$$\begin{aligned} \Phi \circ f - \Phi(c) &\leq \Phi\left(f + c(\mu(X) - 1)\right) - \Phi\left(c + c(\mu(X) - 1)\right) \\ &= \Phi\left(f - c + c\mu(X)\right) - \Phi\left(c\mu(X)\right) \quad \text{a.e.} \end{aligned}$$

Because Φ is increasing it follows by (3.1) that

$$\begin{aligned} \Phi \circ f - \Phi(c) &\leq \Phi\left(\int_X (f - c)d\mu + \int_X c\, d\mu \right) - \Phi\left(c\mu(X)\right) \\ &= \Phi(\int_X f\, d\mu) - \Phi\left(c\mu(X)\right). \end{aligned}$$

□

Theorem 3.2. *Let $\ell \geq 0$ be a real number. Let $\Phi : [\ell, \infty) \to \mathbb{R}$ be convex strictly increasing, and such that $\Phi(xy) \geq \Phi(x)\Phi(y)$ for all $x, y, xy \geq \ell$. Let $f, g \in L'(X)$ be such that $f \geq \ell$ and $g \geq 0$ a.e.. Let λ be a real number and such that $\Phi(\lambda) = \int_X g\, d\mu$. Assume that $0 \leq \lambda \leq \mu(X)$, and let (U, c) be an upper-separating pair for f such that $\mu(U) = \lambda$. Assume that $f - c \leq \int\limits_U (f - c)\, d\mu$ a.e. on U. Then either*

$$\int\limits_X (\Phi \circ f) g\, d\mu \leq \Phi\left(\int\limits_U f\, d\mu\right) \quad \text{or} \quad \int\limits_U g\, d\mu \geq 1.$$

Proof.

$$\begin{aligned}
\int\limits_X (\Phi \circ f) g\, d\mu &= \int\limits_U (\Phi \circ f) g\, d\mu + \int\limits_{X \setminus U} (\Phi \circ f) g\, d\mu \\
&\leq \int\limits_U (\Phi \circ f) g\, d\mu + \Phi(c) \int\limits_{X \setminus U} g\, d\mu \\
&= \int\limits_U (\Phi \circ f) g\, d\mu + \Phi(c) \left(\int\limits_X g\, d\mu - \int\limits_U g\, d\mu\right) \\
&= \int\limits_U g\left(\Phi \circ f - \Phi(c)\right) d\mu + \Phi(c)\Phi(\lambda).
\end{aligned}$$

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By Lemma 3.1

$$\int_X (\Phi \circ f) g \, d\mu \le \left[\Phi\left(\int_U f \, d\mu \right) - \Phi(c\lambda) \right] \int_U g \, d\mu + \Phi(c)\Phi(\lambda)$$
$$\le \left[\Phi\left(\int_U f \, d\mu \right) - \Phi(c\lambda) \right] \int_U g \, d\mu + \Phi(c\lambda).$$

It follows that

(3.2)
$$\int_X (\Phi \circ f) g \, d\mu - \Phi\left(\int_U f \, d\mu \right) \le \left[\Phi\left(\int_U f \, d\mu \right) - \Phi(c\lambda) \right] \left(\int_U g d\mu - 1 \right).$$

Since (U, c) is upper-separating for f, $f \ge c$ on U. Hence

$$\int_U f \, d\mu \ge c\lambda \quad \text{and therefore} \quad \Phi\left(\int_U f \, d\mu \right) - \Phi(c\lambda) \ge 0.$$

Assume first that

$$\Phi\left(\int_U f \, d\mu \right) - \Phi(c\lambda) = 0, \quad \text{then} \quad \Phi\left(\int_U f \, d\mu \right) = \Phi\left(\int_U c \, d\mu \right).$$

Since Φ is strictly increasing,

$$\int_U f \, d\mu = \int_U c \, d\mu, \quad \text{hence} \quad \int_U (f - c) d\mu = 0.$$

Since $f \geq c$ on U, we obtain that $f = c$ a.e. on U. Then

$$\Phi\left(\int_U f\,d\mu\right) - \int_X (\Phi \circ f) g\,d\mu = \Phi\left(\int_U c\,d\mu\right) - \int_X (\Phi \circ f) g\,d\mu$$
$$= \Phi(c\lambda) - \int_X (\Phi \circ f) g\,d\mu$$
$$\geq \Phi(c)\Phi(\lambda) - \int_X (\Phi \circ f) g\,d\mu.$$

Since (U, c) is upper-separating for f, we obtain that $f = c$ a.e. on U and $f \leq c$ a.e. on $X \backslash U$. Hence $f \leq c$ a.e. on X. It follows that

$$\Phi\left(\int_U f\,d\mu\right) - \int_X (\Phi \circ f) g\,d\mu \geq \Phi(c)\Phi(\lambda) - \int_X \Phi(c) g\,d\mu$$
$$= \Phi(c)\left[\Phi(\lambda) - \int_X g\,d\mu\right] = 0.$$

This proves Theorem 1.6 in the case $\Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda) = 0$.

Assume now that $\Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda) > 0$, then equation 3.2 implies that either

$$\int_X (\Phi \circ f) g \, d\mu - \Phi \left(\int_U f \, d\mu \right) \leq 0 \quad \text{or} \quad \int_U g \, d\mu \geq 1.$$

□

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