ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING THE AL-OBOUDI DIFFERENTIAL OPERATOR

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Abstract:	In this paper we introduce a new subclass of normalized analytic functions in the open unit disc which is defined by the Al-Oboudi differential operator. A coefficient inequality, extreme points and integral mean inequalities of a differential		Page 1	0
	operator for this class are given. We investigate various subordination results		Go E	3ad
	for the subclass of analytic functions and obtain sufficient conditions for univa- lent close-to-starlikeness. We also discuss the boundedness properties associated with partial sums of functions in the class. Several interesting connections with		Full S	cre
	the class of close-to-starlike and close-to-convex functions are also pointed out.		Clo	se
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1. Introduction and Preliminaries

Let A denote the class of normalized functions f defined by

(1.1)
$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For $f \in A$, [1] has introduced the following differential operator.

$$D^0 f(z) = f(z)$$

(1.3)
$$D^1 f(z) = (1 - \delta) f(z) + \delta z f'(z) = D_{\delta} f(z), \qquad \delta \ge 0$$

(1.4)
$$D^n f(z) = D_{\delta}(D^{n-1}f(z)), \qquad (n \in \mathbb{N}).$$

For f(z) given by (1.1), we notice from (1.3) and (1.4) that

(1.5)
$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

For $\delta = 1$ we obtain the Sălăgean operator [11].

Definition 1.1. A function f in A is said to be starlike of order α $(0 \le \alpha < 1)$ in U, that is, $f \in S^*(\alpha)$, if and only if

(1.6)
$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U)$$



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Definition 1.2. A function f in A is said to be convex of order α $(0 \le \alpha < 1)$ in U, that is, $f \in K(\alpha)$, if and only if

(1.7)
$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U).$$

Definition 1.3. A function f in A is said to be close-to-convex in U, of order α , that is, $f \in C(\alpha)$, if and only if

(1.8)
$$\operatorname{Re}\{f'(z)\} > \alpha \quad (z \in U)$$

Definition 1.4. A function f in A is said to be close-to-starlike of order α ($0 \le \alpha < 1$) in U, that is, $f \in CS^*(\alpha)$, if and only if

(1.9)
$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \alpha \qquad (z \in U \setminus \{0\})$$

We note that the classes S, $S^*(0) = S^*$, K(0) = K, C(0) = C, $CS^*(0) = CS^*$ are the well known classes of univalent, starlike, convex, close-to-convex and close-to-starlike functions in U, respectively. It is also clear that

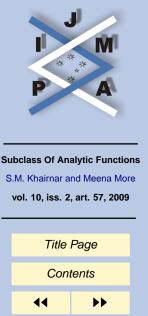
- (i) $f \in K(\alpha)$ if and only if $zf' \in S^*(\alpha)$;
- (ii) $K(\alpha) \subset S^*(\alpha) \subset C(\alpha) \subset S$.

Definition 1.5. For two functions f and g analytic in U, we say that the function f(z) is subordinate to g(z) in U, and write

$$(1.10) f(z) \prec g(z) (z \in U)$$

if there exists a Schwarz function w(z), analytic in U with w(0) = 0 and |w(z)| < 1 such that

(1.11) $f(z) = g(w(z)) \quad (z \in U).$



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In particular, if the function g is univalent in U, the above subordination is equivalent to

(1.12)
$$f(0) = g(0), \quad f(U) \subset g(U).$$

Littlewood [7] in 1925 has proved the following subordination theorem which we state as a lemma.

Lemma 1.6. Let f and g be analytic in the unit disc, and suppose $g \prec f$. Then for 0 ,

(1.13)
$$\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \le \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \qquad (0 \le r < 1, p > 0).$$

Strict inequality holds for 0 < r < 1 unless f is constant or $w(z) = \alpha z$, $|\alpha| = 1$.

Definition 1.7. Let $n \in \mathbb{N} \cup \{0\}$ and $\lambda \ge 0$. Let $D_{\lambda}^{n}f$ denote the operator defined by $D_{\lambda}^{n}: A \to A$ such that

(1.14)
$$D_{\lambda}^{n}f(z) = (1-\lambda)S^{n}f(z) + \lambda R^{n}f(z) \quad z \in U$$

where $S^n f$ is the Sălăgean differential operator and $R^n f$ is the Ruscheweyh differential operator defined by $R^n : A \to A$ such that

$$R^{0}f(z) = f(z), R^{1}f(z) = zf'(z)$$

with recurrence relation given by

(1.15)
$$(n+1)R^{n+1}f(z) = z[R^n f(z)]' + nR^n f(z) \ (z \in U).$$

For $f \in A$ *given by* (1.1)

(1.16)
$$R^{n}f(z) = z + \sum_{k=2}^{\infty} {}^{n}C_{n+k-1}a_{k}z^{k} \qquad (z \in U).$$



Notice that D_{λ}^{n} is a linear operator and for $f \in A$ defined by (1.1), we have

(1.17)
$$D_{\lambda}^{n}f(z) = z + \sum_{k=2}^{\infty} [(1-\lambda)k^{n} + \lambda^{-n}C_{n+k-1}]a_{k}z^{k}.$$

It is observed that for n = 0,

$$D^{0}_{\lambda}f(z) = (1-\lambda)S^{0}f(z) + \lambda R^{0}f(z) = f(z) = S^{0}f(z) = R^{0}f(z),$$

and for n = 1

$$D^{1}_{\lambda}f(z) = (1-\lambda)S^{1}f(z) + \lambda R^{1}f(z) = zf'(z) = S^{1}f(z) = R^{1}f(z).$$

Definition 1.8. Let $K(\gamma, \mu, m, \beta)$ denote the subclass of A consisting of functions f which satisfy the inequality

(1.18)
$$\left|\frac{1}{\gamma}\left((1-\mu)\frac{D^m f}{z} + \mu(D^m f)' - 1\right)\right| < \beta$$

where $z \in U, \gamma \in \mathbb{C} \setminus \{0\}, 0 < \beta \leq 1, 0 \leq \mu \leq 1, m \in \mathbb{N}_0$ and D^m is as defined in (1.5).

Remark 1. For $\gamma = 1$, $\mu = 1$, m = 0, we obtain the class of close-to-convex functions of order $(1 - \beta)$. For the values $\gamma = 1$, $\mu = 0$, m = 0, we obtain the class of close-to-starlike functions of order $(1 - \beta)$.

Let

$$T(\eta, f) = (1 - \eta) \frac{f(z)}{z} + \eta \ f'(z) \quad (z \in U \setminus \{0\})$$

for η real and $f \in A$. Define

$$T_{\eta} := \{ f \in A : \operatorname{Re}\{T(\eta, f)\} > 0 \}.$$

We note that T_{η} can be derived from the class $K(\gamma, \mu, m, \beta)$ by replacing μ by η and $D^m f$ by f.



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2. Coefficient Inequalities, Growth and Distortion Theorems

Here we first give a sufficient condition for $f \in A$ to belong to the class $K(\gamma, \mu, m, \beta)$.

Theorem 2.1. Let $f(z) \in A$ satisfy

(2.1)
$$\sum_{k=2}^{\infty} (1+(k-1)\mu)(1+(k-1)\delta)^m |a_k| \le |\gamma|\beta,$$

where $\gamma \in \mathbb{C} \setminus \{0\}, 0 < \beta \leq 1, 0 \leq \mu \leq 1, m \in \mathbb{N}_0, \delta \geq 0$. Then $f(z) \in K(\gamma, \mu, m, \beta)$.

Proof. Suppose that (2.1) is true for γ ($\gamma \in \mathbb{C} \setminus \{0\}$), β ($0 < \beta \le 1$), μ ($0 \le \mu \le 1$), $m \in \mathbb{N}_0$, and δ ($\delta \ge 0$) for $f(z) \in A$. Using (1.5) for |z| = 1, we have

$$\left| (1-\mu)\frac{D^m f}{z} + \mu (D^m f)' - 1 \right| \le \sum_{k=2}^{\infty} (1+(k-1)\mu)(1+(k-1)\delta)^m | a_k \le |\gamma| \beta.$$

Thus by Definition 1.8 $f(z) \in K(\gamma, \mu, m, \beta)$.

Notice that the function given by

(2.2)
$$f(z) = z + \sum_{k=2}^{\infty} \frac{|\gamma|\beta}{(1+(k-1)\mu)(1+(k-1)\delta)^m} z^k$$

belongs to the class $K(\gamma, \mu, m, \beta)$ and plays the role of extremal function for the result (2.1).



in pure and applied mathematics We denote by $\tilde{K}(\gamma,\mu,m,\beta)\subseteq K(\gamma,\mu,m,\beta)$ the functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

where the Taylor-Maclaurin coefficients satisfy inequality (2.1).

Next we state the growth and distortion theorems for the class $K(\gamma, \mu, m, \beta)$. The results follow easily on applying Theorem 2.1, therefore, we omit the proof.

Theorem 2.2. Let the function f(z) defined by (1.1) be in the class $\tilde{K}(\gamma, \mu, m, \beta)$. Then

(2.3)
$$|z| - \frac{|\gamma|\beta}{(1+\mu)(1+\delta)^m} |z|^2 \le |f(z)| \le |z| + \frac{|\gamma|\beta}{(1+\mu)(1+\delta)^m} |z|^2.$$

The equality in (2.3) is attained for the function f(z) *given by*

(2.4)
$$f(z) = z + \frac{|\gamma|\beta}{(1+\mu)(1+\delta)^m} z^2.$$

Theorem 2.3. Let the function f(z) defined by (1.1) be in the class $\tilde{K}(\gamma, \mu, m, \beta)$. Then

(2.5)
$$1 - \frac{2|\gamma|\beta}{(1+\mu)(1+\delta)^m}|z| \le |f'(z)| \le 1 + \frac{2|\gamma|\beta}{(1+\mu)(1+\delta)^m}|z|.$$

The equality in (2.5) is attained for the function f(z) given by (2.4).

In view of Remark 1, Theorem 2.2 and Theorem 2.3 would yield the corresponding distortion properties for the class of close-to-convex and close-to-starlike functions.



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3. Extreme Points

Now we determine the extreme points of the class $\tilde{K}(\gamma, \mu, m, \beta)$. *Remark* 2. For $\gamma \in \mathbb{C} \setminus \{0\}, 0 < \beta \leq 1, 0 \leq \mu \leq 1, m \in \mathbb{N}_0$ and $\delta \geq 0$ the following functions are in the class $\tilde{K}(\gamma, \mu, m, \beta)$

$$f_1(z) = z + \frac{\beta |\gamma|}{(1+\mu)(1+\delta)^m} z^2 \qquad (z \in U);$$

$$f_2(z) = z + \frac{\beta |\gamma|}{(1+2\mu)(1+2\delta)^m} z^3 \qquad (z \in U);$$

$$f_3(z) = z + \frac{1}{(1+\mu)(1+\delta)^m} z^2 + \frac{(|\gamma|\beta - 1)}{(1+2\mu)(1+2\delta)^m} z^3 \qquad (z \in U).$$

Theorem 3.1. Let $f_1(z) = z$ and

(3.1)
$$f_k(z) = z + \frac{|\gamma|\beta}{(1 + (k-1)\mu)(1 + (k-1)\delta)^m} z^k \qquad (k \ge 2)$$

Then $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$, if and only if it can be expressed in the form

(3.2)
$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

where $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z + \sum_{k=2}^{\infty} \lambda_k \frac{|\gamma|\beta}{(1 + (k-1)\mu)(1 + (k-1)\delta)^m} z^k.$$



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Then

$$\sum_{k=2}^{\infty} (1+(k-1)\mu)(1+(k-1)\delta)^m \frac{|\gamma|\beta}{(1+(k-1)\mu)(1+(k-1)\delta)^m} \lambda_k$$
$$= |\gamma|\beta \sum_{k=2}^{\infty} \lambda_k$$
$$\leq |\gamma|\beta (1-\lambda_1)$$
$$\leq |\gamma|\beta.$$

Thus, in view of Theorem 2.1, $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$. Conversely, suppose that $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$. Setting

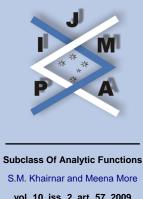
$$\lambda_{k} = \frac{(1 + (k - 1)\mu)(1 + (k - 1)\delta)^{m}}{|\gamma|\beta} a_{k} \text{ and } \lambda_{1} = 1 - \sum_{k=2}^{\infty} \lambda_{k},$$

we obtain

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

Corollary 3.2. The extreme points of $\tilde{K}(\gamma, \mu, m, \beta)$ are the functions $f_1(z) = z$ and

$$f_k(z) = z + \frac{|\gamma|\beta}{(1 + (k-1)\mu)(1 + (k-1)\delta)^m} z^k \qquad (k = 2, 3, \dots).$$



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4. Integral Mean Inequalities for a Differential Operator

Theorem 4.1. Let $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$ and suppose that

(4.1)
$$\sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{-n}C_{n+k-1}]|a_k| \le \frac{|\gamma|\beta[(1-\lambda)j^n + \lambda^{-n}C_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^m}.$$

Also, let the function

(4.2)
$$f_j(z) = z + \frac{|\gamma|\beta}{(1+\mu(j-1))(1+\delta(j-1))^m} z^j \qquad (j \ge 2).$$

If there exists an analytic function w(z) given by

$$w(z)^{j-1} = \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n+\lambda^{-n}C_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{-n}C_{n+k-1}]a_k z^{k-1}]$$

then for $z = re^{i\theta}$ with 0 < r < 1,

$$\int_0^{2\pi} |D_\lambda^n f(z)|^p d\theta \le \int_0^{2\pi} |D_\lambda^n f_j(z)|^p d\theta \qquad (0 \le \lambda \le 1, p > 0)$$

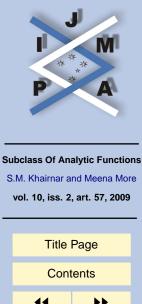
for the differential operator defined in (1.17).

Proof. By Definition 1.7 and by virtue of relation (1.17), we have

(4.3)
$$D_{\lambda}^{n}f(z) = z + \sum_{k=2}^{\infty} [(1-\lambda)k^{n} + \lambda^{-n}C_{n+k-1}]a_{k}z^{k}.$$

Likewise,

(4.4)
$$D_{\lambda}^{n} f_{j}(z) = z + \frac{|\gamma|\beta[(1-\lambda)j^{n} + \lambda^{-n}C_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^{m}} z^{j}.$$





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For $z = re^{i\theta}$, 0 < r < 1, we need to show that

$$(4.5) \quad \int_{0}^{2\pi} \left| 1 + \sum_{k=2}^{\infty} \left[(1-\lambda)k^{n} + \lambda^{-n}C_{n+k-1} \right] a_{k} z^{k-1} \right|^{p} d\theta \\ \leq \int_{0}^{2\pi} \left| 1 + \frac{|\gamma|\beta[(1-\lambda)j^{n} + \lambda^{-n}C_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^{m}} z^{j-1} \right| d\theta \quad (p>0).$$

By applying Littlewood's subordination theorem, it would be sufficient to show that

(4.6)
$$1 + \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{-n}C_{n+k-1}]a_k z^{k-1} \prec 1 + \frac{|\gamma|\beta[(1-\lambda)j^n + \lambda^{-n}C_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^m} z^{j-1}.$$

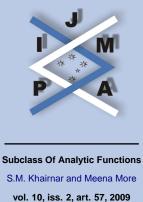
Set

$$1 + \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^n C_{n+k-1}]a_k z^{k-1} = 1 + \frac{|\gamma|\beta[(1-\lambda)j^n + \lambda^n C_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^m}w(z)^{j-1}.$$

We note that

$$(4.7) \quad (w(z))^{j-1} = \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n+\lambda^{-n}C_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{-n}C_{n+k-1}]a_k z^{k-1},$$

and w(0) = 0. Moreover, we prove that the analytic function w(z) satisfies |w(z)| < |w(z)| < 0





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 $1, z \in U$

$$\begin{split} |w(z)|^{j-1} &\leq \left| \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n+\lambda^{-n}C_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n+\lambda^{-n}C_{n+k-1}]a_k z^{k-1} \right| \\ &\leq \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n+\lambda^{-n}C_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n+\lambda^{-n}C_{n+k-1}]|a_k||z|^{k-1} \\ &\leq |z| \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n+\lambda^{-n}C_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n+\lambda^{-n}C_{n+k-1}]|a_k| \\ &\leq |z| < 1 \text{ by hypothesis (4.1).} \end{split}$$

This completes the proof of Theorem 4.1.

As a particular case of Theorem 4.1, we can derive the following result when n = 0. That is, for $D_{\lambda}^{0} f(z) = f(z)$.

Corollary 4.2. Let $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$ be given by (1.1), then for $z = re^{i\theta}$ (0 < r < 1)

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \le \int_0^{2\pi} |f_j(re^{i\theta})|^p d\theta \qquad (p>0),$$

where

$$f_j(z) = z + \frac{|\gamma|\beta}{(1+\mu(j-1))(1+\delta(j-1))^m} z^j \qquad (j \ge 2).$$

We conclude this section by observing that by specializing the parameters in Theorem 4.1, several integral mean inequalities can be deduced for $S^n f(z)$, $R^n f(z)$, the class of close-to-convex functions and the class of close-to-starlike functions as mentioned in Remark 1.



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5. Subordination Results for the Class $T(\eta, f)$

In proving the main subordination results we need the following lemma due to [8, p. 132].

Lemma 5.1. Let q be univalent in U and θ and ϕ be analytic in a domain D containing q(U), with $\phi(w) \neq 0$, when $w \in q(U)$. Set

$$Q(z) = zq'(z) \cdot \phi[q(z)], \qquad h(z) = \theta[q(z)] + Q(z)$$

and suppose that either:

(i) Q is starlike or

(ii) h is convex.

In addition, assume that

iii)
$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + z\frac{Q'(z)}{Q(z)}\right) > 0.$$

If P is analytic in U, with $P(0) = q(0), P(U) \subset D$ and
 $\theta[P(z)] = zP'(z) \cdot \phi[P(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)] = h(z)$

then $P \prec q$, and q is the best dominant.

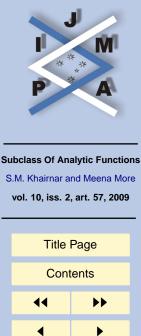
Lemma 5.2. Let $q \in H = \{f \in A : f(z) = 1 + b_1 z + b_2 z^2 + \cdots\}$ be univalent and satisfy the following conditions: q(z) is convex and

(5.1)
$$\operatorname{Re}\left\{\left(\frac{1}{\eta}+1\right)+\frac{zq''(z)}{q'(z)}\right\}>0$$

for $\eta \neq 0$ and all $z \in U$. For $P \in H$ in U if

(5.2)
$$P(z) + \eta z P'(z) \prec q(z) + \eta z q'(z),$$

then $P \prec q$ and q is the best dominant.



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Proof. For $\eta \neq 0$ a real number, we define θ and ϕ by

(5.3)
$$\theta(w) := w, \quad \phi(w) := \eta, \quad D = \{w : w \neq 0\}$$

in Lemma 5.1. Then the functions

$$Q(z) = zq'(z)\phi(q(z)) = \eta zq'(z)$$

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \eta zq'(z)$$

Using (5.1), we notice that Q(z) is starlike in U and $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for all $z \in U$ and $\eta \neq 0$.

Thus the hypotheses of Lemma 5.1 are satisfied. Therefore, from (5.2) it follows that $P \prec q$ and q is the best dominant.

Theorem 5.3. Let $q \in H$ be univalent and satisfy the condition (5.1) in Lemma 5.2. For $D^m f$ if

(5.4)
$$T(\eta, D^m f) \prec q(z) + \eta z q'(z),$$

then $\frac{D^m f(z)}{z} \prec q(z)$ and q(z) is the best dominant.

Proof. Substituting $P(z) = \frac{D^m f(z)}{z}$, where P(0) = 1, we have $P(z) + \eta z P'(z) = T(\eta, D^m f)$.

Thus using (5.4) and Lemma 5.2, we get the required result.

Corollary 5.4. Let $q \in H$ be univalent and satisfy the conditions (5.1) in Lemma 5.2. For $f \in A$, if $T(\eta, f) \prec q(z) + \eta z q'(z)$, then $\frac{f(z)}{z} \prec q(z)$ and q is the best dominant.

Proof. By substituting m = 0 in Theorem 5.3 we obtain Corollary 5.4.



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 \square

Corollary 5.5. Let $q \in H$ be univalent and convex for all $z \in U$. For $P \in H$ in U if

(5.5)
$$P(z) + zP'(z) \prec q(z) + zq'(z),$$

then $P \prec q$, and q is the best dominant.

Proof. Take $\eta = 1$ in Lemma 5.2.

Corollary 5.6. Let $q \in S$ be convex. For $f \in A$ if

 $f'(z) \prec q(z) + zq'(z),$

then $\frac{f(z)}{z} \prec q(z)$ and q is the best dominant. Proof. Take $\eta = 1$ in Corollary 5.4.

Corollary 5.7. Let $q \in S$ satisfy

$$T(\eta, f) \prec \frac{1 + 2(\eta - \alpha - \eta \alpha)z - (1 - 2\alpha)z^2}{(1 - z)^2}$$

where $f \in A$. Then $\frac{f(z)}{z} \in CS^*(\alpha)$ and q is the best dominant.

Proof. Take $q(z) = \frac{1+(1-2\alpha)z}{1-z}$ in Corollary 5.4. Then it follows that

$$\frac{f(z)}{z} \prec \frac{1 + (1 - 2\alpha)z}{1 - z},$$

which is equivalent to $\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \alpha$. Therefore

$$\frac{f(z)}{z} \in CS^*(\alpha).$$



Corollary 5.8. Let $q \in S$ satisfy

$$f'(z) \prec \frac{1 + 2(1 - 2\alpha)z - (1 - 2\alpha)z^2}{(1 - z)^2},$$

where $f \in A$. Then $\frac{f(z)}{z} \in CS^*(\alpha)$ and q is the best dominant.

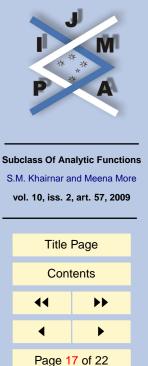
Proof. Substituting $\eta = 1$ in Corollary 5.7, we get the desired result.

Corollary 5.9. Let $q \in S$ satisfy

$$f'(z) \prec \frac{1+2z-z^2}{(1-z)^2},$$

where $f \in A$. Then $f(z) \in CS^*$ and q is the best dominant.

Proof. Take $\alpha = 0$ in Corollary 5.8.



 \square

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6. Partial Sums

In line with the earlier works of Silverman [12] and Silvia [13] on the partial sums of analytic functions, we investigate in this section the partial sums of functions in the class $K(\gamma, \mu, m, \beta)$. We obtain sharp lower bounds for the ratios of the real part of f(z) to $f_N(z)$ and f'(z) to $f'_N(z)$.

Theorem 6.1. Let f(z) of the form (1.1) belong to $K(\gamma, \mu, m, \beta)$ and $h(N+1, \gamma, \mu, m, \beta) \ge 1$. Then

(6.1)
$$\operatorname{Re}\left(\frac{f(z)}{f_N(z)}\right) \ge 1 - \frac{1}{h(N+1,\gamma,\mu,m,\beta)}$$

and

(6.2)
$$\operatorname{Re}\left(\frac{f_N(z)}{f(z)}\right) \ge \frac{h(N+1,\gamma,\mu,m,\beta)}{h(N+1,\gamma,\mu,m,\beta)+1},$$

where

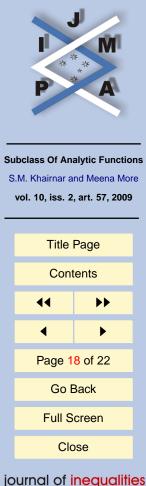
(6.3)
$$h(k,\gamma,\mu,m,\beta) = \frac{(1+(k-1)\mu)(1+(k-1)\delta)^m}{|\gamma|\beta}.$$

The result is sharp for every N, with extremal functions given by

(6.4)
$$f(z) = z + \frac{1}{h(N+1,\gamma,\mu,m,\beta)} z^{N+1} \qquad (N \in \mathbb{N} \setminus \{1\}).$$

Proof. To prove (6.1), it is sufficient to show that

$$h(N+1,\gamma,\mu,m,\beta)\left[\frac{f(z)}{f_N(z)} - \left(1 - \frac{1}{h(N+1,\gamma,\mu,m,\beta)}\right)\right] \prec \frac{1+z}{1-z} \quad (z \in U).$$



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By the subordination property (1.11), we can write

$$h(N+1,\gamma,\mu,m,\beta) \left[\frac{1 + \sum_{k=2}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{N} a_k z^{k-1}} - \left(1 - \frac{1}{h(N+1,\gamma,\mu,m,\beta)} \right) \right] = \frac{1 + w(z)}{1 - w(z)}$$

Notice that w(0) = 0 and

$$|w(z)| \le \frac{h(N+1,\gamma,\mu,m,\beta)\sum_{k=N+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{N}|a_k| - h(N+1,\gamma,\mu,m,\beta)\sum_{k=N+1}^{\infty}|a_k|}$$

|w(z)| < 1 if and only if

$$\sum_{k=2}^{N} |a_k| + h(N+1, \gamma, \mu, m, \beta) \sum_{k=N+1}^{\infty} |a_k| \le 1.$$

In view of (2.1), we can equivalently show that

$$\sum_{k=2}^{N} (h(k,\gamma,\mu,m,\beta) - 1)|a_k| + \sum_{k=N+1}^{\infty} ((h(k,\gamma,\mu,m,\beta) - h(N+1,\gamma,\mu,m,\beta))|a_k| \ge 0.$$

The above inequality holds because $h(k, \gamma, \mu, m, \beta)$ is a non-decreasing sequence. This completes the proof of (6.1). Finally, it is observed that equality in (6.1) is attained for the function given by (6.4) when $z = re^{2\pi i/N}$ as $r \to 1^-$. The proof of (6.2) is similar to that of (6.1), and is hence omitted.



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Using a similar method, we can prove the following theorem.

Theorem 6.2. Let f(z) of the form (1.1) belong to $K(\gamma, \mu, m, \beta)$, and $h(N + 1, \gamma, \mu, m, \beta) \ge N + 1$. Then

$$\operatorname{Re}\left(\frac{f'(z)}{f'_{N}(z)}\right) \ge 1 - \frac{N+1}{h(N+1,\gamma,\mu,m,\beta)}$$

and

$$\operatorname{Re}\left(\frac{f_N'(z)}{f'(z)}\right) \ge \frac{h(N+1,\gamma,\mu,m,\beta)}{N+1+h(N+1,\gamma,\mu,m,\beta)}$$

where $h(N + 1, \gamma, \mu, m, \beta)$ is given by (6.3). The result is sharp for every N, with extremal functions given by (6.4).



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