

# Journal of Inequalities in Pure and Applied Mathematics

## A NEW EXTENSION OF MONOTONE SEQUENCES AND ITS APPLICATIONS

L. LEINDLER

Bolyai Institute  
University of Szeged  
Aradi vértanúk tere 1  
H-6720 Szeged, Hungary.

*EMail:* [leindler@math.u-szeged.hu](mailto:leindler@math.u-szeged.hu)

©2000 Victoria University  
ISSN (electronic): 1443-5756  
371-05



---

volume 7, issue 1, article 39,  
2006.

*Received 20 December, 2005;  
accepted 21 January, 2006.*

*Communicated by: A.G. Babenko*

---

Abstract

Contents

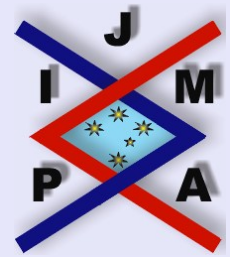


Home Page

Go Back

Close

Quit



## Abstract

We define a new class of numerical sequences. This class is wider than any one of the classical or recently defined new classes of sequences of monotone type. Because of this generality we can generalize only the sufficient part of the classical Chaundy-Jolliffe theorem on the uniform convergence of sine series. We also present two further theorems having conditions of sufficient type.

*2000 Mathematics Subject Classification:* 26A15, 40-99, 40A05, 42A16.

*Key words:* Monotone sequences, Sequence of  $\gamma$  group bounded variation, Sine series.

This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant Nos. T042462 and TS44782.

## Contents

|   |                    |    |
|---|--------------------|----|
| 1 | Introduction ..... | 3  |
| 2 | Theorems .....     | 6  |
| 3 | Lemmas .....       | 9  |
| 4 | Proofs .....       | 11 |
|   | References .....   |    |

---

### A New Extension of Monotone Sequences and its Applications

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 17

# 1. Introduction

In [3] we defined a subclass of the quasimonotone sequences ( $c_n \leq K c_m$ ,  $n \geq m$ ), which is much larger than that of the monotone sequences and not comparable to the class of the classical quasimonotone sequences (see [6]). For this new class we have extended several results proved earlier only for monotone, quasimonotone or classical quasimonotone sequences. The definition of this class reads as follows: A null-sequence  $\mathbf{c}$  ( $c_n \rightarrow 0$ ) belongs to the family of *sequences of rest bounded variation* (in brief,  $\mathbf{c} \in RBVS$ ) if

$$(1.1) \quad \sum_{n=m}^{\infty} |\Delta c_n| \leq K c_m \quad (\Delta c_n = c_n - c_{n+1})$$

holds for all  $m$ , where  $K = K(\mathbf{c})$  is a constant depending only on  $\mathbf{c}$ . Hereafter  $K$  will designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

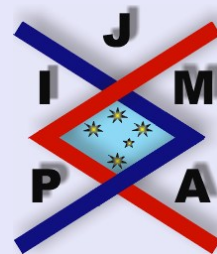
Recently, in [7], we defined a new class of sequences as follows:

Let  $\gamma := \{\gamma_n\}$  be a positive sequence. A null-sequence  $\mathbf{c}$  of *real numbers* satisfying the inequalities

$$(1.2) \quad \sum_{n=m}^{\infty} |\Delta c_n| \leq K \gamma_m$$

is said to be a *sequence of  $\gamma$  rest bounded variation* ( $\gamma RBVS$ ).

We emphasize that the class  $\gamma RBVS$  is no longer a subclass of the quasimonotone sequences. Namely, a sequence  $\mathbf{c}$  satisfying (1.2) may have infinitely many zero and negative terms, as well; but this is not the case if  $\mathbf{c}$  satisfies (1.1).



---

A New Extension of Monotone Sequences and its Applications

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 3 of 17

Very recently Le and Zhou [2] defined another new class of sequences using the following curious definition:

If there exists a natural number  $N$  such that

$$(1.3) \quad \sum_{n=m}^{2m} |\Delta c_n| \leq K \max_{m \leq n < m+N} |c_n|$$

holds for all  $m$ , then  $\mathbf{c}$  belongs to the class  $GBVS$ , in other words,  $\mathbf{c}$  is a *sequence of group bounded variation*.

The class  $GBVS$  is an ingenious generalization of  $RBVS$ , moreover it is wider than the class of the classical quasimonotone sequences  $(c_{n+1} \leq c_n (1 + \frac{\alpha}{n}))$ , too.

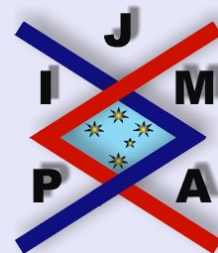
In [2], among others, they verified that the monotonicity condition in the classical theorem of Chaundy and Jolliffe [1] can be replaced by their condition (1.3). Herewith they improved our result, namely that in [5], we verified this by condition (1.1).

The aim of the present work is to unify the advantages of the definitions (1.2) and (1.3). We define a further new class of sequences, to be denoted by  $\gamma GBVS$ , which is wider than any one of the classes  $GBVS$  and  $\gamma RBVS$ .

A null-sequence  $\mathbf{c}$  belongs to  $\gamma GBVS$  if

$$(1.4) \quad \sum_{n=m}^{2m} |\Delta c_n| \leq K \gamma_m, \quad m = 1, 2, \dots$$

holds, where  $\gamma$  is a given sequence of nonnegative numbers.




---

## A New Extension of Monotone Sequences and its Applications

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 4 of 17

---

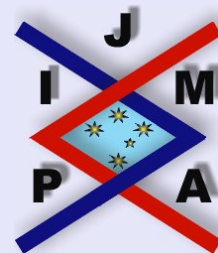
We underline that the sequence  $\gamma$  satisfying (1.4) may have infinitely many zero terms, too; but not in (1.2). We also emphasize that the condition (1.4) gives the greatest freedom for the terms of the sequences  $c$  and  $\gamma$ .

As a first application we shall give a sufficient condition for the uniform convergence of the series

$$(1.5) \quad \sum_{n=1}^{\infty} b_n \sin nx,$$

where  $\mathbf{b} := \{b_n\}$  belongs to a certain class of  $\gamma GBVS$ .

Utilizing the benefits of the sequences of  $\gamma GBVS$  we present two further generalizations of theorems proved earlier for sequences of  $\gamma RBVS$ .




---

**A New Extension of Monotone Sequences and its Applications**

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 5 of 17

## 2. Theorems

We verify the following theorems:

**Theorem 2.1.** *Let  $\gamma := \{\gamma_n\}$  be a sequence of nonnegative numbers satisfying the condition  $\gamma_n = o(n^{-1})$ . If a sequence  $\mathbf{b} := \{b_n\} \in \gamma GBVS$ , then the series (1.5) is uniformly convergent, and consequently its sum function  $f(x)$  is continuous.*

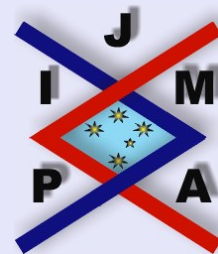
Compare Theorem 2.1 to the mentioned theorem of Chaundy and Jolliffe and two theorems of ours [5, Theorem A and Theorem 1] and [8, Theorem 1]. The cited theorems proved their statements for monotone sequences,  $\mathbf{b} \in RBVS$  and  $\mathbf{b} \in \gamma RBVS$ , respectively.

**Remark 1.** *It is easy to see that if  $b_n = n^{-1}$  and  $\gamma_n = n^{-1}$ , then  $\{b_n\} \in \gamma GBVS$  and the series (1.5) does not converge uniformly. This shows that the assumption  $\gamma_n = o(n^{-1})$  cannot be weakened generally.*

**Theorem 2.2.** *Let  $\beta := \{\eta_n\}$  be a sequence of nonnegative numbers satisfying the condition  $\eta_n = O(n^{-1})$ . If a sequence  $\mathbf{b} := \{b_n\} \in \beta RBVS$ , then the partial sums of the series (1.5) are uniformly bounded.*

We note that for a monotone null-sequence  $\mathbf{b}$ , moreover for  $\mathbf{b} \in RBVS$  and  $\mathbf{b} \in \gamma RBVS$ , the assertion of Theorem 2.2 can be found in [10, Chapter V, §1], in [5, Theorem 2] and [8, Theorem 2].

Before formulating Theorem 2.3 we recall the following definition. A sequence  $\beta := \{\beta_n\}$  of positive numbers is called quasi geometrically increasing (decreasing) if there exist natural numbers  $\mu$  and  $K = K(\beta) \geq 1$  such that for



---

A New Extension of Monotone Sequences and its Applications

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 6 of 17

all natural numbers  $n$

$$\beta_{n+\mu} \geq 2\beta_n \text{ and } \beta_n \leq K \beta_{n+1} \quad \left( \beta_{n+\mu} \leq \frac{1}{2}\beta_n \text{ and } \beta_{n+1} \leq K \beta_n \right).$$

**Theorem 2.3.** *If  $\mathbf{c} := \{c_n\} \in \beta$  GBVS, or belongs to  $\gamma$  GBVS, where  $\beta$  and  $\gamma$  have the same meaning as in Theorems 2.1 and 2.2, furthermore the sequence  $\{n_m\}$  is quasi geometrically increasing, then the estimates*

$$(2.1) \quad \sum_{j=1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx \right| \leq K(\mathbf{c}, \{n_m\}),$$

or

$$(2.2) \quad \sum_{j=m}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx \right| = o(1), \quad m \rightarrow \infty,$$

hold uniformly in  $x$ , respectively.

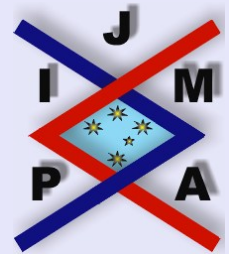
The root of (2.1) goes back to Telyakovskii [9, Theorem 2] and two generalizations of it can be found in [5] and [8].

We note that, in general, (2.1) does not imply (2.2), see the Remark in [8].

It is clear that the “smallest” class  $\gamma$  GBVS which includes a given sequence  $\mathbf{c} := \{c_n\}$  is the one, where

$$\gamma_n := \sum_{k=n}^{2n} |\Delta c_k|, \quad n = 1, 2, \dots$$

In regard to this, it is plain, that our theorems convey the following consequence.




---

**A New Extension of Monotone Sequences and its Applications**

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 7 of 17

---

**Corollary 2.4.** *The assertions of our theorems for an individual sequence  $\mathbf{b}$  hold true under the assumptions*

$$(2.3) \quad \sum_{k=n}^{2n} |\Delta b_k| = o(n^{-1})$$

and

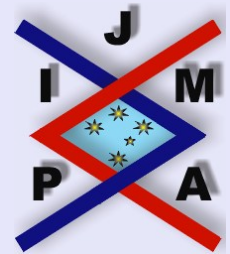
$$\sum_{k=n}^{2n} |\Delta b_k| = O(n^{-1}),$$

respectively.

However, in my view, our theorems give a better perspicuity than Corollary 2.4 does; the arrangement of the proofs are more convenient with our method, furthermore the assumptions of Corollary 2.4 give conditions only for an individual sequence, and not for a class of sequences.

We also remark that e.g. the condition (2.3) is not a necessary one for uniform convergence. See the series

$$\sum_{n=1}^{\infty} 2^{-n} \sin 2^n x.$$




---

**A New Extension of Monotone Sequences and its Applications**

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 8 of 17



### 3. Lemmas

**Lemma 3.1** ([4]). For any positive sequence  $\{\beta_n\}$  the inequalities

$$\sum_{n=m}^{\infty} \beta_n \leq K \beta_m, \quad m = 1, 2, \dots; K \geq 1,$$

hold if and only if the sequence  $\{\beta_n\}$  is quasi geometrically decreasing.

**Lemma 3.2.** Let  $\rho := \{\rho_n\}$  be a nonnegative sequence with  $\rho_n = O(n^{-1})$ , and let  $\delta := \{\delta_n\}$  belong to  $\rho$ GBVS. If a complex series  $\sum_{n=1}^{\infty} a_n$  satisfies the Abel condition, i.e., if there exists a constant  $A$  such that for all  $m \geq 1$ ,

$$\left| \sum_{n=1}^m a_n \right| \leq A,$$

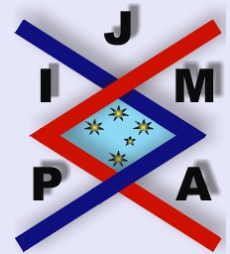
then for any  $\mu \geq m$ ,

$$(3.1) \quad \left| \sum_{n=m}^{\mu} a_n \delta_n \right| \leq 6K(\rho)A \varepsilon_m m^{-1},$$

where  $K(\rho)$  denotes the constant appearing in the definition of  $\rho$ GBVS, furthermore

$$\varepsilon_n := \sup_{k \geq n} k \rho_k.$$

Consequently, if  $\varepsilon_m = o(m)$ , then the series  $\sum_{n=1}^{\infty} a_n \delta_n$  converges.



A New Extension of Monotone Sequences and its Applications

L. Leindler

Title Page

Contents



Go Back

Close

Quit

Page 9 of 17

*Proof.* First we show that

$$(3.2) \quad |\delta_m| \leq \sum_{n=m}^{\infty} |\Delta \delta_n| \leq 2K(\rho)\varepsilon_m m^{-1}.$$

Since  $\delta_n$  tends to zero, the first inequality in (3.2) is obvious; and because  $n \rho_n$  is bounded, thus  $\delta \in \rho GBVS$  implies that

$$(3.3) \quad \begin{aligned} \sum_{n=m}^{\infty} |\Delta \delta_n| &\leq \sum_{\ell=0}^{\infty} \sum_{n=2^\ell m}^{2^{\ell+1}m} |\Delta \delta_n| \leq \sum_{\ell=0}^{\infty} K(\rho)\rho 2^{\ell m} \\ &\leq K(\rho) \sum_{\ell=0}^{\infty} \varepsilon_m (2^\ell m)^{-1} = 2K(\rho)m^{-1}\varepsilon_m, \end{aligned}$$

and this proves (3.2).

Next we verify (3.1). Using the notation

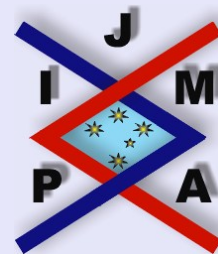
$$\alpha_n := \sum_{k=1}^n a_k,$$

(3.2) and the assumptions of Lemma 3.2, we get that

$$\begin{aligned} \left| \sum_{n=m}^{\mu} a_n \delta_n \right| &= \left| \sum_{n=m}^{\mu-1} \alpha_n (\delta_n - \delta_{n+1}) + \alpha_\mu \delta_\mu - \alpha_{m-1} \delta_m \right| \\ &\leq A \left( \sum_{n=m}^{\mu-1} |\Delta \delta_n| + |\delta_\mu| + |\delta_m| \right) \leq 6AK(\rho)\varepsilon_m m^{-1}, \end{aligned}$$

which proves (3.1).

The proof is complete. □




---

A New Extension of Monotone Sequences and its Applications

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 10 of 17

## 4. Proofs

*Proof of Theorem 2.1.* Denote

$$\varepsilon_n := \sup_{k \geq n} k \gamma_k \quad \text{and} \quad r_n(x) := \sum_{k=n}^{\infty} b_k \sin kx.$$

In view of the assumption  $\gamma_m = o(m^{-1})$  we have that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus it is sufficient to verify that

$$(4.1) \quad |r_n(x)| \leq K \varepsilon_n$$

holds for all  $n$ .

Since  $r_n(k\pi) = 0$  it suffices to prove (4.1) for  $0 < x < \pi$ .

Let  $N$  be the integer for which

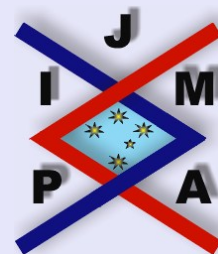
$$(4.2) \quad \frac{\pi}{N+1} < x \leq \frac{\pi}{N}.$$

First we show that if  $k \geq n$  then

$$(4.3) \quad k|b_k| \leq K \varepsilon_n, \quad n = 1, 2, \dots$$

Since  $b_m$  and  $m \gamma_m$  tend to zero, thus the assumption  $\mathbf{b} \in \gamma GBVS$  implies that

$$(4.4) \quad \begin{aligned} |b_k| &\leq \sum_{i=k}^{2k-1} |\Delta b_i| + |b_{2k}| \leq \sum_{\ell=0}^1 \sum_{i=2^\ell k}^{2^{\ell+1}k-1} |\Delta b_i| + |b_{4k}| \leq \dots \\ &\leq K \sum_{\ell=0}^{\infty} \gamma_{2^\ell k} =: \sigma_k. \end{aligned}$$



A New Extension of Monotone Sequences and its Applications

L. Leindler

Title Page

Contents



Go Back

Close

Quit

Page 11 of 17

By the definition of  $\varepsilon_n$  and  $k \geq n$  we have that

$$2^\ell k \gamma_{2^\ell k} \leq \varepsilon_n, \quad \ell = 1, 2, \dots,$$

thus it is clear that

$$\sigma_k \leq 2K\varepsilon_n/k;$$

this and (4.4) proves (4.3).

Now we turn back to the proof of (4.1). Let

$$r_n(x) = \left( \sum_{k=n}^{n+N-1} + \sum_{k=n+N}^{\infty} \right) b_k \sin kx =: r_n^{(1)}(x) + r_n^{(2)}(x).$$

Then, by (4.2) and (4.3),

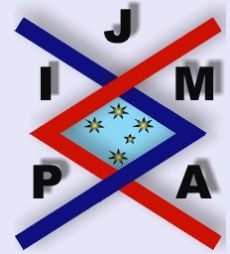
$$(4.5) \quad |r_n^{(1)}(x)| \leq x \sum_{k=n}^{n+N-1} k|b_k| \leq K x N \varepsilon_n \leq K \pi \varepsilon_n.$$

A similar consideration as in (3.3) gives that for any  $m \geq n$

$$\sum_{k=m}^{\infty} |\Delta b_k| \leq K \varepsilon_n/m.$$

Using this, (4.2), (4.3) and the well-known inequality

$$D_n(x) := \left| \sum_{k=1}^n \sin kx \right| \leq \frac{\pi}{x},$$




---

**A New Extension of Monotone Sequences and its Applications**

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 12 of 17

---

furthermore summing by parts, we get that

$$(4.6) \quad \begin{aligned} |r_n^{(2)}(x)| &\leq \sum_{k=n+N}^{\infty} |\Delta b_k| D_k(x) + |b_{n+N}| D_{n+N-1}(x) \\ &\leq 2K \frac{\varepsilon_n}{n+N} \frac{\pi}{x} \leq 2K \varepsilon_n. \end{aligned}$$

The inequalities (4.5) and (4.6) imply (4.1), that is, the series (1.5) is uniformly convergent.

The proof is complete.  $\square$

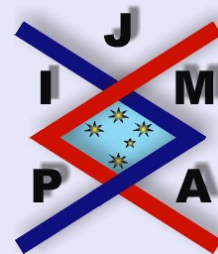
*Proof of Theorem 2.2.* In the proof of Theorems 2.2 and 2.3 we shall use the notations of the proof of Theorem 2.1. The condition  $\eta_n = O(n^{-1})$  implies that the sequence  $\{\varepsilon_n\}$  is bounded, i.e.  $\varepsilon_n \leq K$ . This, (4.2) and (4.3) imply that for any  $m \leq N$

$$\left| \sum_{k=1}^m b_k \sin kx \right| \leq \sum_{k=1}^N |b_k| kx \leq K x N \leq K \pi,$$

furthermore, if  $m > N$  then, by (4.1),

$$\left| \sum_{k=N+1}^m b_k \sin kx \right| \leq |r_{N+1}(x)| + |r_{m+1}(x)| \leq 2K \varepsilon_1.$$

The last two estimates clearly prove Theorem 2.2.  $\square$




---

A New Extension of Monotone Sequences and its Applications

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 13 of 17

*Proof of Theorem 2.3.* First we verify (2.1). Let us suppose that

$$(4.7) \quad n_i \leq N < n_{i+1}.$$

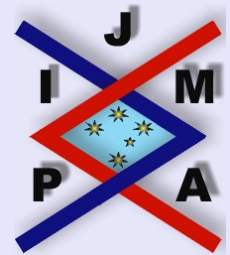
Since  $\mathbf{c} \in \beta GBVS$  and  $\eta_n = O(n^{-1})$ , we get, as in the proof of Theorem 2.2 with  $c_n$  in place of  $b_n$ , that

$$(4.8) \quad \sum_{j=1}^{i-1} \left| \sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx \right| + \left| \sum_{k=n_i}^N c_k \sin kx \right| \\ \leq \sum_{j=1}^{i-1} \sum_{k=n_j}^{n_{j+1}-1} |c_k| kx + \sum_{k=n_i}^N |c_k| kx \leq K\pi.$$

Next applying Lemma 3.2 with  $\rho = \beta$ ,  $\delta_n = c_n$  and  $a_n = \sin nx$ , we get that

$$(4.9) \quad \sigma_N^* := \left| \sum_{k=N+1}^{n_{i+1}-1} c_k \sin kx \right| + \sum_{j=i+1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx \right| \\ \leq K \left\{ \varepsilon_{N+1} (N+1)^{-1} x^{-1} + x^{-1} \sum_{j=i+1}^{\infty} \varepsilon_{n_j} n_j^{-1} \right\} \\ \leq K \left\{ \varepsilon_N + N \varepsilon_N \sum_{j=i+1}^{\infty} n_j^{-1} \right\} \leq K \varepsilon_N \left\{ 1 + N \sum_{j=i+1}^{\infty} n_j^{-1} \right\}.$$

Since the sequence  $\{n_j\}$  is quasi geometrically increasing, so  $\{n_j^{-1}\}$  is quasi



A New Extension of Monotone Sequences and its Applications

L. Leindler

Title Page

Contents



Go Back

Close

Quit

Page 14 of 17

geometrically decreasing, therefore, Lemma 3.1 and (4.7) imply that

$$(4.10) \quad \sum_{j=i+1}^{\infty} n_j^{-1} \leq K N^{-1},$$

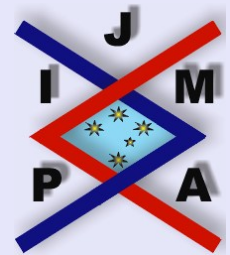
whence, by (4.9) and  $\eta_n = O(n^{-1})$ ,

$$(4.11) \quad \sigma_N^* \leq K \varepsilon_N < \infty$$

follows. Herewith (2.1) is proved.

If  $\mathbf{c} \in \gamma GBVS$  then, by  $\gamma_n = o(n^{-1})$ ,  $\varepsilon_n \rightarrow 0$ , thus, with  $m$  in place of  $N$ , (4.9), (4.10) and (4.11) immediately verify (2.2).

The proof is complete. □




---

**A New Extension of Monotone Sequences and its Applications**

L. Leindler

---

Title Page

Contents



Go Back

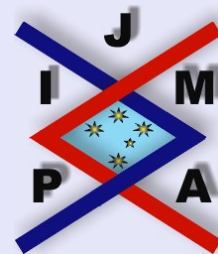
Close

Quit

Page 15 of 17

## References

- [1] T.W. CHAUDY AND A.E. JOLLIFFE, The uniform convergence of a certain class of trigonometric series, *Proc. London Math. Soc.*, **15** (1916), 214–216.
- [2] R.J. LE AND S.P. ZHOU, A new condition for uniform convergence of certain trigonometric series, *Acta Math. Hungar.*, **10**(1-2) (2005), 161–169.
- [3] L. LEINDLER, Embedding results pertaining to strong approximation of Fourier series. II, *Analysis Math.*, **23** (1997), 223–240.
- [4] L. LEINDLER, On the utility of power-monotone sequences, *Publ. Math. Debrecen*, **55**(1-2) (1999), 169–176.
- [5] L. LEINDLER, On the uniform convergence and boundedness of a certain class of sine series, *Analysis Math.*, **27** (2001), 279–285.
- [6] L. LEINDLER, A new class of numerical sequences and its applications to sine and cosine series, *Analysis Math.*, **28** (2002), 279–286.
- [7] L. LEINDLER, Embedding results regarding strong approximation of sine series, *Acta Sci. Math.* (Szeged), **71** (2005), 91–103.
- [8] L. LEINDLER, A note on the uniform convergence and boundedness of a new class of sine series, *Analysis Math.*, **31** (2005), 269–275.
- [9] S.A. TELYAKOVSKIĬ, On partial sums of Fourier series of functions of bounded variation, *Proc. Steklov Inst. Math.*, **219** (1997), 372–381.



---

A New Extension of Monotone Sequences and its Applications

L. Leindler

---

Title Page

Contents



Go Back

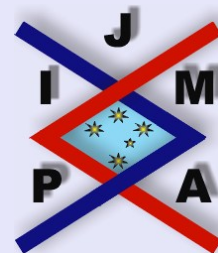
Close

Quit

Page 16 of 17



[10] A. ZYGMUND, *Trigonometric Series*. Vol. I, University Press (Cambridge, 1959).



---

**A New Extension of Monotone Sequences and its Applications**

L. Leindler

---

Title Page

Contents



Go Back

Close

Quit

Page 17 of 17