Journal of Inequalities in Pure and Applied Mathematics

A NEW EXTENSION OF MONOTONE SEQUENCES AND ITS APPLICATIONS

L. LEINDLER

Bolyai Institute
University of Szeged
Aradi vértanúk tere 1
H-6720 Szeged, Hungary.

EMail: leindler@math.u-szeged.hu

ISSN (electronic): 1443-5756
371-05

volume 7, issue 1, article 39,
2006.

*Received 20 December, 2005;
accepted 21 January, 2006.*

Communicated by: A.G. Babenko

Abstract

We define a new class of numerical sequences. This class is wider than any one of the classical or recently defined new classes of sequences of monotone type. Because of this generality we can generalize only the sufficient part of the classical Chaundy-Jolliffe theorem on the uniform convergence of sine series. We also present two further theorems having conditions of sufficient type.

2000 Mathematics Subject Classification: 26A15, 40-99, 40A05, 42A16.
Key words: Monotone sequences, Sequence of γ group bounded variation, Sine series.

This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant Nos. T042462 and TS44782.

Contents

1. Introduction

In [3] we defined a subclass of the quasimonotone sequences ($c_n \le K\, c_m,\ n \ge m$), which is much larger than that of the monotone sequences and not comparable to the class of the classical quasimonotone sequences (see [6]). For this new class we have extended several results proved earlier only for monotone, quasimonotone or classical quasimonotone sequences. The definition of this class reads as follows: A null-sequence $\mathbf{c}$ ($c_n \to 0$) belongs to the family of *sequences of rest bounded variation* (in brief, $\mathbf{c} \in RBVS$) if

$$\sum_{n=m}^{\infty} |\Delta\, c_n| \le K\, c_m \qquad (\Delta\, c_n = c_n - c_{n+1}) \tag{1.1}$$

holds for all m, where $K = K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$. Hereafter K will designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

Recently, in [7], we defined a new class of sequences as follows:

Let $\gamma := \{\gamma_n\}$ be a positive sequence. A null-sequence $\mathbf{c}$ of *real numbers* satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta\, c_n| \le K\, \gamma_m \tag{1.2}$$

is said to be a *sequence of* γ *rest bounded variation* ($\gamma\, RBVS$).

We emphasize that the class $\gamma\, RBVS$ is no longer a subclass of the quasimonotone sequences. Namely, a sequence $\mathbf{c}$ satisfying (1.2) may have infinitely many zero and negative terms, as well; but this is not the case if $\mathbf{c}$ satisfies (1.1).

Very recently Le and Zhou [2] defined another new class of sequences using the following curious definition:

If there exists a natural number N such that

$$\sum_{n=m}^{2m} |\Delta\, c_n| \leq K \max_{m \leq n < m+N} |c_n| \tag{1.3}$$

holds for all m, then $\mathbf{c}$ belongs to the class $GBVS$, in other words, $\mathbf{c}$ is a *sequence of group bounded variation.*

The class $GBVS$ is an ingenious generalization of $RBVS$, moreover it is wider than the class of the classical quasimonotone sequences $\left(c_{n+1} \leq c_n \left(1 + \frac{\alpha}{n}\right)\right)$, too.

In [2], among others, they verified that the monotonicity condition in the classical theorem of Chaundy and Jolliffe [1] can be replaced by their condition (1.3). Herewith they improved our result, namely that in [5], we verified this by condition (1.1).

The aim of the present work is to unify the advantages of the definitions (1.2) and (1.3). We define a further new class of sequences, to be denoted by $\gamma\, GBVS$, which is wider than any one of the classes $GBVS$ and $\gamma\, RBVS$.

A null-sequence $\mathbf{c}$ belongs to $\gamma\, GBVS$ if

$$\sum_{n=m}^{2m} |\Delta\, c_n| \leq K\, \gamma_m, \qquad m = 1, 2, \dots \tag{1.4}$$

holds, where γ is a given sequence of nonnegative numbers.

We underline that the sequence γ satisfying (1.4) may have infinitely many zero terms, too; but not in (1.2). We also emphasize that the condition (1.4) gives the greatest freedom for the terms of the sequences $\mathbf{c}$ and γ.

As a first application we shall give a sufficient condition for the uniform convergence of the series

$$\sum_{n=1}^{\infty} b_n \sin nx, \tag{1.5}$$

where $\mathbf{b} := \{b_n\}$ belongs to a certain class of $\gamma\, GBVS$.

Utilizing the benefits of the sequences of $\gamma\, GBVS$ we present two further generalizations of theorems proved earlier for sequences of $\gamma\, RBVS$.

2. Theorems

We verify the following theorems:

Theorem 2.1. *Let $\gamma := \{\gamma_n\}$ be a sequence of nonnegative numbers satisfying the condition $\gamma_n = o(n^{-1})$. If a sequence $\mathbf{b} := \{b_n\} \in \gamma\, GBVS$, then the series (1.5) is uniformly convergent, and consequently its sum function $f(x)$ is continuous.*

Compare Theorem 2.1 to the mentioned theorem of Chaundy and Jolliffe and two theorems of ours [5, Theorem A and Theorem 1] and [8, Theorem 1]. The cited theorems proved their statements for monotone sequences, $\mathbf{b} \in RBVS$ and $\mathbf{b} \in \gamma\, RBVS$, respectively.

Remark 1. *It is easy to see that if $b_n = n^{-1}$ and $\gamma_n = n^{-1}$, then $\{b_n\} \in \gamma\, GBVS$ and the series (1.5) does not converge uniformly. This shows that the assumption $\gamma_n = o(n^{-1})$ cannot be weakened generally.*

Theorem 2.2. *Let $\beta := \{\eta_n\}$ be a sequence of nonnegative numbers satisfying the condition $\eta_n = O(n^{-1})$. If a sequence $\mathbf{b} := \{b_n\} \in \beta\, RBVS$, then the partial sums of the series (1.5) are uniformly bounded.*

We note that for a monotone null-sequence $\mathbf{b}$, moreover for $\mathbf{b} \in RBVS$ and $\mathbf{b} \in \gamma\, RBVS$, the assertion of Theorem 2.2 can be found in [10, Chapter V, §1], in [5, Theorem 2] and [8, Theorem 2].

Before formulating Theorem 2.3 we recall the following definition. A sequence $\beta := \{\beta_n\}$ of positive numbers is called quasi geometrically increasing (decreasing) if there exist natural numbers μ and $K = K(\beta) \geq 1$ such that for all natural numbers n

$$\beta_{n+\mu} \geq 2\beta_n \text{ and } \beta_n \leq K\,\beta_{n+1} \qquad \left(\beta_{n+\mu} \leq \frac{1}{2}\beta_n \text{ and } \beta_{n+1} \leq K\,\beta_n\right).$$

Theorem 2.3. *If $\mathbf{c} := \{c_n\} \in \beta\,GBVS$, or belongs to $\gamma\,GBVS$, where β and γ have the same meaning as in Theorems 2.1 and 2.2, furthermore the sequence $\{n_m\}$ is quasi geometrically increasing, then the estimates*

$$\sum_{j=1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} c_k \, \sin kx \right| \leq K(\mathbf{c}, \{n_m\}), \tag{2.1}$$

or

$$\sum_{j=m}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} c_k \, \sin kx \right| = o(1), \qquad m \to \infty, \tag{2.2}$$

hold uniformly in x, respectively.

The root of (2.1) goes back to Telyakovskiĭ [9, Theorem 2] and two generalizations of it can be found in [5] and [8].

We note that, in general, (2.1) does not imply (2.2), see the Remark in [8].

It is clear that the "smallest" class $\gamma\,GBVS$ which includes a given sequence $\mathbf{c} := \{c_n\}$ is the one, where

$$\gamma_n := \sum_{k=n}^{2n} |\Delta\, c_k|, \quad n = 1, 2, \ldots$$

In regard to this, it is plain, that our theorems convey the following consequence.

Corollary 2.4. *The assertions of our theorems for an individual sequence* $\mathbf{b}$ *hold true under the assumptions*

$$\sum_{k=n}^{2n} |\Delta\, b_k| = o(n^{-1}) \tag{2.3}$$

and

$$\sum_{k=n}^{2n} |\Delta\, b_k| = O(n^{-1}),$$

respectively.

However, in my view, our theorems give a better perspicuity than Corollary 2.4 does; the arrangement of the proofs are more convenient with our method, furthermore the assumptions of Corollary 2.4 give conditions only for an individual sequence, and not for a class of sequences.

We also remark that e.g. the condition (2.3) is not a necessary one for uniform convergence. See the series

$$\sum_{n=1}^{\infty} 2^{-n} \sin 2^n x.$$

3. Lemmas

Lemma 3.1 ([4]). *For any positive sequence $\{\beta_n\}$ the inequalities*

$$\sum_{n=m}^{\infty} \beta_n \le K\,\beta_m, \qquad m = 1, 2, \dots;\ K \ge 1,$$

hold if and only if the sequence $\{\beta_n\}$ is quasi geometrically decreasing.

Lemma 3.2. *Let $\rho := \{\rho_n\}$ be a nonnegative sequence with $\rho_n = O(n^{-1})$, and let $\delta := \{\delta_n\}$ belong to $\rho\,GBVS$. If a complex series $\sum_{n=1}^{\infty} a_n$ satisfies the Abel condition, i.e., if there exists a constant A such that for all $m \ge 1$,*

$$\left| \sum_{n=1}^{m} a_n \right| \le A,$$

then for any $\mu \ge m$,

$$\left| \sum_{n=m}^{\mu} a_n\,\delta_n \right| \le 6K(\rho)A\,\varepsilon_m m^{-1}, \tag{3.1}$$

where $K(\rho)$ denotes the constant appearing in the definition of $\rho\,GBVS$, furthermore

$$\varepsilon_n := \sup_{k \ge n} k\,\rho_k.$$

Consequently, if $\varepsilon_m = o(m)$, then the series $\sum_{n=1}^{\infty} a_n\,\delta_n$ converges.

Proof. First we show that

$$|\delta_m| \leq \sum_{n=m}^{\infty} |\Delta\,\delta_n| \leq 2K(\rho)\varepsilon_m m^{-1}. \tag{3.2}$$

Since δ_n tends to zero, the first inequality in (3.2) is obvious; and because $n\,\rho_n$ is bounded, thus $\delta \in \rho\,GBVS$ implies that

$$\sum_{n=m}^{\infty} |\Delta\,\delta_n| \leq \sum_{\ell=0}^{\infty} \sum_{n=2^\ell m}^{2^{\ell+1}m} |\Delta\,\delta_n| \leq \sum_{\ell=0}^{\infty} K(\rho)\rho_{2^\ell m}$$

$$\leq K(\rho) \sum_{\ell=0}^{\infty} \varepsilon_m (2^\ell m)^{-1} = 2K(\rho)m^{-1}\varepsilon_m, \tag{3.3}$$

and this proves (3.2).

Next we verify (3.1). Using the notation

$$\alpha_n := \sum_{k=1}^{n} a_k,$$

(3.2) and the assumptions of Lemma 3.2, we get that

$$\left|\sum_{n=m}^{\mu} a_n\,\delta_n\right| = \left|\sum_{n=m}^{\mu-1} \alpha_n(\delta_n - \delta_{n+1}) + \alpha_\mu\,\delta_\mu - \alpha_{m-1}\delta_m\right|$$

$$\leq A\left(\sum_{n=m}^{\mu-1} |\Delta\,\delta_n| + |\delta_\mu| + |\delta_m|\right) \leq 6AK(\rho)\varepsilon_m m^{-1},$$

which proves (3.1).

The proof is complete. □

4. Proofs

Proof of Theorem 2.1. Denote

$$\varepsilon_n := \sup_{k\geq n} k\,\gamma_k \quad \text{and} \quad r_n(x) := \sum_{k=n}^{\infty} b_k \sin kx.$$

In view of the assumption $\gamma_m = o(m^{-1})$ we have that $\varepsilon_n \to 0$ as $n \to \infty$. Thus it is sufficient to verify that

$$|r_n(x)| \leq K\,\varepsilon_n \tag{4.1}$$

holds for all n.

Since $r_n(k\pi) = 0$ it suffices to prove (4.1) for $0 < x < \pi$.

Let N be the integer for which

$$\frac{\pi}{N+1} < x \leq \frac{\pi}{N}. \tag{4.2}$$

First we show that if $k \geq n$ then

$$k|b_k| \leq K\,\varepsilon_n, \quad n = 1, 2, \ldots \tag{4.3}$$

Since b_m and $m\,\gamma_m$ tend to zero, thus the assumption $\mathbf{b} \in \gamma\, GBVS$ implies that

$$|b_k| \leq \sum_{i=k}^{2k-1} |\Delta\, b_i| + |b_{2k}| \leq \sum_{\ell=0}^{1} \sum_{i=2^\ell k}^{2^{\ell+1}k-1} |\Delta\, b_i| + |b_{4k}| \leq \cdots$$

$$\leq K \sum_{\ell=0}^{\infty} \gamma_{2^\ell k} =: \sigma_k. \tag{4.4}$$

By the definition of ε_n and $k \ge n$ we have that

$$2^\ell k\, \gamma_{2^\ell k} \le \varepsilon_n, \quad \ell = 1, 2, \dots,$$

thus it is clear that

$$\sigma_k \le 2K\varepsilon_n/k;$$

this and (4.4) proves (4.3).

Now we turn back to the proof of (4.1). Let

$$r_n(x) = \left(\sum_{k=n}^{n+N-1} + \sum_{k=n+N}^{\infty} \right) b_k \sin kx =: r_n^{(1)}(x) + r_n^{(2)}(x).$$

Then, by (4.2) and (4.3),

$$|r_n^{(1)}(x)| \le x \sum_{k=n}^{n+N-1} k|b_k| \le K\, x\, N\, \varepsilon_n \le K\, \pi\, \varepsilon_n. \tag{4.5}$$

A similar consideration as in (3.3) gives that for any $m \ge n$

$$\sum_{k=m}^{\infty} |\Delta\, b_k| \le K\, \varepsilon_n/m.$$

Using this, (4.2), (4.3) and the well-known inequality

$$D_n(x) := \left| \sum_{k=1}^{n} \sin kx \right| \le \frac{\pi}{x},$$

furthermore summing by parts, we get that

$$|r_n^{(2)}(x)| \leq \sum_{k=n+N}^{\infty} |\Delta\, b_k| D_k(x) + |b_{n+N}| D_{n+N-1}(x)$$

$$\leq 2K \frac{\varepsilon_n}{n+N} \frac{\pi}{x} \leq 2K\, \varepsilon_n. \tag{4.6}$$

The inequalities (4.5) and (4.6) imply (4.1), that is, the series (1.5) is uniformly convergent.

The proof is complete. □

Proof of Theorem 2.2. In the proof of Theorems 2.2 and 2.3 we shall use the notations of the proof of Theorem 2.1. The condition $\eta_n = O(n^{-1})$ implies that the sequence $\{\varepsilon_n\}$ is bounded, i.e. $\varepsilon_n \leq K$. This, (4.2) and (4.3) imply that for any $m \leq N$

$$\left| \sum_{k=1}^{m} b_k \sin kx \right| \leq \sum_{k=1}^{N} |b_k| kx \leq K\, x\, N \leq K\, \pi,$$

furthermore, if $m > N$ then, by (4.1),

$$\left| \sum_{k=N+1}^{m} b_k \sin kx \right| \leq |r_{N+1}(x)| + |r_{m+1}(x)| \leq 2K\, \varepsilon_1.$$

The last two estimates clearly prove Theorem 2.2. □

Proof of Theorem 2.3. First we verify (2.1). Let us suppose that

$$n_i \le N < n_{i+1}. \tag{4.7}$$

Since $\mathbf{c} \in \beta GBVS$ and $\eta_n = O(n^{-1})$, we get, as in the proof of Theorem 2.2 with c_n in place of b_n, that

$$\sum_{j=1}^{i-1} \left| \sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx \right| + \left| \sum_{k=n_i}^{N} c_k \sin kx \right| \le \sum_{j=1}^{i-1} \sum_{k=n_j}^{n_{j+1}-1} |c_k| kx + \sum_{k=n_i}^{N} |c_k| kx \le K\pi. \tag{4.8}$$

Next applying Lemma 3.2 with $\rho = \beta,\ \delta_n = c_n$ and $a_n = \sin nx$, we get that

$$\begin{aligned}\sigma_N^* &:= \left| \sum_{k=N+1}^{n_{i+1}-1} c_k \sin kx \right| + \sum_{j=i+1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx \right| \\ &\le K \left\{ \varepsilon_{N+1}(N+1)^{-1} x^{-1} + x^{-1} \sum_{j=i+1}^{\infty} \varepsilon_{n_j} n_j^{-1} \right\} \\ &\le K \left\{ \varepsilon_N + N\, \varepsilon_N \sum_{j=i+1}^{\infty} n_j^{-1} \right\} \le K\, \varepsilon_N \left\{ 1 + N \sum_{j=i+1}^{\infty} n_j^{-1} \right\}. \end{aligned} \tag{4.9}$$

Since the sequence $\{n_j\}$ is quasi geometrically increasing, so $\{n_j^{-1}\}$ is quasi geometrically decreasing, therefore, Lemma 3.1 and (4.7) imply that

$$\sum_{j=i+1}^{\infty} n_j^{-1} \leq K\, N^{-1}, \tag{4.10}$$

whence, by (4.9) and $\eta_n = O(n^{-1})$,

$$\sigma_N^* \leq K\, \varepsilon_N < \infty \tag{4.11}$$

follows. Herewith (2.1) is proved.

If $\mathbf{c} \in \gamma\, GBVS$ then, by $\gamma_n = o(n^{-1}),\ \varepsilon_n \to 0$, thus, with m in place of N, (4.9), (4.10) and (4.11) immediately verify (2.2).

The proof is complete. $\square$

References

[1] T.W. CHAUNDY AND A.E. JOLLIFFE, The uniform convergence of a certain class of trigonometric series, *Proc. London Math. Soc.*, **15** (1916), 214–216.

[2] R.J. LE AND S.P. ZHOU, A new condition for uniform convergence of certain trigonometric series, *Acta Math. Hungar.*, **10**(1-2) (2005), 161–169.

[3] L. LEINDLER, Embedding results pertaining to strong approximation of Fourier series. II, *Analysis Math.*, **23** (1997), 223–240.

[4] L. LEINDLER, On the utility of power-monotone sequences, *Publ. Math. Debrecen*, **55**(1-2) (1999), 169–176.

[5] L. LEINDLER, On the uniform convergence and boundedness of a certain class of sine series, *Analysis Math.*, **27** (2001), 279–285.

[6] L. LEINDLER, A new class of numerical sequences and its applications to sine and cosine series, *Analysis Math.*, **28** (2002), 279–286.

[7] L. LEINDLER, Embedding results regarding strong approximation of sine series, *Acta Sci. Math.* (Szeged), **71** (2005), 91–103.

[8] L. LEINDLER, A note on the uniform convergence and boundedness of a new class of sine series, *Analysis Math.*, **31** (2005), 269–275.

[9] S.A. TELYAKOVSKIǏ, On partial sums of Fourier series of functions of bounded variation, *Proc. Steklov Inst. Math.*, **219** (1997), 372–381.

[10] A. ZYGMUND, *Trigonometric Series*. Vol. I, University Press (Cambridge, 1959).