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NEGATIVE RESULTS CONCERNING FOURIER SERIES ON THE COMPLETE PRODUCT OF $\mathcal{S}_3$

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ABSTRACT. The aim of this paper is to continue the studies about convergence in L^p-norm of the Fourier series based on representative product systems on the complete product of finite groups. We restrict our attention to bounded groups with unbounded sequence Ψ. The most simple example of this groups is the complete product of $\mathcal{S}_3$. In this case we proved the existence of an $1 < p < 2$ number for which exists an $f \in L^p$ such that its n-th partial sum of Fourier series S_n do not converge to the function f in L^p-norm. In this paper we extend this "negative" result for all $1 < p < \infty$ and $p \neq 2$ numbers.

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In Section 1 we introduce basic concepts in the study of representative product systems and Fourier analysis. We also introduce the system with which we work on the complete product of $\mathcal{S}_3$, i.e. the symmetric group on 3 elements (see [2]). Section 2 extends the definition of the sequence Ψ for all $p \geq 1$. Finally, we use the results of Section 2 to study the convergence in the L^p-norm ($p \geq 1$) of the Fourier series on bounded groups with unbounded sequence Ψ, supposing all the same finite groups appearing in the product of G have the same system φ at all of their occurrences. These results appear in Section 3 and they complete the statement proved by G. Gát and the author of this paper in [2] for the complete product of $\mathcal{S}_3$. There have been similar results proved with respect to Walsh-like systems in [4] and [5].

Throughout this work denote by $\mathbb{N}$, $\mathbb{P}$, $\mathbb{C}$ the set of nonnegative, positive integers and complex numbers, respectively. The notation which we have used in this paper is similar to [3].

1. REPRESENTATIVE PRODUCT SYSTEMS

Let $m := (m_k,\ k \in \mathbb{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k a finite group with order m_k, $(k \in \mathbb{N})$. Suppose that each group has discrete topology and normalized Haar measure μ_k. Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures (μ). Thus each $x \in G$ consists of sequences $x := (x_0, x_1, \ldots)$, where $x_k \in G_k$, $(k \in \mathbb{N})$. We call this sequence the *expansion* of

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x. The compact totally disconnected group G is called a *bounded group* if the sequence m is bounded.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbb{N}$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \le n_k < m_k$, $n_k \in \mathbb{N}$. This allows us to say that the sequence $(n_0, n_1, \dots)$ is the expansion of n with respect to m.

Denote by Σ_k the dual object of the finite group G_k ($k \in \mathbb{N}$). Thus each $\sigma \in \Sigma_k$ is a set of continuous irreducible unitary representations of G_k which are equivalent to some fixed representation $U^{(\sigma)}$. Let d_σ be the dimension of its representation space and let $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$ be a fixed but arbitrary orthonormal basis in the representation space. The functions

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \zeta_i, \zeta_j \rangle \qquad (i, j \in \{1, \dots, d_\sigma\},\ x \in G_k)$$

are called the coordinate functions for $U^{(\sigma)}$ and the basis $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$. In this manner for each $\sigma \in \Sigma_k$ we obtain d_σ^2 number of coordinate functions, in total m_k number of functions for the whole dual object of G_k. The L^2-norm of these functions is $1/\sqrt{d_\sigma}$.

Let $\{\varphi_k^s : 0 \le s < m_k\}$ be the set of all *normalized coordinate functions* of the group G_k and suppose that $\varphi_k^0 \equiv 1$. Thus for every $0 \le s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, \dots, d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k).$$

Let ψ be the product system of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \dots)$. Thus we say that ψ is the *representative product system* of φ. The Weyl-Peter's theorem (see [3]) ensures that the system ψ is orthonormal and complete on $L^2(G)$.

The functions ψ_n $(n \in \mathbb{N})$ are not necessarily uniformly bounded, so define

$$\Psi_k := \max_{n < M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbb{N}).$$

It seems that the boundedness of the sequence Ψ plays an important role in the norm convergence of Fourier series.

For an integrable complex function f defined in G we define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_{G_m} f \overline{\psi}_k \, d\mu \quad (k \in \mathbb{N}), \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbb{P}).$$

According to the theorem of Banach-Steinhauss, $S_n f \to f$ as $n \to \infty$ in the L^p norm for $f \in L^p(G)$ if and only if there exists a $C_p > 0$ such that

$$\|S_n f\|_p \le C_p \|f\|_p \quad (f \in L^p(G)).$$

Thus, we say that the operator S_n is of *type* (p, p). Since the system ψ forms an orthonormal base in the Hilbert space $L^2(G)$, it is obvious that S_n is of type $(2, 2)$.

The representative product systems are the generalization of the well known Walsh-Paley and Vilenkin systems. Indeed, we obtain the Walsh-Paley system if $m_k = 2$ and $G_k := \mathcal{Z}_2$, the cyclic group of order 2 for all $k \in \mathbb{N}$. Moreover, we obtain the Vilenkin systems if the sequence m is an arbitrary sequence of integers greater than 1 and $G_k := \mathcal{Z}_{m_k}$, the cyclic group of order m_k for all $k \in \mathbb{N}$.

Let $m_k = 6$ for all $k \in \mathbb{N}$ and $\mathcal{S}_3$ be the *symmetric group* on 3 elements. Let $G_k := \mathcal{S}_3$ for all $k \in \mathbb{N}$. $\mathcal{S}_3$ has two characters and a 2-dimensional representation. Using a calculation of the matrices corresponding to the 2-dimensional representation we construct the functions φ_k^s. In the notation the index k is omitted because all of the groups G_k are the same.

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

Notice that the functions φ_k^s can take the value 0, and the product system of φ is not uniformly bounded. These facts encumber the study of these systems. On the other hand, $\max_{0\le s<6} \|\varphi^s\|_1\|\varphi^s\|_\infty = \frac{4}{3}$, thus $\Psi_k = \left(\frac{4}{3}\right)^k \to \infty$ if $k \to \infty$. More examples of representative product systems have appeared in [2] and [7].

2. THE SEQUENCE OF FUNCTIONS $\Psi_k(p)$

We extend the definition of the sequence Ψ for all $p \ge 1$ as follows:

$$\Psi_k(p) := \max_{n<M_k} \|\psi_n\|_p\|\psi_n\|_q \qquad \left(p \ge 1,\ \frac{1}{p}+\frac{1}{q}=1,\ k\in\mathbb{N}\right)$$

(if $p=1$ then $q=\infty$). Notice that $\Psi_k = \Psi_k(1)$ for all $k \in \mathbb{N}$. Clearly, the functions $\Psi_k(p)$ can be written in the form

$$\begin{aligned}\Psi_k(p) &= \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_p\|\varphi_i^s\|_q \\ &=: \prod_{i=0}^{k-1} \Upsilon_i(p) \quad \left(p\ge 1,\ \frac{1}{p}+\frac{1}{q}=1,\ k\in\mathbb{N}\right).\end{aligned}$$

Therefore, we study the product $\|f\|_p\|f\|_q$ for normalized functions on finite groups. In this regard we use the Hölder inequality (see [3, p. 137]). First, we prove the following lemma.

Lemma 2.1. *Let G be a finite group with discrete topology and normalized Haar measure μ, and let f be a normalized complex valued function on G ($\|f\|_2 = 1$). Thus,*

(1) *if $\|f\|_1\|f\|_\infty = 1$, then $\|f\|_p\|f\|_q = 1$ for all $p \ge 1$ and $\frac{1}{p}+\frac{1}{q}=1$.*
(2) *if $\|f\|_1\|f\|_\infty > 1$, then $\|f\|_p\|f\|_q > 1$ for all $p \ge 1$, $p \ne 2$ and $\frac{1}{p}+\frac{1}{q}=1$.*

Proof.

(1) The conditions imply the equality

$$\int_G |f|\, d\mu \cdot \|f\|_\infty = 1 = \int_G |f|^2\, d\mu.$$

Let $f_0 := \frac{f}{\|f\|_\infty}$. Then

$$|f_0(x)| \le 1 \quad (x \in G) \tag{2.1}$$

and

$$\int_G |f_0|\,d\mu = \int_G |f_0|^2\,d\mu. \tag{2.2}$$

Thus by (2.1) we obtain $|f_0(x)| - |f_0(x)|^2 \geq 0$ $(x \in G)$ and by (2.2) we have

$$\int_G |f_0| - |f_0|^2\,d\mu = 0.$$

Hence $|f_0(x)| = |f_0(x)|^2$ for all $x \in G$. Thus, we have $|f_0(x)| = 1$ or $|f_0(x)| = 0$ for all $x \in G$, therefore $|f(x)| = \|f\|_\infty$ or $|f(x)| = 0$ for all $x \in G$. For this reason we obtain an equality in the Hölder inequality for all $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and the equality

$$1 = \int_G |f|^2\,d\mu = \|f\|_p \|f\|_q$$

holds.

(2) Suppose there is a $1 < p < 2$ such that

$$\|f\|_p \|f\|_q = 1 = \int_G |f|^2\,d\mu.$$

Then the equality in the Hölder inequality holds. For this reason there are nonnegative numbers A and B not both 0 such that

$$A|f(x)|^p = B|f(x)|^q \quad (x \in G).$$

Thus, there is a $c > 0$ such that $|f| = c$ or $|f| = 0$ for all $x \in G$ $(c = \|f\|_\infty)$. Then $|f| \cdot \|f\|_\infty = |f|^2$. Integrating boths part of the last equation we have $\|f\|_1 \|f\|_\infty = 1$. We obtain a contradiction.

□

However, the following lemma states much more.

Lemma 2.2. *Let G be a finite group with discrete topology and normalized Haar measure μ, and let f be a complex valued function on G. Thus, the function $\Psi(p) := \|f\|_p \|f\|_q$ $(\frac{1}{p} + \frac{1}{q} = 1)$ is a monotone decreasing function on the interval $[1, 2]$.*

Proof. Let $f_0 := \frac{f}{\|f\|_\infty}$. Then $\Psi(p) = \|f\|_\infty^2 \|f_0\|_p \|f_0\|_q$. Let m be the order of the group G. We take the elements of G in the order, $G = \{g_1,\ g_2, \ldots,\ g_m\}$, to obtain the numbers

$$a_i := |f_0(g_i)| \leq 1 \qquad (i = 1, \ldots, m),$$

with which we write

$$\Psi(p) = \frac{\|f\|_\infty^2}{m} \left(\sum_{i=1}^m a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^m a_i^q \right)^{\frac{1}{q}}.$$

Since $q = \frac{p}{p-1}$, we have

$$\frac{\partial q}{\partial p} = -\frac{1}{(p-1)^2} = -\frac{q^2}{p^2}.$$

Therefore,

$$\frac{\partial \Psi}{\partial p} = \Psi(p) \left[-\frac{1}{p^2} \log \left(\sum_{i=1}^m a_i^p \right) + \frac{1}{p} \frac{\sum_{i=1}^m a_i^p \log a_i}{\sum_{i=1}^m a_i^p} \right] + \Psi(p) \left[-\frac{1}{q^2} \log \left(\sum_{i=1}^m a_i^q \right) + \frac{1}{q} \frac{\sum_{i=1}^m a_i^q \log a_i}{\sum_{i=1}^m a_i^q} \right] \left(-\frac{q^2}{p^2} \right).$$

The condition $1 < p < 2$ ensures that

$$-\frac{1}{q}\cdot\frac{q^2}{p^2} = -\frac{1}{p(p-1)} < -\frac{1}{p},$$

from which we have

$$\frac{1}{\Psi(p)}\frac{\partial\Psi}{\partial p} \le \frac{1}{p^2}\left[\log\left(\sum_{i=1}^m a_i^q\right) - \log\left(\sum_{i=1}^m a_i^p\right)\right] + \frac{1}{p}\left[\frac{\sum_{i=1}^m a_i^p\log a_i}{\sum_{i=1}^m a_i^p} - \frac{\sum_{i=1}^m a_i^q\log a_i}{\sum_{i=1}^m a_i^q}\right].$$

Both addends in the sum above are not positive. Indeed, the facts $a_i \le 1$ for all $1 \le i \le m$ and $p < q$ imply that $a_i^q \le a_i^p$ for all $1 \le i \le m$, from which it is clear that

$$\log\left(\sum_{i=1}^m a_i^q\right) - \log\left(\sum_{i=1}^m a_i^p\right) \le 0. \tag{2.3}$$

Secondly,

$$h(x) := \frac{\sum_{i=1}^m a_i^x\log a_i}{\sum_{i=1}^m a_i^x}$$

is a monotone increasing function. Indeed,

$$\begin{aligned} h'(x) =& \frac{\left(\sum_{i=1}^m a_i^x\log^2 a_i\right)\sum_{i=1}^m a_i^x - \left(\sum_{i=1}^m a_i^x\log a_i\right)^2}{\left(\sum_{i=1}^m a_i^x\right)^2} \\ =& \frac{\sum_{i,j=1}^m a_i^x a_j^x(\log a_i - \log a_j)^2}{\left(\sum_{i=1}^m a_i^x\right)^2} \ge 0. \end{aligned}$$

Consequently, we have

$$\frac{\sum_{i=1}^m a_i^p\log a_i}{\sum_{i=1}^m a_i^p} - \frac{\sum_{i=1}^m a_i^q\log a_i}{\sum_{i=1}^m a_i^q} \le 0. \tag{2.4}$$

By (2.3) and (2.4) we obtain $\frac{\partial\Psi}{\partial p} \le 0$ for all $1 < p < 2$, which completes the proof of the lemma. □

We can apply Lemma 2.1 and Lemma 2.2 to obtain similar properties for $\Upsilon_k(p)$ and $\Psi_k(p)$ because these functions are the maximum value and the product of finite functions satisfying the conditions of the two lemmas. Consequently, we obtain:

Theorem 2.3. *Let G_k be a coordinate group of G such that $\|\varphi_k^s\|_1 = 1$ for all $s < m_k$. Then $\Upsilon_k(p) \equiv 1$. Otherwise, the function $\Upsilon_k(p)$ is a strictly monotone decreasing function on the interval $[1,2]$.*

The function $\Psi_k(p) \equiv 1$ if $\|\varphi_i^s\|_1 = 1$ for all $s < m_i$ and $i \le k$. Otherwise, the function $\Psi_k(p)$ is a strictly monotone decreasing function on the interval $[1,2]$.

It is important to remark that the functions $\Upsilon_k(p)$ and $\Psi_k(p)$ are monotone increasing if $p > 2$. It follows from the property $\Upsilon_k(p) = \Upsilon_k\left(\frac{p}{p-1}\right)$. In order to illustrate these properties we plot the values of $\Upsilon(p)$ for the group $\mathcal{S}_3$.

3. NEGATIVE RESULTS

Theorem 3.1. *Let p be a fixed number on the interval $(1,2)$ and $\frac{1}{p}+\frac{1}{q} = 1$. If G is a group with unbounded sequence $\Psi_k(p)$, then the operator S_n is not of type (p,p) or (q,q).*

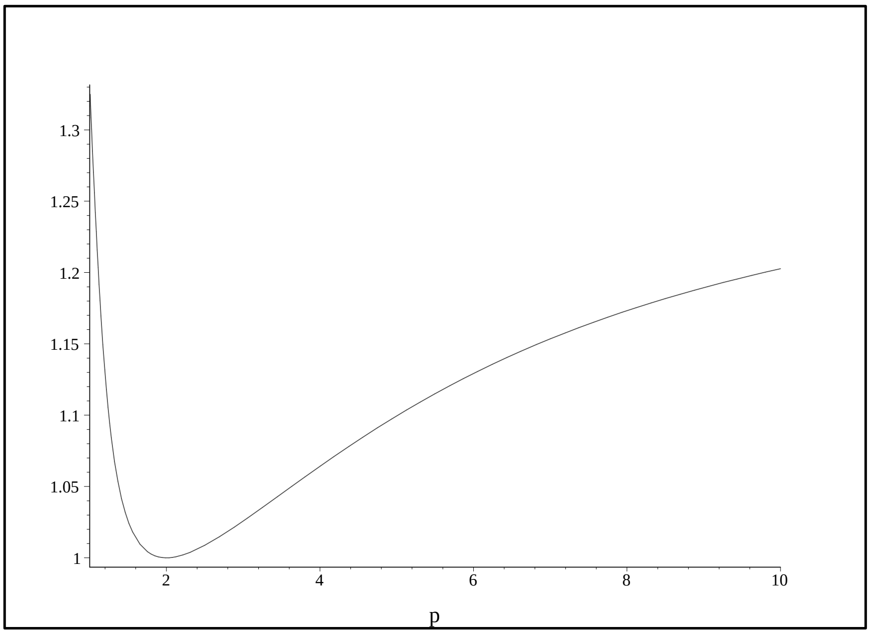

Figure 2.1: Values of $\Upsilon(p)$ for the group $\mathcal{S}_3$

Proof. To prove this theorem, choose $i_k < m_k$ the index for which the normalized coordinate function $\varphi_k^{i_k}$ of the finite group G_k satisfies

$$\left\|\varphi_k^{i_k}\right\|_p \left\|\varphi_k^{i_k}\right\|_q = \max_{s<m_k} \|\varphi_k^s\|_p \|\varphi_k^s\|_q .$$

Define

$$f_k(x) := \varphi_k^{i_k}(x) \left|\varphi_k^{i_k}(x)\right|^{q-2} \quad (x \in G_k).$$

Thus, $|f_k(x)|^p = |\varphi_k^{i_k}(x)|^q$ and $f_k(x)\overline{\varphi}_k^{i_k}(x) = |\varphi_k^{i_k}(x)|^q \in \mathbb{R}^+$ if $\varphi_k^{i_k}(x) \neq 0$. Hence both equalities hold in Hölder's inequality. For this reason

$$\left|\int_{G_k} f_k \overline{\varphi}_k^{i_k}\, d\mu_k\right| \left\|\varphi_k^{i_k}\right\|_p = \|f_k\|_p \left\|\varphi_k^{i_k}\right\|_q \left\|\varphi_k^{i_k}\right\|_p . \tag{3.1}$$

If k is an arbitrary positive integer and $n := \sum_{j=0}^{k-1} i_j M_j$, then define $F_k \in L^p(G)$ by

$$F_k(x) := \prod_{j=0}^{k-1} f_j(x_j) \qquad (x = (x_0, x_1, \dots) \in G).$$

Since $\|F_k\|_p = \prod_{j=0}^{k-1} \|f_j\|_p$, it follows from (3.1) that

$$\begin{aligned} \|S_{n+1}F_k - S_nF_k\|_p &= \left|\int_G F_k \overline{\psi}_n d\mu\right| \|\psi_n\|_p \\ &= \prod_{j=0}^{k-1} \left|\int_G f_j \overline{\varphi}_j^s d\mu_j\right| \|\varphi_j^s\|_p \geq \Psi_k(p)\|F_k\|_p. \end{aligned} \tag{3.2}$$

On the other hand, if S_n is of type (p,p), then there exists a $C_p > 0$ such that

$$\|S_{n+1}F_k - S_nF_k\|_p \leq \|S_{n+1}F_k\|_p + \|S_nF_k\|_p \leq 2C_p\|F_k\|_p$$

for each $k > 0$, which contradicts (3.2) because the sequence $\Psi_k(p)$ is not bounded. For this reason, the operators S_n are not uniformly of type (p,p). By a duality argument (see [6]) the operators S_n cannot be uniformly of type (q,q). This completes the proof of the theorem. $\square$

By Theorem 3.1 we obtain:

Theorem 3.2. *Let G be a bounded group and suppose that all the same finite groups appearing in the product of G have the same system φ at all of their occurrences. If the sequence Ψ is unbounded, then the operator S_n is not of type (p,p) for all $p \neq 2$.*

Proof. If the sequence $\Psi_k = \Psi_k(1)$ is not bounded, there exists a finite group F with system $\{\varphi^s : 0 \le s < |F|\}$ ($|F|$ is the order of the group F) which appears infinitely many times in the product of G and

$$\Upsilon(1) := \max_{s<|F|} \|\varphi^s\|_1 \|\varphi^s\|_\infty > 1.$$

Hence by Theorem 2.3 we have

$$\Upsilon(p) := \max_{s<|F|} \|\varphi^s\|_p \|\varphi^s\|_q > 1$$

for all $p \neq 2$. Denote by $l(k)$ the number of times the group F appears in the first k coordinates of G. Thus $l(k) \to \infty$ if $k \to \infty$ and

$$\Psi_k(p) \ge \prod_{i=1}^{l(k)} \Upsilon(p) \to \infty \qquad \text{if } k \to \infty,$$

for all $p \neq 2$. Consequently, the group G satisfies the conditions of Theorem 3.1 for all $1 < p < 2$. This completes the proof of the theorem. $\square$

Corollary 3.3. *If G is the complete product of $\mathcal{S}_3$ with the system φ appearing in Section 2, then the operator S_n is not of type (p,p) for all $p \neq 2$.*

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