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POWER-MONOTONE SEQUENCES AND FOURIER SERIES WITH POSITIVE COEFFICIENTS.

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ABSTRACT. J. Németh has extended several basic theorems of R. P. Boas Jr. pertaining to Fourier series with positive coefficients from Lipschitz classes to generalized Lipschitz classes. The goal of the present work is to find the common root of known results of this type and to establish two theorems that are generalizations of Németh's results. Our results can be considered as sample examples showing the utility of the notion of power-monotone sequences in a new research field.

Key words and phrases: Fourier series, Fourier coefficients, Lipschitz classes, modulus of continuity, cosine and sine series, quasi power-monotone sequences.

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1. INTRODUCTION

The notion of the power-monotone sequences, as far as we know, appeared first in the paper of A. A. Konyushkov [7], where he proved that the following classical inequality of Hardy and Littlewood [5]

$$\sum_{n=1}^{\infty} n^{-c}\Big(\sum_{k=1}^{n} a_k\Big)^p \le K(p,c)\sum_{n=1}^{\infty} n^{p-c}a_n^p, \quad a_n \ge 0, p \ge 1, c > 1, \tag{1.1}$$

can be reversed if $n^\tau a_n \downarrow$ $(\tau < 0)$, i.e. if the sequence $\{a_n\}$ is τ-power-monotone decreasing.

In [8], among others, we generalized (1.1) as follows

$$\sum_{n=1}^{\infty} \lambda_n\Big(\sum_{k=1}^{n} a_k\Big)^p \le p^p\sum_{n=1}^{\infty} \lambda_n^{1-p}\Big(\sum_{k=n}^{\infty}\lambda_k\Big)^p a_k^p, \quad p \ge 1, \lambda_n > 0. \tag{1.2}$$

The reader can discover a large number of very interesting classical and modern inequalities of Hardy-Littlewood type in the eminent papers of G. Bennett [1, 2, 3].

The author ([10] see also [9]) also proved that the converse of inequality (1.2) holds if and only if the sequence $\{\lambda_n\}$ is nearly geometric in nature. That is, if it is quasi geometrically

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monotone. This was achieved without requiring additional conditions on the nonnegative sequence $\{a_n\}$.

Recently, it was found that the quasi power-monotone sequences and the quasi geometrically monotone sequences are closely interlinked; furthermore, these sequences have appeared in the generalizations of several classical results, sometimes only implicitly.

Very recently, we have also observed that the quasi power-monotone sequences have implicitly emerged in the investigation of Fourier series with nonnegative coefficients. See for example the papers by R. P. Boas Jr. [4] and J. Németh [13]. Both Boas and Németh proved several interesting results. Boas' theorems treat the connection of the nonnegative Fourier coefficients to the classical Lipschitz classes (Lip α, $0 < \alpha \leq 1$), and Németh extends the Boas results to the so called generalized Lipschitz classes.

We can recall some of these theorems only after recollecting some definitions, and this will clear up the notions used loosely above. But before doing this we present the aim of our work.

The object of our paper is to uncover the common root of the results mentioned above and show that quasi power-monotone sequences play a crucial role in the analysis. Furthermore, we shall formulate the generalizations of two theorems of J. Németh as sample examples. We also claim that by using our method some further generalizations can be proved.

2. Notions and notations

Before formulating the known and new results we recall some definitions and notations.

Let $\omega(\delta)$ be a modulus of continuity, i.e. a nondecreasing function on the interval $[0, 2\pi]$ having the properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$.

Denote $\omega(f;\delta)$ the modulus of continuity of a function f.

Let $\Omega_\alpha(0 \leq \alpha \leq 1)$ denote the set of the moduli of continuity $\omega(\delta) = \omega_\alpha(\delta)$ having the following properties:

(1) for any $\alpha' > \alpha$ there exists a natural number $\mu = \mu(\alpha')$ such that

$$2^{\mu\alpha'}\omega_\alpha(2^{-n-\mu}) > 2\omega_\alpha(2^{-n}) \quad \text{holds for all} \quad n(\geq 1), \tag{2.1}$$

(2) for every natural number ν there exists a natural number $N := N(\nu)$ such that

$$2^{\nu\alpha}\omega_\alpha(2^{-n-\nu}) \leq 2\omega_\alpha(2^{-n}), \quad \text{if} \quad n > N. \tag{2.2}$$

For any $\omega_\alpha \in \Omega_\alpha$ the class H^{ω_α}, i.e.

$$H^{\omega_\alpha} := \{f : \omega(f,\delta) = O(\omega_\alpha(\delta))\},$$

will be called a *generalized Lipschitz class* denoted by Lip ω_α.

We note that a class Lip ω_α can be larger, but also smaller than the class Lip α, depending on the considered modulus of continuity $\omega_\alpha(\delta)$.

We shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is *quasi β-power-monotone increasing (decreasing)* if there exists a natural number $N := N(\beta,\gamma)$ and constant $K := K(\beta,\gamma) \geq 1$ such that

$$Kn^\beta\gamma_n \geq m^\beta\gamma_m \quad (n^\beta\gamma_n \leq Km^\beta\gamma_m) \tag{2.3}$$

holds for any $n \geq m \geq N$.

Here and in the sequel, K and K_i denote positive constants that are not necessarily the same at each occurrence.

If (2.3) holds with $\beta = 0$ then we omit the attribute "β" in the equation.

Furthermore, we shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is *quasi geometrically increasing (decreasing)* if there exists natural numbers $\mu := \mu(\gamma)$, $N := N(\gamma)$ and a constant

$K := K(\gamma) \geq 1$ such that

$$\gamma_{n+\mu} \geq 2\gamma_n \quad \text{and} \quad \gamma_n \leq K\gamma_{n+1} \qquad \left(\gamma_{n+\mu} \leq \frac{1}{2}\gamma_n \quad \text{and} \quad \gamma_{n+1} \leq K\gamma_n\right) \tag{2.4}$$

hold for all $n \geq N$.

Finally a sequence $\{\gamma_n\}$ will be called *bounded by blocks* if the inequalities

$$\alpha_1\Gamma_m^{(k)} \leq \gamma_n \leq \alpha_2\Gamma_M^{(k)}, \quad 0 < \alpha_1 \leq \alpha_2 < \infty$$

hold for any $2^k \leq n \leq 2^{k+1}, k = 1, 2, \ldots$, where

$$\Gamma_m^{(k)} := \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \quad \text{and} \quad \Gamma_M^{(k)} := \max(\gamma_{2^k}, \gamma_{2^{k+1}}).$$

3. THEOREMS AND COMMENTS

To begin, we recall two theorems of J. Németh [13].

Theorem 3.1. *Let $\lambda_n \geq 0$ be the Fourier sine or cosine coefficients of φ. Then $\varphi \in$ Lip ω_γ, $0 < \gamma < 1$, if and only if*

$$\sum_{k=n}^{\infty} \lambda_k = O\left(\omega_\gamma\left(\frac{1}{n}\right)\right), \tag{3.1}$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega_\gamma\left(\frac{1}{n}\right)\right). \tag{3.2}$$

Theorem 3.2. *If $\lambda_n \geq 0$ are the Fourier sine coefficients of φ, then $\varphi \in$ Lip ω_1 if and only if*

$$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega_1\left(\frac{1}{n}\right)\right). \tag{3.3}$$

In the special case $\omega_\gamma(\delta) \equiv \delta^\gamma$ $(0 < \gamma \leq 1)$, these theorems reduce to the classical results of Boas [4]. Again, observe that in general, the class Lip ω_γ can be larger (or smaller) than the class Lip γ.

For completeness, we add that in a notable paper by M. and S. Izumi [6], their Theorem 1 is very similar to Theorem 3.1. The difference being the form of the conditions and notation used. The notions used by Németh show an undoubted similarity to that of the classical Lipschitz classes, therefore we use these notions and notations in the present paper. We also omit the discussion of Izumi's result.

As noted above, the quasi power-monotone sequences and the quasi geometrically monotone sequences are closely interlinked. A result showing this strong connection is the following (see [11], Corollary 1).

Proposition 3.3. *A positive sequence $\{\delta_n\}$ bounded by blocks is quasi ε-power-monotone increasing (decreasing) with a certain negative (positive) exponent ε if and only if the sequence $\{\delta_{2^n}\}$ is quasi geometrically increasing (decreasing).*

We note that if a sequence $\{\gamma_n\}$ is either quasi ε-power-monotone increasing or decreasing, then it is also bounded by blocks. In the following sections we shall use this remark and the cited Proposition several times. We now proceed to formulate our new theorems.

Theorem 3.4. *Assume that a given positive sequence $\{\gamma_n\}$ has the following properties. There exists a positive ε such that:*

(P_+) *the sequence $\{n^\varepsilon\gamma_n\}$ is quasi monotone decreasing and*

(P_-) *the sequence $\{n^{1-\varepsilon}\gamma_n\}$ is quasi monotone increasing.*

If $\lambda_n \geq 0$ are the Fourier sine or cosine coefficients of a function φ, then

$$\omega\left(\varphi, \frac{1}{n}\right) = O(\gamma_n) \tag{3.4}$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O(\gamma_n), \tag{3.5}$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O(n\gamma_n). \tag{3.6}$$

Theorem 3.5. *If $\lambda_n \geq 0$ are the Fourier sine coefficients of φ and the sequence $\{\gamma_n\}$ has the property (P_+), then (3.4) holds if and only if (3.6) is true.*

A simple consideration shows that Theorem 3.4 includes Theorem 3.1. Namely, setting $\gamma_n := \omega_\gamma(\frac{1}{n})$, and keeping in mind that $0 < \gamma < 1$, then Proposition 3.3 and the property (2.2) of $\omega_\gamma(\delta)$ imply that the sequence $\{n^\varepsilon \omega_\gamma(\frac{1}{n})\}$ for some small ε has the property (P_+). A similar argument shows that the sequence $\{n^{1-\varepsilon}\omega_\gamma(\frac{1}{n})\}$ satisfies the property (P_-). In this case we use the property (2.1) of $\omega_\gamma(\delta)$ instead of (2.2).

In a similar manner we can verify that Theorem 3.5 includes Theorem 3.2.

We mention that if the sequence

$$\Lambda_n := \sum_{k=n}^{\infty} \lambda_k$$

satisfies the properties (P_+) and (P_-), then, by Theorem 3.4, we have the estimate

$$\omega\left(\varphi, \frac{1}{n}\right) = O(\Lambda_n), \tag{3.7}$$

or equivalently that

$$\sum_{k=1}^{n} k\lambda_k = O(n\Lambda_n) \tag{3.8}$$

holds.

It is easy to see that if the coefficients λ_n are monotone decreasing then (3.8) implies

$$\lambda_n = O(n^{-1}\Lambda_n).$$

Thus, the equivalence of (3.7) and (3.8) can be considered as a generalization of the following classical theorem of G. G. Lorentz [12]

If $\lambda_n \downarrow 0$ and λ_n are the Fourier sine or cosine coefficients of φ, then $\varphi \in Lip\ \alpha$, $0 < \alpha < 1$, if and only if $\lambda_n = O(n^{-1-\alpha})$.

Finally, we comment on the following theorem of J. Németh [13]

If $\lambda_n \geq 0$ are the Fourier sine or cosine coefficients of φ then the conditions

$$\sum_{k=n}^{\infty} \lambda_k = O\left(\omega\left(\frac{1}{n}\right)\right) \tag{3.9}$$

and

$$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{3.10}$$

imply

$$\varphi \in H^{\omega}, \tag{3.11}$$

for arbitrary modulus of continuity ω.

He also showed that neither (3.9) nor (3.10) are sufficient to satisfy (3.11). Theorem 3.4 shows that if the sequence $\{\omega(\frac{1}{n})\}$ itself has the properties (P_+) and (P_-) then (3.9) and (3.10) are equivalent, and both satisfy (3.11). Moreover, given (3.11), both (3.9) and (3.10) can be shown to be true.

As we have verified, the moduli of continuity $\omega_\gamma, 0 < \gamma < 1$, have the properties (P_+) and (P_-).

4. LEMMAS

To prove our theorems we recall one known lemma and generalize two lemmas of [4].

Lemma 4.1. *(*[10]*) For any positive sequence* $\gamma := \{\gamma_n\}$ *the inequalities*

$$\sum_{n=m}^{\infty} \gamma_n \le K\gamma_m \quad (m = 1, 2, \dots; K \ge 1),$$

or

$$\sum_{n=1}^{m} \gamma_n \le K\gamma_m \quad (m = 1, 2, \dots; K \ge 1),$$

hold if and only if the sequence γ *is quasi geometrically decreasing or increasing, respectively.*

Lemma 4.2. *Let* $\mu_n \ge 0$, $\beta_n > 0$ *and* $\delta > 0$. *Assume that there exists a positive* ε *such that the sequence*

$$\{n^{-\varepsilon}\beta_n\} \quad \textit{is quasi monotone increasing}, \tag{4.1}$$

and the sequence

$$\{n^{\varepsilon-\delta}\beta_n\} \quad \textit{is quasi monotone decreasing}. \tag{4.2}$$

Then

$$\sum_{k=1}^{n} k^{\delta}\mu_k = O(\beta_n) \tag{4.3}$$

is equivalent to

$$\sum_{k=n}^{\infty} \mu_k = O(\beta_n n^{-\delta}). \tag{4.4}$$

Proof. By Proposition 3.3, taking into account (4.1) and (4.2), we have that the sequences $\{\beta_{2^n}\}$ and $\{2^{-n\delta}\beta_{2^n}\}$ are quasi geometrically increasing and decreasing, respectively. Thus, by Lemma 4.1, we also have that

$$\sum_{n=1}^{m} \beta_{2^n} = O(\beta_{2^m}) \tag{4.5}$$

and

$$\sum_{n=m}^{\infty} 2^{-n\delta}\beta_{2^n} = O(2^{-m\delta}\beta_{2^m}) \tag{4.6}$$

hold.

To begin, we show that (4.3) implies (4.4). Assume that $2^\nu < n \le 2^{\nu+1}$. Then, by (4.3), (4.6) and (4.2) we have

$$\sum_{k=n}^{\infty} \mu_k \le \sum_{m=\nu}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} \mu_k \le K \sum_{m=\nu}^{\infty} 2^{-m\delta} \beta_{2^{m+1}} \le K_1 2^{-(\nu+1)\delta} \beta_{2^{\gamma+1}} \le K_2 2^{-\nu\delta} \beta_{2^\nu} \le K n^{-\delta} \beta_n.$$

The proof of the implication (4.4) $\Rightarrow$ (4.3) runs similarly. Namely, by (4.4), (4.5), (4.1) and (4.2), we have

$$\sum_{k=2}^{n} k^\delta \mu_k \le \sum_{m=0}^{\nu} \sum_{k=2^m+1}^{2^{m+1}} k^\delta \mu_k \le K \sum_{m=0}^{\nu} 2^{m\delta} \sum_{k=2^m+1}^{2^{m+1}} \mu_k \le K \sum_{m=0}^{\nu} 2^{m\delta} \beta_{2^m} 2^{-m\delta} \le K \beta_n.$$

□

Lemma 4.3. *Let* $\mu_k \ge 0$, $\sum \mu_k$ *be convergent and* $0 \le \alpha \le 1$. *Moreover, assume that a given positive sequence* $\{\delta_n\}$ *has the following properties. There exists a positive* ε *such that:*
(iii) *the sequence* $\{n^{\varepsilon-\alpha}\delta_n\}$ *is quasi monotone decreasing*
and
(iv) *the sequence* $\{n^{2-\alpha-\varepsilon}\delta_n\}$ *is quasi monotone increasing.*

Finally let

$$\delta(x) := \begin{cases} \delta_n & if \quad x = \dfrac{1}{n}, \quad n \ge 1, \\ \textit{linear on the interval} & [1/(n+1), 1/n]. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} \mu_k (1 - \cos kx) = O(x^\alpha \delta(x)) \quad (x \to 0) \tag{4.7}$$

if and only if

$$\sum_{k=n}^{\infty} \mu_k = O(n^{-\alpha}\delta_n). \tag{4.8}$$

Proof. Under the hypotheses of (iii) and (iv) it is obvious that the sequence

$$\beta_n := n^{2-\alpha}\delta_n \tag{4.9}$$

satisfies the assumptions (4.1) and (4.2) of Lemma 4.2 with $\delta = 2$. Using this we can begin to show the equivalence of (4.7) and (4.8). For (4.7) to imply (4.8) first observe from (4.7) that

$$\sum_{k=1}^{1/x} k^2 \mu_k \frac{1 - \cos kx}{k^2 x^2} \le \sum_{k=1}^{\infty} k^2 \mu_k \frac{1 - \cos kx}{k^2 x^2} = O(x^{\alpha-2}\delta(x)).$$

Hence, since $t^{-2}(1 - \cos t)$ decreases on $(0, 1)$, it follows that

$$\sum_{k=1}^{1/x} k^2 \mu_k = O(x^{\alpha-2}\delta(x)), \tag{4.10}$$

and with $x = 1/n$

$$\sum_{k=1}^{n} k^2 \mu_k = O(n^{2-\alpha}\delta_n). \tag{4.11}$$

Thus, by Lemma 4.2 with $\delta = 2$ and $\beta_n = n^{2-\alpha}\delta_n$, it follows that (4.8) is true. To complete the proof assume (4.8) is true, thus (4.10) and (4.11) also hold. Using Lemma 4.2 with β_n given in (4.9) and $\delta = 2$ we obatin

$$\sum_{k=1}^{\infty} \mu_k (1 - \cos kx) \le \sum_{k=1}^{1/x} + \sum_{k \ge 1/x} \le Kx^2 \sum_{k=1}^{1/x} k^2 \mu_k + K \sum_{k \ge 1/x} \mu_k = O(x^\alpha \delta(x)).$$

This verifies (4.7) (see the argument given at the proof of (4.10)).

Herewith the proof of Lemma 4.3 is complete. □

5. PROOF OF THE THEOREMS

Proof. (of Theorem 3.4). First we show that the statements (3.5) and (3.6) are equivalent. This follows by Lemma 4.2 with $\delta = 1$ and $\beta_n = n\gamma_n$. We can apply Lemma 4.2 in this case, namely the sequence $\{n^{1-\varepsilon}\gamma_n\}$ is quasi monotone increasing and simultaneously the sequence $\{n^{\varepsilon}\gamma_n\}$ is quasi monotone decreasing; see the properties (P_+) and (P_-).

Next, we prove that if $\sum \lambda_n \cos nx$ is the Fourier series of φ and (3.4) holds then (3.5) also holds. The assumption (3.4) clearly implies that

$$|\varphi(x) - \varphi(0)| \le K\gamma(x), \tag{5.1}$$

where

$$\gamma(x) := \begin{cases} \gamma_n \quad if \quad x = \dfrac{1}{n}, \quad n \ge 1, \\ \text{linear on the interval } \; [1/(n+1),\ 1/n]. \end{cases} \tag{5.2}$$

By (P_+) and (5.2), Proposition 3.3 implies that

$$\sum_{n=1}^{\infty} \gamma(2^{-n}) < \infty,$$

whence

$$x^{-1}\gamma(x) \in L(0,1)$$

follows. Thus by (5.1) and Dini's test, the Fourier series of φ converges at $x = 0$, i.e. $\sum \lambda_k < \infty$, whence, by (5.1),

$$\sum_{k=1}^{\infty} \lambda_k (1 - \cos kx) = O(\gamma(x)) \tag{5.3}$$

follows.

Using Lemma 4.3 with $\mu_k = \lambda_k$, $\alpha = 0$ and $\delta_n = \gamma_n$, we have that (5.3) is equivalent to (3.5).

Conversely, assuming that (3.5) holds, then $\sum \lambda_k$ converges; and if λ_n are the Fourier cosine coefficients of φ, we shall show that (3.4) also holds.

We have that

$$
\begin{aligned}
|\varphi(x+2h)-\varphi(x)| &= \Big|\sum_{k=1}^{\infty}\lambda_k(\cos k(x+2h)-\cos kx)\Big| \\
&= 2\Big|\sum_{k=1}^{\infty}\lambda_k \sin k(x+h)\sin kh\Big| \\
&\le 2\sum_{k=1}^{1/h}\lambda_k\sin kh + 2\sum_{k\ge 1/h}\lambda_k \\
&\le 2h\sum_{k=1}^{1/h}k\lambda_k + 2\sum_{k\ge 1/h}\lambda_k.
\end{aligned}\tag{5.4}
$$

Here the second sum is $O(\gamma(h))$ by the assumption (3.5). Utilizing the formerly proved equivalence of (3.5) and (3.6), we clearly have that the first term is also $O(\gamma(h))$. Thus, (3.4) is verified assuming (3.5).

In what follows, Theorem 3.4 is proved for the Fourier cosine series. Let us assume that the Fourier series of φ is $\sum \lambda_n \sin nx$ and that (3.4) holds. Since the Fourier series can be integrated term by term, we have

$$\int_0^x \varphi(t)dt = -\sum_{n=1}^{\infty} n^{-1}\lambda_n(1-\cos nx) = O(x\gamma(x)). \tag{5.5}$$

A consideration similar to that given above shows that we can apply Lemma 4.3 with $\alpha = 1$, $\delta_n = \gamma_n$ and $\mu_k = k^{-1}\lambda_k$. Thus we have that (5.5) is equivalent to

$$\sum_{k=n}^{\infty} k^{-1}\lambda_k = O(n^{-1}\gamma_n).$$

Hence, it follows that

$$\sum_{k=n}^{2n}\lambda_k \le K\gamma_n. \tag{5.6}$$

Since the sequence $\{\gamma_{2^n}\}$ is quasi geometrically decreasing, then (5.6) implies (3.5).

Hence, the necessity of the conditions (3.5) and (3.6) for Fourier sine series have been proved.

Finally, we verify the sufficiency of (3.5) for Fourier sine series. Consider

$$\varphi(x+2h)-\varphi(x) = 2\sum_{n=1}^{\infty}\lambda_n\cos n(x+h)\sin nh. \tag{5.7}$$

It is easy to see that the same estimation as given in (5.4) can also be used in this case. Therefore the proof that (3.5) implies (3.4) is similar to that in the cosine case.

The proof of Theorem 3.4 is thus complete. □

Proof. (of Theorem 3.5). First, assume that the condition (3.6) holds. Using the equality (5.7) and the closing estimate of (5.4) we have

$$|\varphi(x+2h)-\varphi(x)| \le 2h\sum_{k=1}^{1/h}k\lambda_k + 2\sum_{k\ge 1/h}\lambda_k. \tag{5.8}$$

Here, the first term is $O(\gamma(h))$ by the assumption (3.6). To prove the same for the second term we observe that (3.6) implies that

$$\sum_{k=2^m}^{2^{m+1}} \lambda_k \leq K\gamma_{2^m}.$$

In addition, by (P_+) Proposition 3.3 yields that the sequence $\{\gamma_{2^m}\}$ decreases quasi geometrically, thus

$$\sum_{k=n}^{\infty} \lambda_k \leq K\gamma_n.$$

This and the previously obtained partial result, by (5.8), verifies that (3.4) holds.

Conversely, let us assume that (3.4) is true. Then, as before in (5.5), we have

$$\sum_{n=1}^{\infty} n^{-1}\lambda_n(1-\cos nx) = O(x\gamma(x)). \tag{5.9}$$

Furthermore, by (5.9),

$$\begin{aligned}\sum_{k=1}^{1/x} k^{-1}\lambda_k(1-\cos kx) &\equiv x^2 \sum_{k=1}^{1/x} k\lambda_k \frac{1-\cos kx}{k^2x^2} \\ &\leq x^2 \sum_{k=1}^{\infty} k\lambda_k \frac{1-\cos kx}{k^2x^2} \equiv \sum_{k=1}^{\infty} k^{-1}\lambda_k(1-\cos kx) = O(x\gamma(x)),\end{aligned}$$

whence by $x = \frac{1}{n}$ we obtain

$$\sum_{k=1}^{n} k\lambda_k \frac{1-\cos k/n}{(k/n)^2} = O\left(n\gamma\left(\frac{1}{n}\right)\right) = O(n\gamma_n).$$

This shows (see the consideration at (4.10)) that the statement (3.6) holds from (3.4).

The proof of the Theorem 3.5 is thus complete. □

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