SOME INEQUALITIES REGARDING A GENERALIZATION OF EULER'S CONSTANT

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Abstract: The purpose of this paper is to evaluate the limit $\gamma(a)$ of the sequence

$$\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}},$$

where $a\in(0,+\infty)$. We give some lower and upper estimates for

$$\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}-\gamma(a),\quad n\in\mathbb{N}.$$

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1. Introduction

Let $(D_n)_{n\in\mathbb{N}}$ be the sequence defined by $D_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$, for each $n \in \mathbb{N}$. It is well-known that the sequence $(D_n)_{n\in\mathbb{N}}$ is convergent and its limit, usually denoted by γ, is called Euler's constant.

For $D_n - \gamma$, $n \in \mathbb{N}$, many lower and upper estimates have been obtained in the literature. We recall some of them:

- $\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2(n-1)}$, for each $n \in \mathbb{N} \setminus \{1\}$ ([14]);
- $\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}$, for each $n \in \mathbb{N}$ ([8], [19]);
- $\frac{1}{2n+1} < D_n - \gamma < \frac{1}{2n}$, for each $n \in \mathbb{N}$ ([17]);
- $\frac{1}{2n+\frac{2}{5}} < D_n - \gamma < \frac{1}{2n+\frac{1}{3}}$, for each $n \in \mathbb{N}$ ([15], [16]);
- $\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} \leq D_n - \gamma < \frac{1}{2n+\frac{1}{3}}$, for each $n \in \mathbb{N}$ ([16, Editorial comment], [2], [3]).

In Section 2 we present a generalization of Euler's constant as the limit of the sequence

$$\left(\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}}, \quad a \in (0, +\infty),$$

and we denote this limit by $\gamma(a)$.

In Section 3 we give some lower and upper estimates for

$$\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\frac{a+n-1}{a} - \gamma(a), \quad n \in \mathbb{N}.$$

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2. The Number $\gamma(a)$

It is known that the sequence

$$\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}}, \quad a\in(0,+\infty),$$

is convergent (see for example [5, p. 453], [7], where problems in this sense were proposed; [6]; [13]).

The results contained in the following theorem were given in [10].

Theorem 2.1. *Let $a\in(0,+\infty)$. We consider the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined by*

$$x_n=\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n}{a}$$

and

$$y_n=\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a},$$

for each $n\in\mathbb{N}$.

Then:

(i) *the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent to the same number, which we denote by $\gamma(a)$, and satisfy the inequalities $x_n<x_{n+1}<\gamma(a)<y_{n+1}<y_n$, for each $n\in\mathbb{N}$;*

(ii) $0<\frac{1}{a}-\ln\left(1+\frac{1}{a}\right)<\gamma(a)<\frac{1}{a}$;

(iii) $\lim_{n\to\infty} n(\gamma(a)-x_n)=\frac{1}{2}$ *and* $\lim_{n\to\infty} n(y_n-\gamma(a))=\frac{1}{2}$.

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Remark 1. The sequence $(y_n)_{n\in\mathbb{N}}$ from Theorem 2.1, for $a = 1$, becomes the sequence $(D_n)_{n\in\mathbb{N}}$, so $\gamma(1) = \gamma$.

The following theorem was given by the author in [12, Theorem 2.3].

Theorem 2.2. *Let* $a \in (0, +\infty)$. *We consider the sequence* $(u_n)_{n\in\mathbb{N}}$ *defined by* $u_n = y_n - \frac{1}{2(a+n-1)+\frac{1}{3}}$, *for each* $n \in \mathbb{N}$, *where* $(y_n)_{n\in\mathbb{N}}$ *is the sequence from the statement of Theorem 2.1. Also, we specify that* $\gamma(a)$ *is the limit of the sequence* $(y_n)_{n\in\mathbb{N}}$.

Then:

(i) $u_n < u_{n+1} < \gamma(a)$, *for each* $n \in \mathbb{N} \setminus \{1\}$, *and* $\lim\limits_{n\to\infty} n^3(\gamma(a) - u_n) = \frac{1}{72}$;

(ii) $\frac{1}{2(a+n-1)+\frac{11}{28}} < y_n - \gamma(a) < \frac{1}{2(a+n-1)+\frac{1}{3}}$, *for each* $n \in \mathbb{N} \setminus \{1\}$.

Remark 2. The lower estimate from part (ii) of Theorem 2.2 holds for $n = 1$ as well.

Remark 3. The second limit from part (iii) of Theorem 2.1 also follows from part (ii) of Theorem 2.2.

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3. Proving Some Estimates for $y_n - \gamma(a)$ using the Logarithmic Derivative of the Gamma Function

As we already mentioned in Section 1, it is known that ([16, Editorial comment], [2, Theorem 3], [3, Theorem 1.1])

$$\frac{1}{2n + \frac{2\gamma-1}{1-\gamma}} \le D_n - \gamma < \frac{1}{2n + \frac{1}{3}},$$

for each $n \in \mathbb{N}$, the constants $\frac{2\gamma-1}{1-\gamma}$ and $\frac{1}{3}$ being the best possible with this property.

Let $a \in (0, +\infty)$. In a similar way as in the proof given by H. Alzer in [2, Theorem 3], we shall obtain lower and upper estimates for $y_n - \gamma(a)$ $(n \in \mathbb{N})$, where $(y_n)_{n\in\mathbb{N}}$ is the sequence from the statement of Theorem 2.1, the limit of which we denoted by $\gamma(a)$. In order to do this we shall prove, in a similar way as in [3, Lemma 2.1], some finer inequalities than those used by H. Alzer in [2, Theorem 3].

Lemma 3.1. *We have:*

(i) $\psi(x+1) - \ln x > \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}$, *for each* $x \in (0, +\infty)$;

(ii) $\frac{1}{x} - \psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \frac{1}{42x^7} + \frac{1}{30x^9}$, *for each* $x \in (0, +\infty)$.

We specify that the function ψ *is the logarithmic derivative of the gamma function, i.e.* $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, *for each* $x \in (0, +\infty)$.

Proof. (i) It is known (see, for example, [18, p. 116]) that $\ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t}\,dt$, for each $x \in (0, +\infty)$. Also, we shall need the formula

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right) dt,$$

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which holds for each $x \in (0, +\infty)$, known as Gauss' expression of $\psi(x)$ as an infinite integral (see, for example, [18, p. 247]). Having in view the above relations, we are able to write that

$$\psi(x+1) - \ln x = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt}\, dt,$$

for each $x \in (0, +\infty)$.

It is not difficult to verify that

$$\int_0^\infty t^n e^{-xt}\, dt = \frac{n!}{x^{n+1}},$$

for each $n \in \mathbb{N} \cup \{0\}$, any $x \in (0, +\infty)$.

Then we have

$$\begin{aligned}
&\psi(x+1) - \ln x - \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} \\
&= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{t}{12} - \frac{t^3}{720} + \frac{t^5}{30240} \right) e^{-xt}\, dt \\
&= \int_0^\infty \frac{1}{30240t(e^t - 1)} [30240(e^t - 1) - 30240t - 15120t(e^t - 1) + 2520t^2(e^t - 1) \\
&\qquad - 42t^4(e^t - 1) + t^6(e^t - 1)] e^{-xt}\, dt \\
&= \int_0^\infty \frac{1}{30240t(e^t - 1)} \left[30240 \sum_{n=2}^\infty \frac{t^n}{n!} - 15120 \sum_{n=1}^\infty \frac{t^{n+1}}{n!} + 2520 \sum_{n=1}^\infty \frac{t^{n+2}}{n!} \right. \\
&\qquad \left. -42 \sum_{n=1}^\infty \frac{t^{n+4}}{n!} + \sum_{n=1}^\infty \frac{t^{n+6}}{n!} \right] e^{-xt}\, dt
\end{aligned}$$

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$$= \int_0^\infty \frac{\sum_{n=9}^{\infty} \frac{(n-3)(n-5)(n-7)(n-8)(n^2+8n+36)}{n!} t^n}{30240t(e^t-1)} \cdot e^{-xt}\, dt > 0,$$

for each $x \in (0, +\infty)$.

(ii) In part (i) we obtained that

$$\ln x - \psi(x+1) = \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} \right) e^{-xt}\, dt,$$

for each $x \in (0, +\infty)$. Differentiating here we get that

$$\frac{1}{x} - \psi'(x+1) = \int_0^\infty \left(1 - \frac{t}{e^t-1} \right) e^{-xt}\, dt,$$

for each $x \in (0, +\infty)$.

Then we have

$$\frac{1}{x} - \psi'(x+1) - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}$$
$$= \int_0^\infty \left(1 - \frac{t}{e^t-1} - \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \frac{t^6}{30240} - \frac{t^8}{1209600} \right) e^{-xt}\, dt$$
$$= \int_0^\infty \frac{1}{1209600(e^t-1)} [1209600(e^t-1) - 1209600t - 604800t(e^t-1)$$
$$+ 100800t^2(e^t-1) - 1680t^4(e^t-1) + 40t^6(e^t-1) - t^8(e^t-1)] e^{-xt}\, dt$$

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$$= \int_0^\infty \frac{1}{1209600(e^t-1)}\left[1209600\sum_{n=2}^\infty \frac{t^n}{n!} - 604800\sum_{n=1}^\infty \frac{t^{n+1}}{n!}\right.$$

$$\left. +100800\sum_{n=1}^\infty \frac{t^{n+2}}{n!} - 1680\sum_{n=1}^\infty \frac{t^{n+4}}{n!} + 40\sum_{n=1}^\infty \frac{t^{n+6}}{n!} - \sum_{n=1}^\infty \frac{t^{n+8}}{n!}\right] e^{-xt}\,dt$$

$$= -\int_0^\infty \frac{\sum_{n=11}^\infty \frac{(n-3)(n-5)(n-7)(n-9)(n-10)(n+4)(n^2+2n+32)}{n!} t^n}{1209600(e^t-1)} \cdot e^{-xt}\,dt < 0,$$

for each $x \in (0, +\infty)$. □

Remark 4. In fact, these inequalities from Lemma 3.1 come from the asymptotic formulae (see, for example, [1, *pp*. 259, 260])

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{n=1}^\infty \frac{B_{2n}}{2nx^{2n}}$$
$$= \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots$$

and

$$\psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \sum_{n=1}^\infty \frac{B_{2n}}{x^{2n+1}}$$
$$= \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \cdots,$$

where B_{2n} is the Bernoulli number of index $2n$.

Theorem 3.2. *Let $a \in (0, +\infty)$. We consider the sequence $(y_n)_{n\in\mathbb{N}}$ from the statement of Theorem 2.1, the limit of which we denoted by $\gamma(a)$.*

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Then

$$\frac{1}{2(a+n-1)+\alpha} \le y_n - \gamma(a) < \frac{1}{2(a+n-1)+\beta},$$

for each $n \in \mathbb{N} \setminus \{1,2\}$, *with* $\alpha = \frac{1}{y_3-\gamma(a)} - 2(a+2)$ *and* $\beta = \frac{1}{3}$.

Moreover, the constants α *and* β *are the best possible with this property.*

Proof. The inequalities from the statement of the theorem can be rewritten in the form

$$\beta < \frac{1}{y_n - \gamma(a)} - 2(a+n-1) \le \alpha,$$

for each $n \in \mathbb{N} \setminus \{1,2\}$.

Taking into account that $\psi(x+1) = \psi(x) + \frac{1}{x}$, for each $x \in (0,+\infty)$, we can write that

$$\psi(a+n) - \psi(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1},$$

for each $n \in \mathbb{N}$ (see, for example, [1, p. 258]).

It is known that we have the series expansion (see, for example, [9, p. 336])

$$\psi(x) = \ln x - \sum_{k=0}^{\infty} \left[\frac{1}{x+k} - \ln\left(1 + \frac{1}{x+k}\right)\right],$$

for each $x \in (0,+\infty)$. So, we are able to write the following relation between $\gamma(a)$ and the logarithmic derivative of the gamma function:

$$\gamma(a) = \ln a - \psi(a)$$

(see [6, Theorem 7], [11, Theorem 4.1, Remark 4.2]).

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Then

$$y_n - \gamma(a) = \psi(a+n) - \psi(a) - \ln\frac{a+n-1}{a} - [\ln a - \psi(a)]$$
$$= \psi(a+n) - \ln(a+n-1),$$

for each $n \in \mathbb{N}$. It means that, in fact, we have to prove that

$$\beta < \frac{1}{\psi(a+n) - \ln(a+n-1)} - 2(a+n-1) \le \alpha,$$

for each $n \in \mathbb{N} \setminus \{1,2\}$, and that the constants α and β are the best possible with this property.

We consider the function $f : (0,+\infty) \to \mathbb{R}$, defined by

$$f(x) = \frac{1}{\psi(x+1) - \ln x} - 2x,$$

for each $x \in (0,+\infty)$. Differentiating, we get that

$$f'(x) = \frac{\frac{1}{x} - \psi'(x+1) - 2[\psi(x+1) - \ln x]^2}{[\psi(x+1) - \ln x]^2},$$

for each $x \in (0,+\infty)$. Using the inequalities from Lemma 3.1, we are able to write that

$$\frac{1}{x} - \psi'(x+1) - 2[\psi(x+1) - \ln x]^2$$
$$< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \frac{1}{42x^7} + \frac{1}{30x^9} - 2\left(\frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}\right)^2$$
$$= -\frac{1}{72x^4} + \frac{1}{60x^5} + \frac{1}{360x^6} - \frac{1}{63x^7} - \frac{221}{151200x^8} + \frac{1}{30x^9} + \frac{1}{7560x^{10}} - \frac{1}{31752x^{12}}$$
$$=: g(x),$$

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for each $x \in (0, +\infty)$. It is not difficult to verify that $g(x) < 0$, for each $x \in \left[\frac{3}{2}, +\infty\right)$ ($\frac{3}{2}$ not being the best lower value possible with this property). It follows that $f'(x) < 0$, for each $x \in \left[\frac{3}{2}, +\infty\right)$. So, the function f is strictly decreasing on $\left[\frac{3}{2}, +\infty\right)$. This means that the sequence $(f(a+n-1))_{n\geq 3}$ is strictly decreasing. Therefore

$$\begin{aligned}\lim_{k\to\infty} f(a+k-1) &< f(a+n-1) \\ &\leq f(a+2) \\ &= \frac{1}{y_3 - \gamma(a)} - 2(a+2),\end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1, 2\}$.

The asymptotic formula for the function ψ, mentioned in Remark 4, permits us to write that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{1}{6} + O\left(\frac{1}{x^2}\right)}{\frac{1}{2} + O\left(\frac{1}{x}\right)} = \frac{1}{3}.$$

□

Theorem 3.3. *Let $a \in \left[\frac{1}{2}, +\infty\right)$. We consider the sequence $(y_n)_{n\in\mathbb{N}}$ from the statement of Theorem 2.1, the limit of which we denoted by $\gamma(a)$.*

Then

$$\frac{1}{2(a+n-1)+\alpha} \leq y_n - \gamma(a) < \frac{1}{2(a+n-1)+\beta},$$

for each $n \in \mathbb{N} \setminus \{1\}$, with $\alpha = \frac{1}{y_2 - \gamma(a)} - 2(a+1)$ and $\beta = \frac{1}{3}$.

Moreover, the constants α and β are the best possible with this property.

Proof. Since $a \in \left[\frac{1}{2}, +\infty\right)$, it follows that the sequence $(f(a+n-1))_{n\geq 2}$ is strictly decreasing, where f is the function defined in the proof of Theorem 3.2. □

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Theorem 3.4. *Let* $a \in \left[\frac{3}{2}, +\infty\right)$. *We consider the sequence* $(y_n)_{n\in\mathbb{N}}$ *from the statement of Theorem 2.1, the limit of which we denoted by* $\gamma(a)$.

Then

$$\frac{1}{2(a+n-1)+\alpha} \leq y_n - \gamma(a) < \frac{1}{2(a+n-1)+\beta},$$

for each $n \in \mathbb{N}$, *with* $\alpha = \frac{1}{y_1-\gamma(a)} - 2a = \frac{a[2a\gamma(a)-1]}{1-a\gamma(a)}$ *and* $\beta = \frac{1}{3}$.

Moreover, the constants α *and* β *are the best possible with this property.*

Proof. Since $a \in \left[\frac{3}{2}, +\infty\right)$, it follows that the sequence $(f(a+n-1))_{n\in\mathbb{N}}$ is strictly decreasing, where f is the function defined in the proof of Theorem 3.2. □

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