ON SOME MAXIMAL INEQUALITIES FOR DEMISUBMARTINGALES AND $N-$DEMISUPER MARTINGALES

B.L.S. PRAKASA RAO
Department of Mathematics
University of Hyderabad,
India
EMail: blsprsm@uohyd.ernet.in

Received: 20 August, 2007

Accepted: 17 September, 2007

Communicated by: N.S. Barnett

2000 AMS Sub. Class.: Primary 60E15; Secondary 60G48.

Key words: Maximal inequalities, Demisubmartingales, $N-$demisupermartingales.

Abstract: We study maximal inequalities for demisubmartingales and N-demisupermartingales and obtain inequalities between dominated demisubmartingales. A sequence of partial sums of zero mean associated random variables is an example of a demimartingale and a sequence of partial sums of zero mean negatively associated random variables is an example of a N-demimartingale.

Acknowledgements: This work was done while the author was visiting the Department of Mathematics, Indian Institute of Technology, Bombay during May 2007. The author thanks Prof. P. Velliasamy and his colleagues for their invitation and hospitality.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

vol. 8, iss. 4, art. 112, 2007

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{S_n, n \geq 1\}$ be a sequence of random variables defined on it such that $E|S_n| < \infty, n \geq 1$. Suppose that

$$E[(S_{n+1} - S_n) f(S_1, \ldots, S_n)] \geq 0 \tag{1.1}$$

for all coordinate-wise nondecreasing functions f whenever the expectation is defined. Then the sequence $\{S_n, n \geq 1\}$ is called a *demimartingale*. If the inequality (1.1) holds for nonnegative coordinate-wise nondecreasing functions f, then the sequence $\{S_n, n \geq 1\}$ is called a *demisubmartingale*. If

$$E[(S_{n+1} - S_n) f(S_1, \ldots, S_n)] \leq 0 \tag{1.2}$$

for all coordinatewise nondecreasing functions f whenever the expectation is defined, then the sequence $\{S_n, n \geq 1\}$ is called a $N-$*demimartingale*. If the inequality (1.2) holds for nonnegative coordinate-wise nondecreasing functions f, then the sequence $\{S_n, n \geq 1\}$ is called a $N-$*demisupermartingale*.

Remark 1. If the function f in (1.1) is not required to be nondecreasing, then the condition defined by the inequality (1.1) is equivalent to the condition that $\{S_n, n \geq 1\}$ is a martingale with respect to the natural choice of σ-algebras. If the inequality defined by (1.1) holds for all nonnegative functions f, then $\{S_n, n \geq 1\}$ is a submartingale with respect to the natural choice of σ-algebras. A martingale with the natural choice of σ-algebras is a demimartingale as well as a $N-$demimartingale since it satisfies (1.1) as well as (1.2). It can be checked that a submartingale is a demisubmartingale and a supermartingale is an N-demisupermartingale. However there are stochastic processes which are demimartingales but not martingales with respect to the natural choice of σ-algebras (cf. [18]).

The concept of demimartingales and demisubmartingales was introduced by Newman and Wright [11] and the notion of $N-$demimartingales (termed earlier as nega-

journal of inequalities in pure and applied mathematics
issn: 1443-5756

tive demimartingales in [14]) and $N-$demisupermartingales were introduced in [14] and [6].

A set of random variables $X_1, \ldots, X_n$ is said to be *associated* if

$$\mathrm{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0 \tag{1.3}$$

for any two coordinatewise nondecreasing functions f and g whenever the covariance is defined. They are said to be *negatively associated* if

$$\mathrm{Cov}(f(X_i, i \in A), g(X_i, i \in B)) \leq 0 \tag{1.4}$$

for any two disjoint subsets A and B and for any two coordinatewise nondecreasing functions f and g whenever the covariance is defined.

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *associated* (*negatively associated*) if every finite subset of random variables of the sequence is associated (negatively associated).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Maximal Inequalities for Demimartingales and Demisubmartingales

Newman and Wright [11] proved that the partial sums of a sequence of mean zero associated random variables form a demimartingale. We will now discuss some properties of demimartingales and demisubmartingales. The following result is due to Christofides [5].

Theorem 2.1. *Suppose the sequence $\{S_n, n \geq 1\}$ is a demisubmartingale or a demimartingale and $g(\cdot)$ is a nondecreasing convex function. Then the sequence $\{g(S_n), n \geq 1\}$ is a demisubmartingale.*

Let $g(x) = x^+ = \max(0, x)$. Then the function g is nondecreasing and convex. As a special case of the previous result, we get that $\{S_n^+, n \geq 1\}$ is a demisubmartingale. Note that $S_n^+ = \max(0, S_n)$.

Newman and Wright [11] proved the following maximal inequality for demisubmartingales which is an analogue of a maximal inequality for submartingales due to Garsia [8].

Theorem 2.2. *Suppose $\{S_n, n \geq 1\}$ is a demimartingale (demisubmartingale) and $m(\cdot)$ is a nondecreasing (nonnegative and nondecreasing) function with $m(0) = 0$. Let*

$$\begin{aligned} S_{nj} &= j - \textit{th largest of } (S_1, \ldots, S_n) \ \textit{ if } \ j \leq n \\ &= \min(S_1, \ldots, S_n) = S_{n,n} \ \textit{ if } \ j > n. \end{aligned}$$

Then, for any n and j,

$$E\left(\int_0^{S_{nj}} u dm(u)\right) \leq E\left[S_n m(S_{nj})\right].$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

In particular, for any $\lambda > 0$,

$$\lambda\, P(S_{n1} \geq \lambda) \leq \int_{[S_{n1}\geq\lambda]} S_n dP. \tag{2.1}$$

As an application of the above inequality and an upcrossing inequality for demisubmartingales, the following convergence theorem was proved in [11].

Theorem 2.3. *If* $\{S_n, n \geq 1\}$ *is a demisubmartingale and* $\sup_n E|S_n| < \infty$, *then* S_n *converges almost surely to a finite limit.*

Christofides [5] proved a general version of the inequality (2.1) of Theorem 2.2 which is an analogue of Chow's maximal inequality for martingales [3].

Theorem 2.4. *Let* $\{S_n, n \geq 1\}$ *be a demisubmartingale with* $S_0 = 0$. *Let the sequence* $\{c_k, k \geq 1\}$ *be a nonincreasing sequence of positive numbers. Then, for any* $\lambda > 0$,

$$\lambda\, P\left(\max_{1\leq k\leq n} c_k S_k \geq \lambda\right) \leq \sum_{j=1}^{n} c_j E\left(S_j^+ - S_{j-1}^+\right).$$

Wang [16] obtained the following maximal inequality generalizing Theorems 2.2 and 2.4.

Theorem 2.5. *Let* $\{S_n, n \geq 1\}$ *be a demimartingale and* $g(\cdot)$ *be a nonnegative convex function on* $\mathbb{R}$ *with* $g(0) = 0$. *Suppose that* $\{c_i, 1 \leq i \leq n\}$ *is a nonincreasing sequence of positive numbers. Let* $S_n^* = \max_{1\leq i\leq n} c_i g(S_i)$. *Then, for any* $\lambda > 0$,

$$\lambda\, P(S_n^* \geq \lambda) \leq \sum_{i=1}^{n} c_i E\{(g(S_i) - g(S_{i-1}))I[S_n^* \geq \lambda]\}.$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Suppose $\{S_n, n \geq 1\}$ is a nonnegative demimartingale. As a corollary to the above theorem, it can be proved that

$$E(S_n^{max}) \leq \frac{e}{e-1}[1 + E(S_n \log^+ S_n)].$$

For a proof of this inequality, see Corollary 2.1 in [16].

We now discuss a Whittle type inequality for demisubmartingales due to Prakasa Rao [13]. This result generalizes the Kolmogorov inequality and the Hajek-Renyi inequality for independent random variables [17] and is an extension of the results in [5] for demisubmartingales.

Theorem 2.6. *Let $S_0 = 0$ and $\{S_n, n \geq 1\}$ be a demisubmartingale. Let $\phi(\cdot)$ be a nonnegative nondecreasing convex function such that $\phi(0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$. Further suppose that $0 = u_0 < u_1 \leq \cdots \leq u_n$. Then*

$$P(\phi(S_k) \leq \psi(u_k), 1 \leq k \leq n) \geq 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}.$$

As a corollary of the above theorem, it follows that

$$P\left(\sup_{1\leq j\leq n} \frac{\phi(S_j)}{\psi(u_j)} \geq \epsilon\right) \leq \epsilon^{-1} \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}$$

for any $\epsilon > 0$. In particular, for any fixed $n \geq 1$,

$$P\left(\sup_{k\geq n} \frac{\phi(S_k)}{\psi(u_k)} \geq \epsilon\right) \leq \epsilon^{-1} \left[E\left(\frac{\phi(S_n)}{\psi(u_n)}\right) + \sum_{k=n+1}^{\infty} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}\right]$$

for any $\epsilon > 0$. As a consequence of this inequality, we get the following strong law of large numbers for demisubmartingales [13].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 2.7. *Let $S_0 = 0$ and $\{S_n, n \geq 1\}$ be a demisubmartingale. Let $\phi(\cdot)$ be a nonegative nondecreasing convex function such that $\phi(0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$ such that $\psi(u) \to \infty$ as $u \to \infty$. Further suppose that*

$$\sum_{k=1}^{\infty} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)} < \infty$$

for a nondecreasing sequence $u_n \to \infty$ as $n \to \infty$. Then

$$\frac{\phi(S_n)}{\psi(u_n)} \stackrel{a.s}{\to} 0 \text{ as } n \to \infty.$$

Suppose $\{S_n, n \geq 1\}$ is a demisubmartingale. Let $S_n^{\max} = \max_{1 \leq i \leq n} S_i$ and $S_n^{\min} = \min_{1 \leq i \leq n} S_i$. As special cases of Theorem 2.2, we get that

$$\lambda \ P[S_n^{\max} \geq \lambda] \leq \int_{[S_n^{\max} \geq \lambda]} S_n dP \tag{2.2}$$

and

$$\lambda \ P[S_n^{\min} \geq \lambda] \leq \int_{[S_n^{\min} \geq \lambda]} S_n dP \tag{2.3}$$

for any $\lambda > 0$.

The inequality (2.2) can also be obtained directly without using Theorem 2.2 by the standard methods used to prove Kolomogorov's inequality. We now prove a variant of the inequality given by (2.3).

Suppose $\{S_n, n \geq 1\}$ is a demisubmartingale. Let $\lambda > 0$. Let

$$N = \left[\min_{1 \leq k \leq n} S_k < \lambda\right], \quad N_1 = [S_1 < \lambda]$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

and

$$N_k = [S_k < \lambda,\ S_j \geq \lambda,\ 1 \leq j \leq k-1], \quad k > 1.$$

Observe that

$$N = \bigcup_{k=1}^{n} N_k$$

and $N_k \in \mathcal{F}_k = \sigma\{S_1, \ldots, S_k\}$. Furthermore N_k, $1 \leq k \leq n$ are disjoint and

$$N_k \subset \left(\bigcup_{i=1}^{k-1} N_i\right)^c,$$

where A^c denotes the complement of the set A in Ω. Note that

$$\begin{aligned} E(S_1) &= \int_{N_1} S_1 dP + \int_{N_1^c} S_1 dP \\ &\leq \lambda \int_{N_1} dP + \int_{N_1^c} S_2 dP. \end{aligned}$$

The last inequality follows by observing that

$$\begin{aligned} \int_{N_1^c} S_1 dP - \int_{N_1^c} S_2 dP &= \int_{N_1^c} (S_1 - S_2) dP \\ &= E((S_1 - S_2) I[N_1^c]). \end{aligned}$$

Since the indicator function of the set $N_1^c = [S_1 \geq \lambda]$ is a nonnegative nondecreasing function of S_1 and $\{S_k, 1 \leq k \leq n\}$ is a demisubmartingale, it follows that

$$E((S_2 - S_1) I[N_1^c]) \geq 0.$$

vol. 8, iss. 4, art. 112, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Therefore

$$E((S_1 - S_2)I[N_1^c]) \leq 0,$$

which implies that

$$\int_{N_1^c} S_1 dP \leq \int_{N_1^c} S_2 dP.$$

This proves the inequality

$$E(S_1) \leq \lambda \int_{N_1} dP + \int_{N_1^c} S_2 dP$$
$$= \lambda P(N_1) + \int_{N_1^c} S_2 dP.$$

Observe that $N_2 \subset N_1^c$. Hence

$$\int_{N_1^c} S_2 dP = \int_{N_2} S_2 dP + \int_{N_2^c \cap N_1^c} S_2 dP$$
$$\leq \int_{N_2} S_2 dP + \int_{N_2^c \cap N_1^c} S_3 dP$$
$$\leq \lambda\, P(N_2) + \int_{N_2^c \cap N_1^c} S_3 dP.$$

The second inequality in the above chain follows from the observation that the indicator function of the set $N_2^c \cap N_1^c = I[S_1 \geq \lambda, S_2 \geq \lambda]$ is a nonnegative nondecreasing function of S_1, S_2 and the fact that $\{S_k, 1 \leq k \leq n\}$ is a demisubmartingale. By

journal of inequalities in pure and applied mathematics
issn: 1443-5756

repeated application of these arguments, we get that

$$E(S_1) \leq \lambda \sum_{i=1}^{n} P(N_i) + \int_{\cap_{i=1}^{n} N_i^c} S_n dP$$
$$= \lambda\, P(N) + \int_{\Omega} S_n dP - \int_{N} S_n dP.$$

Hence

$$\lambda\, P(N) \geq \int_N S_n dP - \int_{\Omega} (S_n - S_1) dP$$

and we have the following result.

Theorem 2.8. *Suppose that* $\{S_n, n \geq 1\}$ *is a demisubmartingale . Let*

$$N = \left[\min_{1 \leq k \leq n} S_k < \lambda\right]$$

for any $\lambda > 0$. *Then*

$$\lambda\, P(N) \geq \int_N S_n dP - \int_{\Omega} (S_n - S_1) dP. \tag{2.4}$$

In particular, if $\{S_n, n \geq 1\}$ is a demimartingale, then it is easy to check that $E(S_n) = E(S_1)$ for all $n \geq 1$ and hence we have the following result as a corollary to Theorem 2.8.

Theorem 2.9. *Suppose that* $\{S_n, n \geq 1\}$ *is a demimartingale . Let* $N = [\min_{1 \leq k \leq n} S_k < \lambda]$ *for any* $\lambda > 0$. *Then*

$$\lambda\, P(N) \geq \int_N S_n dP. \tag{2.5}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

We now prove some new maximal inequalities for nonnegative demisubmartingales.

Theorem 2.10. *Suppose that* $\{S_n, n \geq 1\}$ *is a positive demimartingale with* $S_1 = 1$. *Let* $\gamma(x) = x - 1 - \log x$ *for* $x > 0$. *Then*

$$\gamma(E\left[S_n^{\max}\right]) \leq E\left[S_n \log S_n\right] \tag{2.6}$$

and

$$\gamma\left(E\left[S_n^{\min}\right]\right) \leq E\left[S_n \log S_n\right]. \tag{2.7}$$

Proof. Note that the function $\gamma(x)$ is a convex function with minimum $\gamma(1) = 0$. Let $I(A)$ denote the indicator function of the set A. Observe that $S_n^{\max} \geq S_1 = 1$ and hence

$$\begin{aligned}
E(S_n^{\max}) - 1 &= \int_0^\infty P[S_n^{\max} \geq \lambda] d\lambda - 1 \\
&= \int_0^1 P[S_n^{\max} \geq \lambda] d\lambda + \int_1^\infty P[S_n^{\max} \geq \lambda] d\lambda - 1 \\
&= \int_1^\infty P[S_n^{\max} \geq \lambda] d\lambda \quad (\text{since } \ S_1 = 1) \\
&\leq \int_1^\infty \left\{ \frac{1}{\lambda} \int_{[S_n^{\max} \geq \lambda]} S_n dP \right\} d\lambda \quad (\text{by } (2.2)) \\
&= E\left(\int_1^\infty \frac{S_n I[S_n^{\max} \geq \lambda]}{\lambda} d\lambda \right) \\
&= E\left(S_n \int_1^{S_n^{\max}} \frac{1}{\lambda} d\lambda \right) \\
&= E(S_n \log(S_n^{\max})).
\end{aligned}$$

Demisubmartingales and $N-$demisuper Martingales
B.L.S. Prakasa Rao
vol. 8, iss. 4, art. 112, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Using the fact that $\gamma(x) \geq 0$ for all $x > 0$, we get that

$$\begin{aligned}
E(S_n^{\max}) - 1 &\leq E\left[S_n\left(\log(S_n^{\max}) + \gamma\left(\frac{S_n^{\max}}{S_n E(S_n^{\max})}\right)\right)\right] \\
&= E\left[S_n\left(\log(S_n^{\max}) + \frac{S_n^{\max}}{S_n E(S_n^{\max})} - 1 - \log\left(\frac{S_n^{\max}}{S_n E(S_n^{\max})}\right)\right)\right] \\
&= 1 - E(S_n) + E(S_n \log S_n) + E(S_n) \log E(S_n^{\max}).
\end{aligned}$$

Rearranging the terms in the above inequality, we obtain

$$\begin{aligned}
\gamma(E(S_n^{\max})) &= E(S_n^{\max}) - 1 - \log E(S_n^{\max}) \\
&\leq 1 - E(S_n) + E(S_n \log S_n) \\
&\qquad + E(S_n) \log E(S_n^{\max}) - \log E(S_n^{\max}) \\
&= E(S_n \log S_n) + (E(S_n) - 1)\left(\log E\left(S_n^{(\max)}\right) - 1\right) \\
&= E(S_n \log S_n)
\end{aligned} \tag{2.8}$$

since $E(S_n) = E(S_1) = 1$ for all $n \geq 1$. This proves the inequality (2.6).

Observe that $0 \leq S_n^{\min} \leq S_1 = 1$, which implies that

$$\begin{aligned}
E(S_n^{\min}) &= \int_0^1 P[S_n^{\min} \geq \lambda] d\lambda \\
&= 1 - \int_0^1 P[S_n^{\min} < \lambda] d\lambda \\
&\leq 1 - \int_0^1 \left\{\frac{1}{\lambda}\int_{[S_n^{\min} < \lambda]} S_n dP\right\} d\lambda \quad \text{(by Theorem 2.9)}
\end{aligned}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$
\begin{aligned}
&= 1 - E\left(\int_0^1 \frac{S_n I[S_n^{\min} < \lambda]}{\lambda} d\lambda\right) \\
&= 1 - E\left(S_n \int_{S_n^{\min}}^1 \frac{1}{\lambda} d\lambda\right) \\
&= 1 + E(S_n \log(S_n^{\min})).
\end{aligned}
$$

Applying arguments similar to those given above to prove the inequality (2.8), we get that

$$
\gamma(E(S_n^{\min})) \leq E(S_n \log S_n) \tag{2.9}
$$

which proves the inequality (2.7). □

The above inequalities for positive demimartingales are analogues of maximal inequalities for nonnegative martingales proved in [9].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Maximal ϕ-inequalities for Nonnegative Demisubmartingales

Let $\mathcal{C}$ denote the class of *Orlicz functions*, that is, unbounded, nondecreasing convex functions $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$. If the right derivative ϕ' is unbounded, then the function ϕ is called a *Young function* and we denote the subclass of such functions by $\mathcal{C}'$. Since

$$\phi(x) = \int_0^x \phi'(s)ds \leq x\phi'(x)$$

by convexity, it follows that

$$p_\phi = \inf_{x>0} \frac{x\phi'(x)}{\phi(x)}$$

and

$$p_\phi^* = \sup_{x>0} \frac{x\phi'(x)}{\phi(x)}$$

are in $[1,\infty]$. The function ϕ is called *moderate* if $p_\phi^* < \infty$, or equivalently, if for some $\lambda > 1$, there exists a finite constant c_λ such that

$$\phi(\lambda x) \leq c_\lambda \phi(x), \quad x \geq 0.$$

An example of such a function is $\phi(x) = x^\alpha$ for $\alpha \in [1,\infty)$. An example of a nonmoderate Orlicz function is $\phi(x) = \exp(x^\alpha) - 1$ for $\alpha \geq 1$.

Let $\mathcal{C}^*$ denote the set of all differentiable $\phi \in \mathcal{C}$ whose derivative is concave or convex and $\mathcal{C}'$ denote the set of $\phi \in \mathcal{C}$ such that $\phi'(x)/x$ is integrable at 0, and thus, in particular $\phi'(0) = 0$. Let $\mathcal{C}_0^* = \mathcal{C}' \cap \mathcal{C}^*$.

Given $\phi \in \mathcal{C}$ and $a \geq 0$, define

$$\Phi_a(x) = \int_a^x \int_a^s \frac{\phi'(r)}{r} dr ds, \quad x > 0.$$

vol. 8, iss. 4, art. 112, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

It can be seen that the function $\Phi_a I_{[a,\infty)} \in \mathcal{C}$ for any $a > 0$, where I_A denotes the indicator function of the set A. If $\phi \in \mathcal{C}'$, the same holds for $\Phi \equiv \Phi_0$. If $\phi \in \mathcal{C}_0^*$, then $\Phi \in \mathcal{C}_0^*$. Furthermore, if ϕ' is concave or convex, the same holds for

$$\Phi'(x) = \int_0^x \frac{\phi'(r)}{r} dr,$$

and hence $\phi \in \mathcal{C}_0^*$ implies that $\Phi \in \mathcal{C}_0^*$. It can be checked that ϕ and Φ are related through the diferential equation

$$x\Phi'(x) - \Phi(x) = \phi(x), \quad x \geq 0$$

under the initial conditions $\phi(0) = \phi'(0) = \Phi(0) = \Phi'(0) = 0$. If $\phi(x) = x^p$ for some $p > 1$, then $\Phi(x) = x^p/(p-1)$. For instance, if $\phi(x) = x^2$, then $\Phi(x) = x^2$. If $\phi(x) = x$, then $\Phi(x) \equiv \infty$ but $\Phi_1(x) = x \log x - x + 1$. It is known that if $\phi \in \mathcal{C}'$ with $p_\phi > 1$, then the function ϕ satisfies the inequalities

$$\Phi(x) \leq \frac{1}{p_\phi - 1}\phi(x), \quad x \geq 0.$$

Furthermore, if ϕ is moderate, that is $p_\phi^* < \infty$, then

$$\Phi(x) \geq \frac{1}{p_\phi^* - 1}\phi(x), \quad x \geq 0.$$

The brief introduction for properties of Orlicz functions given here is based on [2].

We now prove some maximal ϕ-inequalities for nonnegative demisubmartingales following the techniques in [2].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 3.1. *Let* $\{S_n, n \geq 1\}$ *be a nonnegative demisubmartingale and let* $\phi \in \mathcal{C}$*. Then*

$$P\left(S_n^{\max} \geq t\right) \leq \frac{\lambda}{(1-\lambda)t} \int_t^\infty P(S_n > \lambda s) ds \tag{3.1}$$
$$= \frac{\lambda}{(1-\lambda)t} E\left(\frac{S_n}{\lambda} - t\right)^+$$

for all $n \geq 1, t > 0$ *and* $0 < \lambda < 1$*. Furthermore,*

$$E[\phi(S_n^{\max})] \tag{3.2}$$
$$\leq \phi(b) + \frac{\lambda}{1-\lambda} \int_{[S_n > \lambda b]} \left(\Phi_a\left(\frac{S_n}{\lambda}\right) - \Phi_a(b) - \Phi_a'(b)\left(\frac{S_n}{\lambda} - b\right)\right) dP$$

for all $n \geq 1, a > 0, b > 0$ *and* $0 < \lambda < 1$*. If* $\phi'(x)/x$ *is integrable at 0, that is,* $\phi \in \mathcal{C}'$*, then the inequality (3.2) holds for* $b = 0$*.*

Proof. Let $t > 0$ and $0 < \lambda < 1$. Inequality (2.2) implies that

$$P\left(S_n^{\max} \geq t\right) \leq \frac{1}{t} \int_{[S_n^{\max} \geq t]} S_n dP \tag{3.3}$$
$$= \frac{1}{t} \int_0^\infty P[S_n^{\max} \geq t, S_n > s] ds$$
$$\leq \frac{1}{t} \int_0^{\lambda t} P[S_n^{\max} \geq t] ds + \frac{1}{t} \int_{\lambda t}^\infty P[S_n > s] ds$$
$$\leq \lambda P[S_n^{\max} \geq t] ds + \frac{\lambda}{t} \int_t^\infty P[S_n > \lambda s] ds.$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Rearranging the last inequality, we get that

$$\begin{aligned} P\left(S_n^{\max} \geq t\right) &\leq \frac{\lambda}{(1-\lambda)t} \int_t^{\infty} P(S_n > \lambda s)ds \\ &= \frac{\lambda}{(1-\lambda)t} E\left(\frac{S_n}{\lambda} - t\right)^+ \end{aligned}$$

for all $n \geq 1, t > 0$ and $0 < \lambda < 1$ proving the inequality (3.1) in Theorem 3.1. Let $b > 0$. Then

$$\begin{aligned} E[\phi(S_n^{\max})] &= \int_0^{\infty} \phi'(t)P(S_n^{\max} > t)dt \\ &= \int_0^{b} \phi'(t)P(S_n^{\max} > t)dt + \int_b^{\infty} \phi'(t)P(S_n^{\max} > t)dt \\ &\leq \phi(b) + \int_b^{\infty} \phi'(t)P(S_n^{\max} > t)dt \\ &\leq \phi(b) + \frac{\lambda}{1-\lambda} \int_b^{\infty} \frac{\phi'(t)}{t} \left[\int_t^{\infty} P(S_n > \lambda s)ds\right] dt \quad \text{(by (3.1))} \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_b^{\infty} \left(\int_b^{s} \frac{\phi'(t)}{t} dt\right) P(S_n > \lambda s)ds \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_b^{\infty} (\Phi_a'(s) - \Phi_a'(b))P(S_n > \lambda s)ds \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_{[S_n > \lambda b]} \left(\Phi_a\left(\frac{S_n}{\lambda}\right) - \Phi_a(b) - \Phi_a'(b)\left(\frac{S_n}{\lambda} - b\right)\right) dP \end{aligned}$$

for all $n \geq 1$, $b > 0$, $t > 0$, $0 < \lambda < 1$ and $a > 0$.The value of a can be chosen to be 0 if $\phi'(x)/x$ is integrable at 0. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

As special cases of the above result, we obtain the following inequalities by choosing $b = a$ in (3.2). Observe that $\Phi_a(a) = \Phi'_a(a) = 0$.

Theorem 3.2. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale and let $\phi \in \mathcal{C}$. Then*

$$E[\phi(S_n^{\max})] \leq \phi(a) + \frac{\lambda}{1-\lambda} E\left[\Phi_a\left(\frac{S_n}{\lambda}\right)\right] \tag{3.4}$$

for all $a \geq 0$, $0 < \lambda < 1$ and $n \geq 1$. Let $\lambda = \frac{1}{2}$ in (3.4). Then

$$E[\phi(S_n^{\max})] \leq \phi(a) + E[\Phi_a(2S_n)] \tag{3.5}$$

for all $a \geq 0$ and $n \geq 1$.

The following lemma is due to Alsmeyer and Rosler [2].

Lemma 3.3. *Let X and Y be nonnegative random variables satisfying the inequality*

$$t\, P(Y \geq t) \leq E(X I_{[Y \geq t]})$$

for all $t \geq 0$. Then

$$E[\phi(Y)] \leq E[\phi(q_\phi X)] \tag{3.6}$$

for any Orlicz function ϕ, where $q_\phi = \frac{p_\phi}{p_\phi - 1}$ and $p_\phi = \inf_{x>0} \frac{x\phi'(x)}{\phi(x)}$.

This lemma follows as an application of the Choquet decomposition

$$\phi(x) = \int_{[0,\infty)} (x-t)^+ \phi'(dt), \quad x \geq 0.$$

In view of the inequality (2.2), we can apply the above lemma to the random variables $X = S_n$ and $Y = S_n^{\max}$ to obtain the following result.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 3.4. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale and let $\phi \in \mathcal{C}$ with $p_\phi > 1$. Then*

$$E[\phi(S_n^{\max})] \leq E[\phi(q_\phi S_n)] \tag{3.7}$$

for all $n \geq 1$.

Theorem 3.5. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Suppose that the function $\phi \in \mathcal{C}$ is moderate. Then*

$$E[\phi(S_n^{\max})] \leq E[\phi(q_\phi S_n)] \leq q_\Phi^{p_\phi^*} E[\phi(S_n)]. \tag{3.8}$$

The first part of the inequality (3.8) of Theorem 3.5 follows from Theorem 3.4. The last part of the inequality follows from the observation that if $\phi \in \mathcal{C}$ is moderate, that is,

$$p_\phi^* = \sup_{x>0} \frac{x\phi'(x)}{\phi(x)} < \infty,$$

then

$$\phi(\lambda x) \leq \lambda^{p_\phi^*} \phi(x)$$

for all $\lambda > 1$ and $x > 0$ (see [2, equation (1.10)]).

Theorem 3.6. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Suppose ϕ is a nonnegative nondecreasing function on $[0, \infty)$ such that $\phi^{1/\gamma}$ is also nondecreasing and convex for some $\gamma > 1$. Then*

$$E[\phi(S_n^{\max})] \leq \left(\frac{\gamma}{\gamma - 1}\right)^\gamma E[\phi(S_n)]. \tag{3.9}$$

Proof. The inequality

$$\lambda P(S_n^{\max} \geq \lambda) \leq \int_{[S_n^{\max} \geq \lambda]} S_n dP$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

given in (2.2) implies that

$$E[(S_n^{\max})^p] \le \left(\frac{p}{p-1}\right)^p E(S_n^p), \quad p > 1 \tag{3.10}$$

by an application of the Holder inequality (cf. [4, p. 255]). Note that the sequence $\{[\phi(S_n)]^{1/\gamma}, n \ge 1\}$ is a nonnegative demisubmartingale by Lemma 2.1 of [5]. Applying the inequality (3.10) for the sequence $\{[\phi(S_n)]^{1/\gamma}, n \ge 1\}$ and choosing $p = \gamma$ in that inequality, we get that

$$E[\phi(S_n^{\max})] \le \left(\frac{\gamma}{\gamma-1}\right)^\gamma E[\phi(S_n)]. \tag{3.11}$$

for all $\gamma > 1$. □

Examples of functions ϕ satisfying the conditions stated in Theorem 3.6 are $\phi(x) = x^p[\log(1+x)]^r$ for $p > 1$ and $r \ge 0$ and $\phi(x) = e^{rx}$ for $r > 0$. Applying the result in Theorem 3.6 for the function $\phi(x) = e^{rx}, r > 0$, we obtain the following inequality.

Theorem 3.7. *Let* $\{S_n, n \ge 1\}$ *be a nonnegative demisubmartingale. Then*

$$E[e^{rS_n^{\max}}] \le eE[e^{rS_n}], \quad r > 0. \tag{3.12}$$

Proof. Applying the result stated in Theorem 3.6 to the function $\phi(x) = e^{rx}$, we get that

$$E[e^{rS_n^{\max}}] \le \left(\frac{\gamma}{\gamma-1}\right)^\gamma E[e^{rS_n}] \tag{3.13}$$

for any $\gamma > 1$. Let $\gamma \to \infty$. Then

$$\left(\frac{\gamma}{\gamma-1}\right)^\gamma \downarrow e$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

and we get that

$$E[e^{rS_n^{\max}}] \leq eE[e^{rS_n}], \quad r > 0. \tag{3.14}$$

□

The next result deals with maximal inequalities for functions $\phi \in \mathcal{C}$ which are k times differentiable with the k-th derivative $\phi^{(k)} \in \mathcal{C}$ for some $k \geq 1$.

Theorem 3.8. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Let $\phi \in \mathcal{C}$ which is differentiable k times with the k-th derivative $\phi^{(k)} \in \mathcal{C}$ for some $k \geq 1$.Then*

$$E[\phi(S_n^{\max})] \leq \left(\frac{k+1}{k}\right)^{k+1} E[\phi(S_n)]. \tag{3.15}$$

Proof. The proof follows the arguments given in [2] following the inequality (3.9). We present the proof here for completeness. Note that

$$\phi(x) = \int_{[0,\infty)} (x-t)^+ Q_\phi(dt),$$

where

$$Q_\phi(dt) = \phi'(0)\delta_0 + \phi'(dt)$$

and δ_0 is the Kronecker delta function. Hence, if $\phi' \in \mathcal{C}$, then

$$\begin{aligned}
\phi(x) &= \int_0^x \phi'(y)dy = \int_0^x \int_{[0,\infty)} (y-t)^+ Q_{\phi'}(dt)dy \\
&= \int_{[0,\infty)} \int_0^x (y-t)^+ dy Q_{\phi'}(dt) = \int_{[0,\infty)} \frac{((x-t)^+)^2}{2} Q_{\phi'}(dt).
\end{aligned} \tag{3.16}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

An inductive argument shows that

$$\phi(x) = \int_{[0,\infty)} \frac{((x-t)^+)^{k+1}}{(k+1)!} Q_{\phi^{(k)}}(dt) \tag{3.17}$$

for any $\phi \in \mathcal{C}$ such that $\phi^{(k)} \in \mathcal{C}$. Let

$$\phi_{k,t}(x) = \frac{((x-t)^+)^{k+1}}{(k+1)!}$$

for any $k \geq 1$ and $t \geq 0$. Note that the function $[\phi_{k,t}(x)]^{1/(k+1)}$ is nonnegative, convex and nondecreasing in x for any $k \geq 1$ and $t \geq 0$. Hence the process $\{[\phi_{k,t}(S_n)]^{1/(k+1)}, n \geq 1\}$ is a nonnegative demisubmartingale by [5]. Following the arguments given to prove (3.10), we obtain that

$$E(([\phi_{k,t}(S_n^{\max})]^{1/(k+1)})^{k+1}) \leq \left(\frac{k+1}{k}\right)^{k+1} E(([\phi_{k,t}(S_n)]^{1/(k+1)})^{k+1})$$

which implies that

$$E[\phi_{k,t}(S_n^{\max})] \leq \left(\frac{k+1}{k}\right)^{k+1} E[\phi_{k,t}(S_n)]. \tag{3.18}$$

Hence

$$\begin{aligned} E[\phi(S_n^{\max})] &= \int_{[0,\infty)} E[\phi_{k,t}(S_n^{\max})] Q_{\phi^{(k)}}(dt) \quad \text{(by (3.17))} \\ &\leq \left(\frac{k+1}{k}\right)^{k+1} \int_{[0,\infty)} E[\phi_{k,t}(S_n)] Q_{\phi^{(k)}}(dt) \quad \text{(by (3.18))} \\ &= \left(\frac{k+1}{k}\right)^{k+1} E[\phi(S_n)] \end{aligned} \tag{3.19}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

which proves the theorem. □

We now consider a special case of the maximal inequality derived in (3.2) of Theorem 3.1. Let $\phi(x) = x$. Then $\Phi_1(x) = x\log x - x + 1$ and $\Phi_1'(x) = \log x$. The inequality (3.2) reduces to

$$\begin{aligned} E[S_n^{\max}] &\le b + \frac{\lambda}{1-\lambda}\int_{[S_n>\lambda b]}\left(\frac{S_n}{\lambda}\log\frac{S_n}{\lambda} - \frac{S_n}{\lambda} + b - (\log b)\frac{S_n}{\lambda}\right)dP \\ &= b + \frac{\lambda}{1-\lambda}\int_{[S_n>\lambda b]}(S_n\log S_n - S_n(\log\lambda + \log b + 1) + \lambda b)dP \end{aligned}$$

for all $b > 0$ and $0 < \lambda < 1$. Let $b > 1$ and $\lambda = \frac{1}{b}$. Then we obtain the inequality

$$E\left[S_n^{\max}\right] \le b + \frac{b}{b-1}E\left[\int_1^{\max(S_n,1)}\log x\, dx\right], \quad b > 1, n \ge 1. \tag{3.20}$$

The value of b which minimizes the term on the right hand side of the equation (3.20) is

$$b^* = 1 + \left(E\left[\int_1^{\max(S_n,1)}\log x\, dx\right]\right)^{\frac{1}{2}}$$

and hence

$$E(S_n^{\max}) \le \left(1 + E\left[\int_1^{\max(S_n,1)}\log x\, dx\right]^{\frac{1}{2}}\right)^2. \tag{3.21}$$

Since

$$\int_1^x \log y dy = x\log^+ x - (x-1), \quad x \ge 1,$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

the inequality (3.20) can be written in the form

$$E(S_n^{\max}) \le b + \frac{b}{b-1}(E(S_n \log^+ S_n) - E(S_n - 1)^+), \quad b > 1, n \ge 1. \tag{3.22}$$

Let $b = E(S_n - 1)^+$ in the equation (3.22). Then we get the maximal inequality

$$E(S_n^{\max}) \le \frac{1 + E(S_n - 1)^+}{E(S_n - 1)^+} E(S_n \log^+ S_n). \tag{3.23}$$

If we choose $b = e$ in the equation (3.22), then we get the maximal inequality

$$E(S_n^{\max}) \le e + \frac{e}{e-1}(E(S_n \log^+ S_n) - E(S_n - 1)^+), \quad b > 1, n \ge 1. \tag{3.24}$$

This inequality gives a better bound than the bound obtained as a consequence of the result stated in Theorem 2.5 (cf. [16]) if $E(S_n - 1)^+ \ge e - 2$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

4. Inequalities for Dominated Demisubmartingales

Let $M_0 = N_0 = 0$ and $\{M_n, n \geq 0\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that

$$E[(M_{n+1} - M_n)f(M_0, \ldots, M_n)|\zeta_n] \geq 0$$

for any nonnegative coordinatewise nondecreasing function f given a filtration $\{\zeta_n, n \geq 0\}$ contained in $\mathcal{F}$. Then the sequence $\{M_n, n \geq 0\}$ is said to be a *strong demisubmartingale* with respect to the filtration $\{\zeta_n, n \geq 0\}$. It is obvious that a strong demisubmartingale is a demisubmartingale in the sense discused earlier.

Definition 4.1. *Let $M_0 = 0 = N_0$. Suppose $\{M_n, n \geq 0\}$ is a strong demisubmartingale with respect to the filtration generated by a demisubmartingale $\{N_n, n \geq 0\}$. The strong demisubmartingale $\{M_n, n \geq 0\}$ is said to be weakly dominated by the demisubmartingale $\{N_n, n \geq 0\}$ if for every nondecreasing convex function $\phi : \mathbb{R}_+ \to \mathbb{R}$, and for any nonnegative coordinatewise nondecreasing function $f : \mathbb{R}^{2n} \to \mathbb{R}$,*

$$E[(\phi(|e_n|) - \phi(|d_n|))f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1}) \\ |N_0, \ldots, N_{n-1}] \geq 0 \quad a.s., \tag{4.1}$$

for all $n \geq 1$ where $d_n = M_n - M_{n-1}$ and $e_n = N_n - N_{n-1}$. We write $M \ll N$ in such a case.

In analogy with the inequalities for dominated martingales developed in [12], we will now prove an inequality for domination between a strong demisubmartingale and a demisubmartingale.

Define the functions $u_{<2}(x, y)$ and $u_{>2}(x, y)$ as in Section 2.1 of [12] for $(x, y) \in \mathbb{R}^2$. We now state a weak-type inequality between dominated demisubmartingales.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 4.2. *Suppose* $\{M_n, n \geq 0\}$ *is a strong demisubmartingale with respect to the filtration generated by the sequence* $\{N_n, n \geq 0\}$ *which is a demisubmartingale. Further suppose that* $M \ll N$*. Then, for any* $\lambda > 0$,

$$\lambda \ \ P(|M_n| \geq \lambda) \leq 6 \ \ E|N_n|, \quad n \geq 0. \tag{4.2}$$

We will at first prove a Lemma which will be used to prove Theorem 4.2.

Lemma 4.3. *Suppose* $\{M_n, n \geq 0\}$ *is a strong demisubmartingale with respect to the filtration generated by the sequence* $\{N_n, n \geq 0\}$ *which is a demisubmartingale. Further suppose that* $M \ll N$*. Then*

$$\begin{aligned} E[u_{<2}(M_n, N_n) f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \\ \geq E[u_{<2}(M_{n-1}, N_{n-1}) f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} E[u_{>2}(M_n, N_n) f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \\ \geq E[u_{>2}(M_{n-1}, N_{n-1}) f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \end{aligned} \tag{4.4}$$

for any nonnegative coordinatewise nondecreasing function $f : \mathbb{R}^{2n} \to \mathbb{R}, n \geq 1.$

Proof. Define $u(x, y)$ where $u = u_{<2}$ or $u = u_{>2}$ as in Section 2.1 of [12]. From the arguments given in [12], it follows that there exist a nonnegative function $A(x, y)$ nondecreasing in x and a nonnegative function $B(x, y)$ nondecreasing in y and a convex nondecreasing function $\phi_{x,y}(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, such that, for any h and k,

$$u(x, y) + A(x, y)h + B(x, y)k + \phi_{x,y}(|k|) - \phi_{x,y}(|h|) \leq u(x + h, y + k). \tag{4.5}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Let $x = M_{n-1}, y = N_{n-1}, h = d_n$ and $k = e_n$. Then, it follows that

$$(4.6)\quad u(M_{n-1}, N_{n-1}) + A(M_{n-1}, N_{n-1})d_n + B(M_{n-1}, N_{n-1})e_n + \phi_{M_{n-1},N_{n-1}}(|e_n|) - \phi_{M_{n-1},N_{n-1}}(|d_n|) \leq u(M_{n-1} + d_n, N_{n-1} + e_n) = u(M_n, N_n).$$

Note that,

$$E[A(M_{n-1}, N_{n-1})d_n f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})|N_0, \dots, N_{n-1}] \geq 0 \text{ a.s.}$$

from the fact that $\{M_n, n \geq 0\}$ is a strong demisubmartingale with respect to the filtration generated by the process $\{N_n, n \geq 0\}$ and that the function

$$A(x_{n-1}, y_{n-1})f(x_0, \dots, x_{n-1}; y_0, \dots, y_{n-1})$$

is a nonnegative coordinatewise nondecreasing function in $x_0, \dots, x_{n-1}$ for any fixed $y_0, \dots, y_{n-1}$. Taking expectation on both sides of the above inequality, we get that

$$(4.7)\qquad E[A(M_{n-1}, N_{n-1})d_n f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \geq 0.$$

Similarly we get that

$$(4.8)\qquad E[B(M_{n-1}, N_{n-1})d_n f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \geq 0.$$

Since the sequence $\{M_n, n \geq 0\}$ is dominated by the sequence $\{N_n, n \geq 0\}$, it follows that

$$(4.9)\quad E[(\phi_{M_{n-1},N_{n-1}}(|e_n|) - \phi_{M_{n-1},N_{n-1}}(|d_n|)) \times f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \geq 0$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

by taking expectation on both sides of (4.1). Combining the relations (4.6) to (4.9), we get that

$$E[u(M_n, N_n)f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \geq E[u(M_{n-1}, N_{n-1})f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})]. \tag{4.10}$$

□

Remark 2. Let $f \equiv 1$. Repeated application of the inequality obtained in Lemma 4.2 shows that

$$E[u(M_n, N_n)] \geq E[u(M_0, N_0)] = 0. \tag{4.11}$$

Proof of Theorem 4.2. Let

$$v(x, y) = 18 \ |y| - I\left[|x| \geq \frac{1}{3}\right].$$

It can be checked that (cf. [12])

$$v(x, y) \geq u_{<2}(x, y). \tag{4.12}$$

Let $\lambda > 0$. It is easy to see that the strong demisubmartingale $\{\frac{M_n}{3\lambda}, n \geq 0\}$ is weakly dominated by the demisubmartingale $\{\frac{N_n}{3\lambda}, n \geq 0\}$. In view of the inequalities (4.7) and (4.8), we get that

$$\begin{aligned} 6 \ E|N_n| - \lambda \ P(|M_n| \geq \lambda) &= \lambda E\left[v\left(\frac{M_n}{3\lambda}, \frac{N_n}{3\lambda}\right)\right] \\ &\geq \lambda E\left[u_{<2}\left(\frac{M_n}{3\lambda}, \frac{N_n}{3\lambda}\right)\right] \geq 0 \end{aligned} \tag{4.13}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

which proves the inequality

$$\lambda \ P(|M_n| \geq \lambda) \leq 6 \ \ E|N_n|, n \geq 0. \tag{4.14}$$

□

Remark 3. It would be interesting if the other results in [12] can be extended in a similar fashion for dominated demisubmartingales. We do not discuss them here.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

5. $N-$demimartingales and $N-$demisupermartingales

The concept of a negative demimartingale, which is now termed as $N-$demimartingale, was introduced in [14] and in [6]. It can be shown that the partial sum $\{S_n, n \geq 1\}$ of mean zero negatively associated random variables $\{X_j, j \geq 1\}$ is a $N-$demimartingale (cf. [6]). This can be seen from the observation

$$E[(S_{n+1} - S_n))f(S_1, \dots, S_n)] = E(X_{n+1} f(S_1, \dots, S_n)] \leq 0$$

for any coordinatewise nondecreasing function f and from the observation that increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated (cf. [10]) and the fact that $\{X_n, n \geq 1\}$ are negatively associated. Suppose U_n is a U-statistic based on negatively associated random variables $\{X_n, n \geq 1\}$ and the product kernel $h(x_1, \dots, x_m) = \prod_{i=1}^{m} g(x_i)$ for some nondecreasing function $g(\cdot)$ with $E(g(X_i)) = 0, 1 \leq i \leq n$. Let

$$T_n = \frac{n!}{(n-m)!m!} U_n, n \geq m.$$

Then the sequence $\{T_n, n \geq m\}$ is a $N-$demimartingale. For a proof, see [6].

The following theorem is due to Christofides [6].

Theorem 5.1. *Suppose $\{S_n, n \geq 1\}$ is a $N-$demisupermartingale. Then, for any $\lambda > 0$,*

$$\lambda\, P\left[\max_{1\leq k\leq n} S_k \geq \lambda\right] \leq E(S_1) - \int_{[\max_{1\leq k\leq n} S_k \geq \lambda]} S_n dP.$$

In particular, the following maximal inequality holds for a nonnegative $N-$ demisupermartingale.

vol. 8, iss. 4, art. 112, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 5.2. *Suppose $\{S_n, n \geq 1\}$ is a nonnegative $N-$demisupermartingale. Then, for any $\lambda > 0$,*

$$\lambda P\left(\max_{1\leq k\leq n} S_k \geq \lambda\right) \leq E(S_1)$$

and

$$\lambda P\left(\max_{k\geq n} S_k \geq \lambda\right) \leq E(S_n).$$

Prakasa Rao [15] gives a Chow type maximal inequality for $N-$demimartingales.

Suppose ϕ is a right continuous decreasing function on $(0, \infty)$ satisfying the condition

$$\lim_{t\to\infty} \phi(t) = 0.$$

Further suppose that ϕ is also integrable on any finite interval $(0, x)$. Let

$$\Phi(x) = \int_0^x \phi(t)dt, \quad x \geq 0.$$

Then the function $\Phi(x)$ is a nonnegative nondecreasing function such that $\Phi(0) = 0$. Further suppose that $\Phi(\infty) = \infty$. Such a function is called a *concave Young function*. Properties of such functions are given in [1]. An example of such a function is $\Phi(x) = x^p$, $0 < p < 1$. Christofides [6] obtained the following maximal inequality.

Theorem 5.3. *Let $\{S_n, n \geq 1\}$ be a nonnegative $N-$demisupermartingale. Let $\Phi(x)$ be a concave Young function and define $\psi(x) = \Phi(x) - x\phi(x)$. Then*

$$E[\psi(S_n^{\max})] \leq E[\Phi(S_1)]. \tag{5.1}$$

Furthermore, if

$$\limsup_{x\to\infty} \frac{x\phi(x)}{\Phi(x)} < 1,$$

journal of inequalities in pure and applied mathematics

issn: 1443-5756

then

$$E[\Phi(S_n^{\max})] \leq c_\Phi(1 + E[\Phi(S_1)]) \tag{5.2}$$

for some constant c_Φ *depending only on the function* Φ.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

6. Remarks

It would be interesting to find whether an upcrossing inequality can be obtained for $N-$ demimartingales and then derive an almost sure convergence theorem for $N-$demisupermartingales. Such results are known for demisubmartingales (see Theorem 2.3).

Wood [18] extended the notion of a discrete time parameter demisubmartingale to a continuous time parameter demisubmartingale following the ideas in [7]. A stochastic process $\{S_t, 0 \leq t \leq T\}$ is said to be a demisubmartingale if for every set $\{t_j, 0 \leq j \leq k\}$, $k \geq 1$ contained in the interval $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_k = T$, the sequence $\{S_{t_j}, 0 \leq j \leq k\}$ forms a demisubmartingale.

Suppose that a stochastic process $\{S_t, 0 \leq t \leq T\}$ is a demisubmartingale in the sense defined above. One can assume that the process is separable in the sense of [7]. It is easy to check that $E(S_\alpha) \leq E(S_\beta)$ whenever $\alpha \leq \beta$ since the constant function $f \equiv 1$ is a nonnegative nondecreasing function and

$$E[(S_\beta - S_\alpha)f(S_0, S_\alpha)] \geq 0.$$

Furthermore, for any $\lambda > 0$,

$$\lambda P\left(\sup_{0\leq t\leq T} S_t \geq \lambda\right) \leq \int_{[\sup_{0\leq t\leq T} S_t \geq \lambda]} S_T dP$$

and

$$\lambda P\left(\inf_{0\leq t\leq T} S_t \leq \lambda\right) \geq \int_{[\inf_{0\leq t\leq T} S_t \leq \lambda]} S_T dP - E(S_T) + E(S_0).$$

In analogy with the above remarks, a continuous time parameter stochastic process $\{S_t, 0 \leq t \leq T\}$ is said to be a $N-$demisupermartingale if for every set $\{t_j, 0 \leq j \leq k\}, k \geq 1$ contained in the interval $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_k = T$, the

journal of inequalities in pure and applied mathematics
issn: 1443-5756

sequence $\{S_{t_j}, 0 \leq j \leq k\}$ forms a $N-$demisupermartingale. Theorems 5.1 and 5.2 can be extended to continuous time parameter $N-$demisupermartingales.

Results on maximal inequalities stated and proved in this paper for demisubmaqrtingales and $N-$sdemisupermartingales generalize maximal inequalities for submartingales and supermartingales respectively. Recall that the class of submartingales is a *proper* subclass of demisubmartingales and the class of supermartingales is a *proper* subclass of $N-$ demisupermartingales with respect to the natural choice of σ-algebras..

vol. 8, iss. 4, art. 112, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] N.K. AGBEKO, Concave function inequalities for sub-(super-)martingales, *Ann. Univ. Sci. Budapest, sect. Math.*, **29** (1986), 9–17.

[2] G. ALSMEYER AND U. ROSLER, Maximal ϕ-inequalities for nonegative submartingales, *Theory Probab. Appl.*, **50** (2006), 118–128.

[3] Y.S. CHOW, A martingale inequality and the law of large numbers, *Proc. Amer. Math. Soc.*, **11** (1960), 107–111.

[4] Y.S. CHOW AND H. TEICHER, *Probability Theory: Independence, Interchangeability and Martingales*, Springer-verlag, New York (1997).

[5] T.C. CHRISTOFIDES, Maximal inequalities for demimartingales and a strong law of large numbers, *Statist. and Probab. Lett.*, **50** (2000), 357–363.

[6] T.C. CHRISTOFIDES, Maximal inequalities for N-demimartingales, *Arch. Inequal. Appl.*, **1** (2003), 387–397.

[7] J.L. DOOB, *Stochastic Processes*, Wiley, New York (1953).

[8] A. GARSIA, On a convex function inequality for submartingales, *Ann. Probab.*, **1** (1973), 171–174.

[9] P. HARREMOËS, Some new maximal inequalities, preprint, *Centrum voor wiskunde en Informatica*, Amsterdam (2007).

[10] K. JOAG-DEV AND F. PROSCHAN, Negative association of random variables with applications, *Ann. Statist.*, **11** (1983), 286–295.

[11] C.M. NEWMAN AND A.L. WRIGHT, Associated random variables and martingale inequalities, *Z. Wahrsch. Verw. Geb.*, **59** (1982), 361–371.

journal of inequalities in pure and applied mathematics

issn: 1443-5756

[12] A. OSEKOWSKI, Inequalities for dominated martingales, *Bernoulli*, **13** (2007), 54–79.

[13] B.L.S. PRAKASA RAO, Whittle type inequality for demisubmartingales, *Proc. Amer. Math. Soc.*, **130** (2002), 3719–3724.

[14] B.L.S. PRAKASA RAO, Negatively associated random variables and inequalities for negative demisubmartingales, preprint, *Indian Statistical Institute*, New Delhi (2002).

[15] B.L.S. PRAKASA RAO, On some inequalities for N-demimartingales, *J. Indian Soc. Agricultural Statist.*, **57** (2004), 208–216.

[16] JIANFENG WANG, Maximal inequalities for associated random variables and demimartingales, *Statist. and Probab. Lett.*, **66** (2004), 347–354.

[17] P. WHITTLE, Refinements of Kolmogorov's inequality, *Theory Probab. Appl.*, **14** (1969), 310–311.

[18] T.E. WOOD, Ph.D. Thesis, University of Virginia (1984).

journal of inequalities in pure and applied mathematics

issn: 1443-5756