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ON ČEBYŠEV TYPE INEQUALITIES INVOLVING FUNCTIONS WHOSE DERIVATIVES BELONG TO L_p SPACES

B.G. PACHPATTE
57 SHRI NIKETAN COLONY
NEAR ABHINAY TALKIES
AURANGABAD 431 001 (MAHARASHTRA)
INDIA
bgpachpatte@gmail.com

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ABSTRACT. In this note we establish new Čebyšev type integral inequalities involving functions whose derivatives belong to L_p spaces via certain integral identities.

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1. INTRODUCTION

One of the classical and important inequalities discovered by P.L.Čebyšev [1] is the following integral inequality (see also [10, p. 207]):

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions whose derivatives $f', g' \in L_\infty[a,b]$ and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.2}$$

which is called the Čebyšev functional, provided the integrals in (1.2) exist.

Because of fundamental importance of (1.1) in analysis and applications, many researchers have given considerable attention to it and a number of extensions, generalizations and variants have appeared in the literature, see [5], [6], [8] – [10] and the references given therein. The main purpose of the present note is to establish new inequalities similar to the inequality (1.1) involving functions whose derivatives belong to L_p spaces. The analysis used in the proofs is based on the integral identities proved in [3] and [2] and our results provide new estimates on these types of inequalities.

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2. STATEMENT OF RESULTS

In what follows $\mathbb{R}$ and $'$ denote the set of real numbers and the derivative of a function. Let $[a,b] \subset \mathbb{R}$, $a < b$; and as usual for any function $h \in L_p[a,b]$, $p > 1$ we define $\|h\|_p = \left(\int_a^b |h(t)|^p\, dt\right)^{\frac{1}{p}}$. We use the following notations to simplify the details of presentation. For suitable functions $f, g : [a,b] \to \mathbb{R}$ we set

$$F = \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right],$$
$$G = \frac{1}{3}\left[\frac{g(a)+g(b)}{2} + 2g\left(\frac{a+b}{2}\right)\right],$$

$$S(f,g) = FG - \frac{1}{b-a}\left[F\int_a^b g(x)\,dx + G\int_a^b f(x)\,dx\right] + \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

$$H(f,g) = \frac{1}{b-a}\int_a^b [Fg(x) + Gf(x)]\,dx - 2\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right).$$

Now, we state our main results as follows.

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a,b]$, $p > 1$. Then we have the inequalities*

$$|T(f,g)| \le \frac{1}{(b-a)^3}\|f'\|_p \|g'\|_p \int_a^b (B(x))^{\frac{2}{q}}\,dx, \tag{2.1}$$

$$|T(f,g)| \le \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\,\|f'\|_p + |f(x)|\,\|g'\|_p\right](B(x))^{\frac{1}{q}}\,dx, \tag{2.2}$$

where

$$B(x) = \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}, \tag{2.3}$$

for $x \in [a,b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The following variants of the inequalities in (2.1) and (2.2) hold.

Theorem 2.2. *Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a,b]$, $p > 1$. Then we have the inequalities*

$$|S(f,g)| \le \frac{1}{(b-a)^2} M^{\frac{2}{q}} \|f'\|_p \|g'\|_p, \tag{2.4}$$

$$|H(f,g)| \le \frac{1}{(b-a)^2} M^{\frac{1}{q}} \int_a^b \left[|g(x)|\,\|f'\|_p + |f(x)|\,\|g'\|_p\right] dx, \tag{2.5}$$

where

$$M=\frac{\left(2^{q+1}+1\right)(b-a)^{q+1}}{3\,(q+1)\,6^{q}}, \tag{2.6}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

3. PROOF OF THEOREM 2.1

From the hypotheses we have the following identities (see [3, 7]):

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b k(x,t)\,f'(t)\,dt, \tag{3.1}$$

$$g(x)-\frac{1}{b-a}\int_a^b g(t)\,dt=\frac{1}{b-a}\int_a^b k(x,t)\,g'(t)\,dt, \tag{3.2}$$

for $x\in[a,b]$, where

$$k(x,t)=\begin{cases} t-a & if\ t\in[a,x] \\ t-b & if\ t\in(x,b] \end{cases}. \tag{3.3}$$

Multiplying the left sides and right sides of (3.1) and (3.2) we have

$$\begin{aligned} f(x)\,g(x)-f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)-g(x)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)& \\ +\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)& \\ =\frac{1}{(b-a)^2}\left(\int_a^b k(x,t)\,f'(t)\,dt\right)\left(\int_a^b k(x,t)\,g'(t)\,dt\right)&. \end{aligned} \tag{3.4}$$

Integrating both sides of (3.4) with respect to x over $[a,b]$ and dividing both sides of the resulting identity by $(b-a)$ we get

$$T(f,g)=\frac{1}{(b-a)^3}\int_a^b\left(\int_a^b k(x,t)\,f'(t)\,dt\right)\left(\int_a^b k(x,t)\,g'(t)\,dt\right)dx. \tag{3.5}$$

From (3.5) and using the properties of modulus and Hölder's integral inequality we have

$$\begin{aligned} |T(f,g)| &\le \frac{1}{(b-a)^3}\int_a^b\left(\int_a^b |k(x,t)|\,|f'(t)|\,dt\right)\left(\int_a^b |k(x,t)|\,|g'(t)|\,dt\right)dx \\ &\le \frac{1}{(b-a)^3}\int_a^b\left(\left\{\int_a^b |k(x,t)|^q\right\}^{\frac{1}{q}}\left\{\int_a^b |f'(t)|^p\,dt\right\}^{\frac{1}{p}}\right) \\ &\qquad\times\left(\left\{\int_a^b |k(x,t)|^q\,dt\right\}^{\frac{1}{q}}\left\{\int_a^b |g'(t)|^p\,dt\right\}^{\frac{1}{p}}\right)dx \\ &= \frac{1}{(b-a)^3}\,\|f'\|_p\,\|g'\|_p\int_a^b\left(\left\{\int_a^b |k(x,t)|^q\right\}^{\frac{1}{q}}\right)^2 dx. \end{aligned} \tag{3.6}$$

A simple calculation shows that (see [4])

$$\begin{aligned}\int_a^b |k(x,t)|^q\,dt &= \int_a^x |t-a|^q\,dt + \int_x^b |t-b|^q\,dt\\ &= \int_a^x (t-a)^q\,dt + \int_x^b (b-t)^q\,dt\\ &= \frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1} = B(x)\,. \end{aligned} \tag{3.7}$$

Using (3.7) in (3.6) we get (2.1).

Multiplying both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we get

$$\begin{aligned} 2f(x)\,g(x) - \left[g(x)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) + f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\right]\\ = g(x)\left(\frac{1}{b-a}\int_a^b k(x,t)\,f'(t)\,dt\right) + f(x)\left(\frac{1}{b-a}\int_a^b k(x,t)\,g'(t)\,dt\right). \end{aligned} \tag{3.8}$$

Integrating both sides of (3.8) with respect to x over $[a,b]$ and rewriting we obtain

$$T(f,g) = \frac{1}{2(b-a)^2}\int_a^b \left[g(x)\int_a^b k(x,t)\,f'(t)\,dt + f(x)\int_a^b k(x,t)\,g'(t)\,dt\right]dx. \tag{3.9}$$

From (3.9) and using the properties of modulus, Hölder's integral inequality and (3.7) we have

$$\begin{aligned}|T(f,g)| &\le \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\int_a^b |k(x,t)|\,|f'(t)|\,dt + |f(x)|\int_a^b |k(x,t)|\,|g'(t)|\,dt\right]dx\\ &\le \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\left\{\int_a^b |k(x,t)|^q\,dt\right\}^{\frac{1}{q}}\left\{\int_a^b |f'(t)|^p\,dt\right\}^{\frac{1}{p}}\right.\\ &\qquad \left. + |f(x)|\left\{\int_a^b |k(x,t)|^q\,dt\right\}^{\frac{1}{q}}\left\{\int_a^b |g'(t)|^p\,dt\right\}^{\frac{1}{p}}\right]dx\\ &= \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\,\|f'\|_p + |f(x)|\,\|g'\|_p\right]\left\{\int_a^b |k(x,t)|^q\,dt\right\}^{\frac{1}{q}}dx\\ &= \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\,\|f'\|_p + |f(x)|\,\|g'\|_p\right](B(x))^{\frac{1}{q}}\,dx. \end{aligned}$$

This is the required inequality in (2.2). The proof is complete.

4. Proof of Theorem 2.2

From the hypotheses we have the following identities (see [2]):

$$F - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b m(x)\,f'(x)\,dx, \tag{4.1}$$

$$G - \frac{1}{b-a}\int_a^b g(x)\,dx = \frac{1}{b-a}\int_a^b m(x)\,g'(x)\,dx, \tag{4.2}$$

where

$$m(x)=\begin{cases} x-\frac{5a+b}{6} & if\ x\in\left[a,\frac{a+b}{2}\right) \\ x-\frac{a+5b}{6} & if\ x\in\left[\frac{a+b}{2},b\right] \end{cases}. \tag{4.3}$$

Multiplying the left sides and right sides of (4.1) and (4.2) we get

$$S(f,g)=\frac{1}{(b-a)^2}\left(\int_a^b m(x)\,f'(x)\,dx\right)\left(\int_a^b m(x)\,g'(x)\,dx\right). \tag{4.4}$$

From (4.4) and using the properties of modulus and Hölder's integral inequality, we have

$$\begin{aligned}
|S(f,g)| &\le \frac{1}{(b-a)^2}\left(\int_a^b |m(x)|\,|f'(x)|\,dx\right)\left(\int_a^b |m(x)|\,|g'(x)|\,dx\right) \\
&\le \frac{1}{(b-a)^2}\left(\left\{\int_a^b |m(x)|^q\,dx\right\}^{\frac{1}{q}}\left\{\int_a^b |f'(x)|^p\,dx\right\}^{\frac{1}{p}}\right) \\
&\qquad \times\left(\left\{\int_a^b |m(x)|^q\,dx\right\}^{\frac{1}{q}}\left\{\int_a^b |g'(x)|^p\,dx\right\}^{\frac{1}{p}}\right) \\
&= \frac{1}{(b-a)^2}\left(\left\{\int_a^b |m(x)|^q\,dx\right\}^{\frac{1}{q}}\right)^2 \|f'\|_p\,\|g'\|_p\,. \qquad (4.5)
\end{aligned}$$

A simple computation gives (see [2])

$$\begin{aligned}
\int_a^b |m(x)|^q\,dx &= \int_a^{\frac{a+b}{2}}\left|x-\frac{5a+b}{6}\right|^q dx+\int_{\frac{a+b}{2}}^b\left|x-\frac{a+5b}{6}\right|^q dx \\
&= \int_a^{\frac{5a+b}{6}}\left(\frac{5a+b}{6}-x\right)^q dx+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left(x-\frac{5a+b}{6}\right)^q dx \\
&\qquad +\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}\left(\frac{a+5b}{6}-x\right)^q dx+\int_{\frac{a+5b}{6}}^{b}\left(x-\frac{a+5b}{6}\right)^q dx \\
&= \frac{1}{q+1}\left[\left(\frac{5a+b}{6}-a\right)^{q+1}+\left(\frac{a+b}{2}-\frac{5a+b}{6}\right)^{q+1}\right. \\
&\qquad \left.+\left(\frac{a+5b}{6}-\frac{a+b}{2}\right)^{q+1}+\left(b-\frac{a+5b}{6}\right)^{q+1}\right] \\
&= \frac{(2^{q+1}+1)(b-a)^{q+1}}{3(q+1)6^q}=M. \qquad (4.6)
\end{aligned}$$

Using (4.6) in (4.5) we get the required inequality in (2.4).

Multiplying both sides of (4.1) and (4.2) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we get

$$\begin{aligned}
Fg(x)+Gf(x)&-\left[g(x)\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)+f(x)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right] \qquad (4.7) \\
&= g(x)\left(\frac{1}{b-a}\int_a^b m(x)\,f'(x)\,dx\right)+f(x)\left(\frac{1}{b-a}\int_a^b m(x)\,g'(x)\,dx\right).
\end{aligned}$$

Integrating both sides of (4.7) with respect to x over $[a,b]$ and dividing both sides of the resulting identity by $(b-a)$ we get

$$H(f,g)=\frac{1}{(b-a)^2}\int_a^b\left[g(x)\int_a^b m(x)f'(x)\,dx+f(x)\int_a^b m(x)g'(x)\,dx\right]dx. \tag{4.8}$$

From (4.8) and using the properties of modulus, Hölder's integral inequality and (4.6) we have

$$\begin{aligned}|H(f,g)|&\le\frac{1}{(b-a)^2}\int_a^b\left[|g(x)|\int_a^b|m(x)|\,|f'(x)|\,dx+|f(x)|\int_a^b|m(x)|\,|g'(x)|\,dx\right]dx\\&\le\frac{1}{(b-a)^2}\int_a^b\left[|g(x)|\left\{\int_a^b|m(x)|^q\,dx\right\}^{\frac{1}{q}}\left\{\int_a^b|f'(x)|^p\,dx\right\}^{\frac{1}{p}}\right.\\&\qquad\left.+|f(x)|\left\{\int_a^b|m(x)|^q\,dx\right\}^{\frac{1}{q}}\left\{\int_a^b|g'(x)|^p\,dx\right\}^{\frac{1}{p}}\right]dx\\&=\frac{1}{(b-a)^2}\int_a^b\left[|g(x)|\,\|f'\|_p+|f(x)|\,\|g'\|_p\right]\left(\int_a^b|m(x)|^q\,dx\right)^{\frac{1}{q}}\\&=\frac{1}{(b-a)^2}M^{\frac{1}{q}}\int_a^b\left[|g(x)|\,\|f'\|_p+|f(x)|\,\|g'\|_p\right]dx.\end{aligned}$$

This is the desired inequality in (2.5). The proof is complete.

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