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CONVEXITY OF WEIGHTED STOLARSKY MEANS

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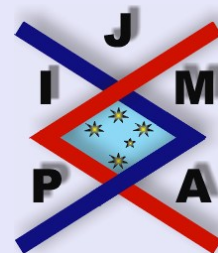


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Abstract

We investigate monotonicity and logarithmic convexity properties of one-parameter family of means

$$F_h(r; a, b; x, y) = E(r, r + h; ax, by) / E(r, r + h; a, b)$$

where E is the Stolarsky mean. Some inequalities between classic means are obtained.

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1. Introduction

Extended mean values of positive numbers x, y introduced by Stolarsky in [6] are defined as

$$(1.1) \quad E(r, s; x, y) = \begin{cases} \left(\frac{r y^s - x^s}{s y^r - x^r} \right)^{\frac{1}{s-r}} & sr(s-r)(x-y) \neq 0, \\ \left(\frac{1}{r} \frac{y^r - x^r}{\log y - \log x} \right)^{\frac{1}{r}} & r(x-y) \neq 0, s = 0, \\ e^{-\frac{1}{r}} \left(\frac{y y^r}{x x^r} \right)^{\frac{1}{y^r - x^r}} & r = s, r(x-y) \neq 0, \\ \sqrt{xy} & r = s = 0, x - y \neq 0, \\ x & x = y. \end{cases}$$

This mean is also called the Stolarsky mean.

In [9] the author extended the Stolarsky means to a four-parameter family of means by adding positive weights a, b :

$$(1.2) \quad F(r, s; a, b; x, y) = \frac{E(r, s; ax, by)}{E(r, s; a, b)}.$$

From the continuity of E it follows that F is continuous in $\mathbb{R}^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$. Our goal in this paper is to investigate the logarithmic convexity of

$$(1.3) \quad F_h(r; a, b; x, y) = F(r, r+h; a, b; x, y).$$

In [1] Horst Alzer investigated the one-parameter mean

$$(1.4) \quad J(r) = J(r; x, y) = E(r, r+1; x, y)$$



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and proved that for $x \neq y$, J is strictly log-convex for $r < -1/2$ and strictly log-concave for $r > -1/2$. He also proved that $J(r)J(-r) \leq J^2(0)$. In [2] he obtained a similar result for the Lehmer means

$$(1.5) \quad L(r) = L(r; x, y) = \frac{x^{r+1} + y^{r+1}}{x^r + y^r}.$$

With an appropriate choice of parameters in (1.2) one can obtain both the one-parameter mean and the Lehmer mean. Namely,

$$J(r; x, y) = F(r, r + 1; 1, 1; x, y)$$

and

$$L(r, x, y) = F(r, r + 1; x, y; x, y).$$

Another example may be the mean created the same way from the Heronian mean

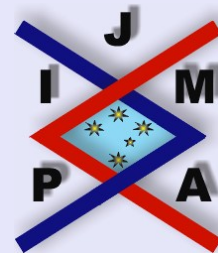
$$(1.6) \quad N(r; x, y) = F(r, r + 1; \sqrt{x}, \sqrt{y}; x, y) = \frac{x^{r+1} + \sqrt{xy}^{r+1} + y^{r+1}}{x^r + \sqrt{xy}^r + y^r}.$$

The following monotonicity properties of weighted Stolarsky means have been established in [9]:

Property 1.1. F increases in x and y .

Property 1.2. F increases in r and s if $(x - y)(a^2x - b^2y) > 0$ and decreases if $(x - y)(a^2x - b^2y) < 0$.

Property 1.3. F increases in a if $(x - y)(r + s) > 0$ and decreases if $(x - y)(r + s) < 0$, F decreases in b if $(x - y)(r + s) > 0$ and increases if $(x - y)(r + s) < 0$.



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Definition 1.1. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be symmetrically convex (concave) with respect to the point r_0 if f is convex (concave) in (r_0, ∞) and for every $t > 0$ $f(r_0 + t) + f(r_0 - t) = 2f(r_0)$.

Definition 1.2. A function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be symmetrically log-convex (log-concave) with respect to the point r_0 if $\log f$ is symmetrically convex (concave) w.r.t. r_0 .

For symmetrically log-convex functions the symmetry condition reads $f(r_0 + t)f(r_0 - t) = f^2(r_0)$. We shall recall now two properties of convex functions.

Property 1.4. If f is convex (concave) then for $h > 0$ the function $g(t) = f(t + h) - f(t)$ is increasing (decreasing). For $h < 0$ the monotonicity of g reverses.

For log-convex f the same holds for $g(t) = f(t + h)/f(t)$.

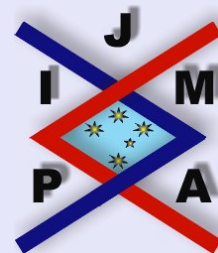
Property 1.5. If f is convex (concave) then for arbitrary x the function $h_x(t) = f(x - t) + f(x + t)$ is increasing (decreasing) in $(0, \infty)$. For log-convex f the same holds for $h_x(t) = f(x - t)f(x + t)$.

The property 1.5 holds also for symmetrically convex (concave) functions:

Lemma 1.6. Let f be symmetrically convex w.r.t. r_0 , and let $x > r_0$. Then the function $h_x(t) = f(x - t) + f(x + t)$ is increasing (decreasing) in $(0, \infty)$. If $x < r_0$ then $h_x(t)$ decreases.

For f symmetrically concave the monotonicity of h_x is reverse.

For the case where f is symmetrically log-convex (log-concave) $h_x(t) = f(x + t)f(x - t)$ is monotone accordingly.



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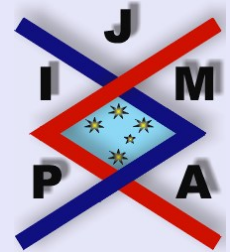
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Proof. We shall prove the lemma for f symmetrically convex and $x > r_0$. For $x < r_0$ or f symmetrically convex the proofs are similar.

Consider two cases:

- $0 < t < x - r_0$. In this case $h_x(t)$ is increasing by Property 1.5.
- $t > x - r_0$. Now $h_x(t) = f(x + t) + f(x - t) = 2f(r_0) + f(x + t) - f(t - x + 2r_0)$ increases by Property 1.4 because $t - x + 2r_0 > r_0$ and $(x + t) - (t - x + 2r_0) > 0$.

□



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2. Main Result

It is obvious that the monotonicity of F_h matches that of F . The main result consists of the following theorem:

Theorem 2.1. *If $(x - y)(a^2x - b^2y) > 0$ (resp. < 0) then $F_h(r)$ is symmetrically log-concave (resp. log-convex) with respect to the point $-h/2$.*

To prove it we need the following

Lemma 2.2. *Let*

$$g(t, A, B) = \frac{A^t \log^2 A}{(A^t - 1)^2} - \frac{B^t \log^2 B}{(B^t - 1)^2}.$$

Then

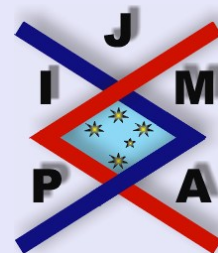
1. $g(t, A, B) = g(\pm t, A^{\pm 1}, B^{\pm 1})$,
2. g is increasing in t on $(0, \infty)$ if $\log^2 A - \log^2 B > 0$ and decreasing otherwise.

Proof. (1) becomes obvious when we write

$$g(t, A, B) = \frac{\log^2 A}{A^t - 2 + A^{-t}} - \frac{\log^2 B}{B^t - 2 + B^{-t}}.$$

From (1) it follows that replacing A, B with A^{-1}, B^{-1} if necessary we may assume that $A, B > 1$. In this case $\text{sgn}(\log^2 A - \log^2 B) = \text{sgn}(A^t - B^t)$.

$$\begin{aligned} \frac{\partial g}{\partial t} &= -\frac{A^t(A^t + 1)\log^3 A}{(A^t - 1)^3} + \frac{B^t(B^t + 1)\log^3 B}{(B^t - 1)^3} \\ &= -\frac{1}{t^3}(\phi(A^t) - \phi(B^t)) = -\frac{1}{t^3}(A^t - B^t)\phi'(\xi), \end{aligned}$$



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where $\xi > 1$ lies between A^t and B^t and

$$\phi(u) = \frac{u(u+1)\log^3 u}{(u-1)^3}.$$

To complete the proof it is enough to show that $\phi'(u) < 0$ for $u > 1$.

$$\phi'(u) = \frac{(u^2 + 4u + 1)\log^2 u}{(u-1)^4} \left[\frac{3(u^2 - 1)}{u^2 + 4u + 1} - \log u \right],$$

so the sign of ϕ' is the same as the sign of $\psi(u) = \frac{3(u^2-1)}{u^2+4u+1} - \log u$. But $\psi(1) = 0$ and $\psi'(u) = -(u-1)^4/u(u^2+4u+1)^2 < 0$, so $\phi(u) < 0$. \square

Proof of Theorem 2.1. First of all note that

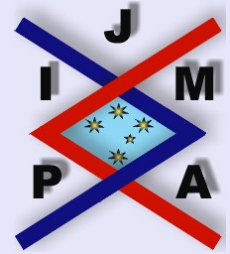
$$\log^2 \frac{ax}{by} - \log^2 \frac{a}{b} = \log \frac{x}{y} \log \frac{a^2x}{b^2y}$$

and because $\operatorname{sgn}(x - y) = \operatorname{sgn} \log \frac{x}{y}$ we see that

$$(2.1) \quad \operatorname{sgn}(x - y)(a^2x - b^2y) = \operatorname{sgn} \left(\log^2 \frac{ax}{by} - \log^2 \frac{a}{b} \right).$$

Let $A = \frac{ax}{by}$ and $B = \frac{a}{b}$. Suppose that $A, B \neq 1$ (in other cases we use a standard continuity argument). $F_h(r)$ can be written as

$$F_h(r) = y \left(\frac{A^{r+h} - 1}{B^{r+h} - 1} \bigg/ \frac{A^r - 1}{B^r - 1} \right)^{\frac{1}{h}},$$



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We show symmetry by performing simple calculations:

$$\begin{aligned}
 & F_h^h(-h/2 - r)F_h^h(-h/2 + r) \\
 &= y^{2h} \frac{A^{h/2-r} - 1}{B^{h/2-r} - 1} \cdot \frac{B^{-h/2-r} - 1}{A^{-h/2-r} - 1} \cdot \frac{A^{h/2+r} - 1}{B^{h/2+r} - 1} \cdot \frac{B^{-h/2+r} - 1}{A^{-h/2+r} - 1} \\
 (2.2) \quad &= y^{2h} \frac{B^{-h}}{A^{-h}} \cdot \frac{A^{h/2-r} - 1}{B^{h/2-r} - 1} \cdot \frac{1 - B^{h/2+r}}{1 - A^{h/2+r}} \cdot \frac{A^{h/2+r} - 1}{B^{h/2+r} - 1} \cdot \frac{1 - B^{h/2-r}}{1 - A^{h/2-r}} \\
 &= y^{2h} \left(\frac{x}{y}\right)^h = (xy)^h = F_h^{2h}(-h/2).
 \end{aligned}$$

Differentiating twice we obtain

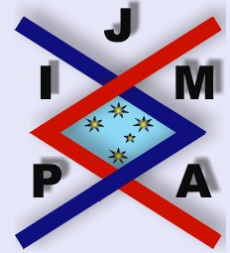
$$\begin{aligned}
 \frac{d^2}{dr^2} \log F_h(r) &= \frac{g(r, A, B) - g(r + h, A, B)}{h} \\
 &= \frac{g(|r|, A, B) - g(|r + h|, A, B)}{h} \quad (\text{by Lemma 2.2 (1)}),
 \end{aligned}$$

hence by Lemma 2.2 (2)

$$\begin{aligned}
 \operatorname{sgn} \frac{d^2}{dr^2} \log F_h(r) &= \operatorname{sgn} h(|r| - |r + h|)(\log^2 A - \log^2 B) \\
 &= \operatorname{sgn}(r + h/2)(x - y)(a^2x - b^2y).
 \end{aligned}$$

The last equation follows from (2.1) and from the fact that the inequality $|r| < |r + h|$ is valid if and only if $r > -h/2$ and $h > 0$ or $r < -h/2$ and $h < 0$. \square

The following theorem is an immediate consequence of Theorem 2.1 and Lemma 1.6.



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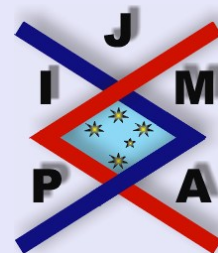


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Theorem 2.3. If $(x - y)(a^2x - b^2y)(r_0 + h/2) > 0$ then the function

$$\Phi(t) = F_h(r_0 - t)F_h(r_0 + t)$$

is decreasing in $(0, \infty)$. In particular for every real t

$$(2.3) \quad F_h(r_0 - t)F_h(r_0 + t) \leq F_h^2(r_0).$$

If $(x - y)(a^2x - b^2y)(r_0 + h/2) < 0$ then $\Phi(t)$ is increasing in $(0, \infty)$. In particular for every real t

$$(2.4) \quad F_h(r_0 - t)F_h(r_0 + t) \geq F_h^2(r_0).$$

The following corollaries are immediate consequences of Theorems 2.1 and 2.3:

Corollary 2.4. For $x \neq y$ the one-parameter mean $J(r)$ defined by (1.4) is log-convex for $r < -1/2$ and log-concave for $r > -1/2$. If $r_0 > -1/2$ then for all real t , $J(r_0 - t)J(r_0 + t) \leq J^2(r_0)$. For $r_0 < -1/2$ the inequality reverses.

Proof. $J(r; x, y) = F_1(r; 1, 1; x, y)$. □

Corollary 2.5. For $x \neq y$ the Lehmer mean $L(r)$ defined by (1.5) is log-convex for $r < -1/2$ and log-concave for $r > -1/2$. If $r_0 > -1/2$ then for all real t , $L(r_0 - t)L(r_0 + t) \leq L^2(r_0)$. For $r_0 < -1/2$ the inequality reverses.

Proof. $L(r; x, y) = F_1(r; x, y; x, y)$. □

Corollary 2.6. For $x \neq y$ the mean $N(r)$ defined by (1.6) is log-convex for $r < -1/2$ and log-concave for $r > -1/2$. If $r_0 > -1/2$ then for all real t , $N(r_0 - t)N(r_0 + t) \leq N^2(r_0)$. For $r_0 < -1/2$ the inequality reverses.

Proof. $N(r; x, y) = F_1(r; \sqrt{x}, \sqrt{y}; x, y)$. □

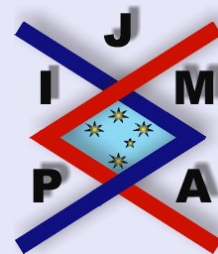
3. Application

In this section we show some inequalities between classic means:

Power means	$A_r = A_r(x, y) = \left(\frac{x^r + y^r}{2} \right)^{\frac{1}{r}},$
Harmonic mean	$H = A_{-1}(x, y) = \frac{2xy}{x + y},$
Geometric mean	$G = A_0(x, y) = \sqrt{xy},$
Logarithmic mean	$L = L(x, y) = \frac{x - y}{\log x - \log y},$
Heronian mean	$N = N(x, y) = \frac{x + \sqrt{xy} + y}{3},$
Arithmetic mean	$A = A_1(x, y) = \frac{x + y}{2},$
Centroidal mean	$T = T(x, y) = \frac{2x^2 + xy + y^2}{3(x + y)},$
Root-mean-square	$R = A_2(x, y) = \sqrt{\frac{x^2 + y^2}{2}},$
Contrharmonic mean	$C = C(x, y) = \frac{x^2 + y^2}{x + y}.$

Corollary 3.1 (Tung-Po Lin inequality [4]).

$$L \leq A_{1/3}.$$



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Proof. By Theorem 2.3

$$F_{1/3}(0; 1, 1, ; x, y)F_{1/3}(2/3; 1, 1; x, y) \leq F_{1/3}^2(1/3; 1, 1; x, y)$$

or

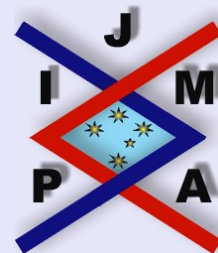
$$\left(3 \frac{\sqrt[3]{x} - \sqrt[3]{y}}{\log x - \log y}\right)^3 \left(\frac{2}{3} \frac{x - y}{\sqrt[3]{x^2} - \sqrt[3]{y^2}}\right)^3 \leq \left(\frac{1}{2} \frac{\sqrt[3]{x^2} - \sqrt[3]{y^2}}{\sqrt[3]{x} - \sqrt[3]{y}}\right)^6.$$

Simplifying we obtain

$$L^3(x, y) \leq A_{1/3}^3(x, y).$$

□

Inequalities in the table below can be shown the same way as above by an appropriate choice of parameters in (2.3) and (2.4).



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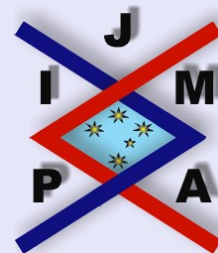


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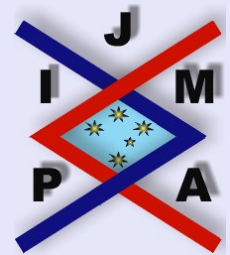
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No	Inequality	h	r_0	t	a	b
1	$L^2 \geq GN$	1/2	0	1	1	1
2	$L^2 \geq HT$	1	0	2	1	1
3	$A_{1/2}^2 \geq AG$	1/2	0	1/2	x	y
4	$A_{1/2}^2 \geq LN$	1/2	1/2	1/2	1	1
5	$N^2 \geq AL$	1	1/2	1/2	1	1
6	$A^2 \geq LT$	1	1	1	1	1
7	$A^2 \geq CH$	1	0	1	x	y
8	$LN \geq AG$	1/2	1/2	1	1	1
9	$GN \geq HT$	1	-1	1/2	x	y
10	$AN \geq TG$	1/2	0	1	x	y
11	$LT \geq HC$	1	1	2	1	1
12	$TA \geq NR$	1	1/2	1/2	x	y
13	$L^3 \geq AG^2$	1	0	1	1	1
14	$L^3 \geq GA_{1/2}^2$	1/2	-1/2	1/2	1	1
15	$N^3 \geq AA_{1/2}^2$	1/2	1	1/2	1	1
16	$T^3 \geq AR^2$	1	2	1	1	1
17	$LN^2 \geq G^2T$	1	1/2	3/2	1	1

Note that 4 is stronger than 3 (due to inequality 8), 14 is stronger than 13 (due to 3). Also, 1 is stronger than 2 because of 9.

References

- [1] H. ALZER, Über eine einparametrische Familie von Mittelwerten, *Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber.*, 1987 (1988), 1–9.
- [2] H. ALZER, Über Lehmers Mittelwertefamilie, *Elem. Math.*, **43** (1988), 50–54.
- [3] E. LEACH AND M. SHOLANDER, Extended mean values, *Amer. Math. Monthly*, **85** (1978), 84–90.
- [4] T.-P. LIN, The power mean and the logarithmic mean, *Amer. Math. Monthly*, **81**(8) (1974), 879–883.
- [5] E. NEUMAN AND Zs. PÁLES, On comparison of Stolarsky and Gini means, *J. Math. Anal. Appl.*, **278** (2003), 274–285.
- [6] K.B. STOLARSKY, Generalizations of the logarithmic mean, *Math. Mag.*, **48** (1975), 87–92.
- [7] FENG QI, Generalized weighted mean values with two parameters, *Proc. Roy. Soc. London Ser. A*, **454** (1998), No. 1978, 2723–2732.
- [8] A. WITKOWSKI, Monotonicity of generalized extended mean values, *Colloq. Math.*, **99**(2) (2004), 203–206. *RGMIA Research Report Collection*, **7**(1) (2004), Art. 12. [ONLINE: <http://rgmia.vu.edu.au/v7n1.html>].
- [9] A. WITKOWSKI, Weighted extended mean values, *Colloq. Math.*, **100**(1) (2004), 111–117. *RGMIA Research Report Collection*, **7**(1) (2004), Art. 6. [ONLINE: <http://rgmia.vu.edu.au/v7n1.html>].



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