

Journal of Inequalities in Pure and Applied Mathematics

FOUR INEQUALITIES SIMILAR TO HARDY-HILBERT'S INTEGRAL INEQUALITY

W.T. SULAIMAN

College of Computer Sciences and Mathematics
University of Mosul, Iraq.

EMail: waadsulaiman@hotmail.com

©2000 Victoria University
ISSN (electronic): 1443-5756
285-05



volume 7, issue 2, article 76,
2006.

*Received 22 September, 2005;
accepted 05 January, 2006.*

Communicated by: B. Yang

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

Four new different types of inequalities similar to Hardy-Hilbert's inequality are given.

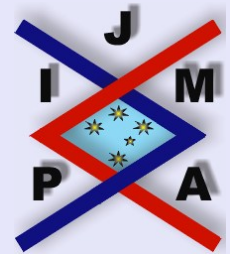
2000 Mathematics Subject Classification: 26D15.

Key words: Hardy-Hilbert integral inequality.

The author is grateful to the referee who read through the paper very carefully and correct many typing mistakes.

Contents

1	Introduction	3
2	New Inequalities	6
	References	



Four Inequalities Similar To Hardy-Hilbert's Integral Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 2 of 18

1. Introduction

Suppose that f and g are real functions, such that $0 < \int_0^\infty f^2(t)dt < \infty$ and $0 < \int_0^\infty g^2(t)dt < \infty$, then

$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} < \pi \left(\int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt \right)^{\frac{1}{2}},$$

where π is best possible. If (a_n) and (b_n) are sequences of real numbers such that $0 < \sum_{n=1}^\infty a_n^2 < \infty$ and $0 < \sum_{n=1}^\infty b_n^2 < \infty$, then

$$(1.2) \quad \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi \left(\sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2 \right)^{\frac{1}{2}}.$$

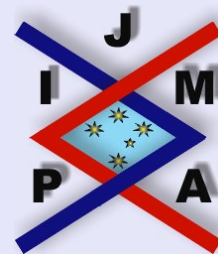
The inequalities (1.1) and (1.2) are called Hilbert's inequalities. These inequalities play an important role in analysis (cf. [1, Chap. 9]). In their recent papers Hu [5] and Gao [3] gave two distinct improvements of (1.1) and Gao [4] gave a strengthened version of (1.2).

The following definitions are given:

$$\varphi_\lambda(r) = \frac{r + \lambda - 2}{r} \quad (r = p, q), \quad k_\lambda(p) = B(\varphi_\lambda(p), \varphi_\lambda(q)),$$

and B is the beta function.

Recently, by introducing some parameters, Yang and Debnath [2] gave the following extensions:



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 3 of 18

Theorem A. If $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, such that

$$0 < \int_0^\infty t^{1-\lambda} f^p(t) dt < \infty \quad \text{and} \quad 0 < \int_0^\infty t^{1-\lambda} g^q(t) dt < \infty,$$

then

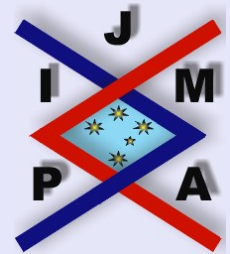
$$(1.3) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax + By)^\lambda} dx dy < \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left(\int_0^\infty x^{1-\lambda} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{1-\lambda} g^q(y) dy \right)^{\frac{1}{q}},$$

where the constant factor $[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}]$ is the best possible.

Theorem B. If $f \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $A, B > 0$ such that $0 < \int_0^\infty t^{1-\lambda} f^p(t) dt < \infty$, then

$$(1.4) \quad \int_0^\infty y^{(\lambda-1)(p-1)} \left(\int_0^\infty \frac{f(x)}{(Ax + By)^\lambda} dx \right)^p dy < \left(\frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \right)^p \int_0^\infty x^{1-\lambda} f^p(x) dx,$$

where the constant factor $[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}]^p$ is the best possible. The inequalities (1.3) and (1.4) are equivalent.



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 4 of 18

Theorem C. If $a_n, b_n > 0$ ($n \in \mathbb{N}$), $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda < 2$, $A, B > 0$ such that

$$0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty,$$

then

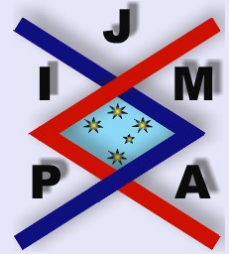
$$(1.5) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(Am + Bn)^\lambda} < \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left(\sum_{n=1}^{\infty} n^{1-\lambda} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{1-\lambda} b_n^q \right)^{\frac{1}{q}},$$

where the constant factor $[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}]$ is the best possible.

Theorem D. If $a_n \geq 0$ ($n \in \mathbb{N}$), $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, $A, B > 0$ such that $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty$, then

$$(1.6) \quad \sum_{n=1}^{\infty} n^{(\lambda-1)(p-1)} \left(\sum_{m=1}^{\infty} \frac{a_m}{(Am + Bn)^\lambda} \right)^p < \left(\frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \right)^p \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p,$$

where the constant factor $[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}]^p$ is the best possible. The inequalities (1.5) and (1.6) are equivalent.



Four Inequalities Similar To Hardy-Hilbert's Integral Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 5 of 18

2. New Inequalities

The aim of this paper is to give the following results:

Theorem 2.1. Let $\ln f$, $\ln g$ be convex for nonnegative functions f and g such that $f(0) = g(0) = 0$, $f(\infty) = g(\infty) = \infty$, $f'(s) \geq 0$, $g'(s) \geq 0$, $s \in \{x^p, y^q\}$. Let $\lambda > \max\{p, q\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$0 < \int_0^\infty \frac{t^{-p^2/q^2} [f(t^p)]^{2-\lambda+p/q}}{[f'(t)]^{\frac{p}{q}}} dt < \infty,$$

$$0 < \int_0^\infty \frac{t^{-q^2/p^2} [g(t^q)]^{2-\lambda+q/p}}{[g'(t)]^{\frac{q}{p}}} dt < \infty,$$

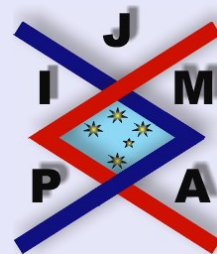
then we have

$$(2.1) \quad \int_0^\infty \int_0^\infty \frac{f(xy)g(xy)}{(f(x^p) + g(y^q))^\lambda} dx dy$$

$$\leq \frac{1}{\sqrt[p]{p}\sqrt[q]{q}} B^{\frac{1}{p}}(p, \lambda - p) B^{\frac{1}{q}}(q, \lambda - q)$$

$$\times \left(\int_0^\infty \frac{t^{-p^2/q^2} [f(t^p)]^{2-\lambda+p/q}}{[f'(t)]^{\frac{p}{q}}} dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty \frac{t^{-q^2/p^2} [g(t^q)]^{2-\lambda+q/p}}{[g'(t)]^{\frac{q}{p}}} dt \right)^{\frac{1}{q}}.$$



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 6 of 18

Proof. Since $\ln f$ is convex and $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$, then

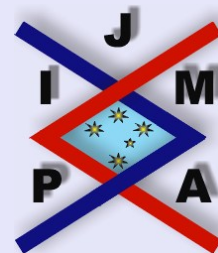
$$f(xy) = e^{\ln f(xy)} \leq e^{\ln f\left(\frac{x^p}{p} + \frac{y^q}{q}\right)} \leq e^{\frac{\ln f(x^p)}{p} + \frac{\ln f(y^q)}{q}} = f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q).$$

Therefore, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(xy)g(xy)}{(f(x^p) + g(y^q))^\lambda} dx dy \\ & \leq \int_0^\infty \int_0^\infty \frac{f^{\frac{1}{p}}(x^p) g^{\frac{1}{q}}(y^q) \frac{[g'(y^q)]^{\frac{1}{p}} y^{\frac{q-1}{p}}}{[f'(x^p)]^{\frac{1}{q}} x^{\frac{p-1}{q}}}}{(f(x^p) + g(y^q))^{\frac{\lambda}{p}}} \frac{f^{\frac{1}{q}}(y^q) g^{\frac{1}{p}}(x^p) \frac{[f'(x^p)]^{\frac{1}{q}} x^{\frac{p-1}{q}}}{[g'(y^q)]^{\frac{1}{p}} y^{\frac{q-1}{p}}}}{(f(x^p) + g(y^q))^{\frac{\lambda}{q}}} dx dy \\ & \leq \left(\int_0^\infty \int_0^\infty \frac{f(x^p) g^{p/q}(y^q) g'(y^q) y^{q-1}}{x^{(p-1)p/q} [f'(x^p)]^{\frac{p}{q}} (f(x^p) + g(y^q))^\lambda} dx dy \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^\infty \int_0^\infty \frac{f(y^q) g^{q/p}(x^p) f'(x^p) x^{p-1}}{y^{(q-1)q/p} [g'(y^q)]^{\frac{q}{p}} (f(x^p) + g(y^q))^\lambda} dx dy \right)^{\frac{1}{q}} \\ & = M^{\frac{1}{p}} N^{\frac{1}{q}}, \quad \text{say.} \end{aligned}$$

Then

$$M = \frac{1}{q} \int_0^\infty \frac{x^{(1-p)p/q} [f(x^p)]^{2-\lambda+p/q}}{[f'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{\left(\frac{g(y^q)}{f(x^p)}\right)^{\frac{p}{q}} g'(y^q) \frac{qy^{q-1}}{f(x^p)}}{\left(1 + \frac{g(y^q)}{f(x^p)}\right)^\lambda} dy$$



Four Inequalities Similar To Hardy-Hilbert's Integral Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 7 of 18

$$\begin{aligned}
&= \frac{1}{q} \int_0^\infty \frac{x^{-p^2/q^2} [f(x^p)]^{2-\lambda+p/q}}{[f'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{u^{p/q}}{(1+u)^\lambda} du \\
&= \frac{1}{q} B(p, \lambda - p) \int_0^\infty \frac{x^{-p^2/q^2} [f(x^p)]^{2-\lambda+p/q}}{[f'(x)]^{\frac{p}{q}}} dx.
\end{aligned}$$

Similarly,

$$N = \frac{1}{p} B(q, \lambda - q) \int_0^\infty \frac{y^{-q^2/p^2} [g(y^q)]^{2-\lambda+q/p}}{[g'(y)]^{\frac{q}{p}}} dy.$$

Therefore

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(xy)g(xy)}{(f(x^p) + g(y^q))^\lambda} dx dy \\
&\leq \frac{1}{\sqrt[p]{p} \sqrt[q]{q}} B^{\frac{1}{p}}(p, \lambda - p) B^{\frac{1}{q}}(q, \lambda - q) \\
&\quad \times \left(\int_0^\infty t^{-p^2/q^2} \frac{[f(t^p)]^{2-\lambda+p/q}}{[f'(t)]^{\frac{p}{q}}} dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty t^{-q^2/p^2} \frac{[g(t^q)]^{2-\lambda+q/p}}{[g'(t)]^{\frac{q}{p}}} dt \right)^{\frac{1}{q}}.
\end{aligned}$$

□

Theorem 2.2. Let f, g, h be nonnegative functions, $h(x, y)$ is homogeneous of order n such that $h(0, 1) = h(1, 0) = 0$, $h(\infty, 1) = h(1, \infty) = \infty$. Let $p > 1$,



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 8 of 18

$$\frac{1}{p} + \frac{1}{q} = 1,$$

$$0 < 1 + \mu - r < \lambda, \quad r \in \left\{ \frac{p}{q}, \frac{q}{p} \right\}, \quad h_x(x, y) \geq 0, \quad h_y(x, y) \geq 0,$$

where $h_x = dh/dx$, $0 < \int_0^\infty t^{p/q} f^p(t) dt < \infty$, $0 < \int_0^\infty t^{q/p} g^q(t) dt < \infty$, then

$$(2.2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)h^\mu(x, y)}{(1 + h(x, y))^\lambda} dx dy$$

$$\leq \frac{1}{n} B^{\frac{1}{p}} \left(1 + \mu - \frac{p}{q}, \lambda - 1 - \mu + \frac{p}{q} \right) B^{\frac{1}{q}} \left(1 + \mu - \frac{q}{p}, \lambda - 1 - \mu - \frac{q}{p} \right)$$

$$\times \left(\int_0^\infty t^{\frac{p}{q-1}} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_0^\infty t^{\frac{q}{p-1}} g^q(t) dt \right)^{\frac{1}{q}}.$$

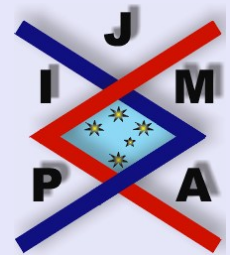
Proof. Observe that

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)h^\mu(x, y)}{(1 + h(x, y))^\lambda} dx dy$$

$$\leq \left(\int_0^\infty \int_0^\infty \frac{f^p(x)h^\mu(x, y)h_y(x, y)}{h_x^{p/q}(x, y)(1 + h(x, y))^\lambda} dx dy \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty \int_0^\infty \frac{g^q(y)h^\mu(x, y)h_x(x, y)}{h_y^{q/p}(x, y)(1 + h(x, y))^\lambda} dx dy \right)^{\frac{1}{q}}$$

$$= M^{\frac{1}{p}} N^{\frac{1}{q}}, \quad \text{say.}$$



**Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality**

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 9 of 18

Then

$$M = \int_0^\infty f^p(x) dx \int_0^\infty \frac{h^\mu(x, y) h_y(x, y)}{h_x^{p/q}(x, y) (1 + h(x, y))^\lambda} dy.$$

Let $y = xv$, $dy = xdv$ and hence

$$h_y(x, y) = \frac{dh(x, xv)}{dy} = x^n \frac{dh(1, v)}{dy} = x^{n-1} \frac{dh(1, v)}{dv} = x^{n-1} h_v(1, v),$$

$$h_x(x, y) = \frac{dh(x, xv)}{dx} = \frac{d}{dx} x^n h(1, v) = n x^{n-1} h(1, v),$$

therefore

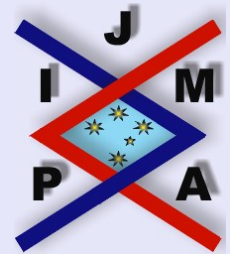
$$\begin{aligned} M &= \frac{1}{n^{p/q}} \int_0^\infty x^{p/q-1} f^p(x) dx \int_0^\infty \frac{[x^n h(1, v)]^{\mu-p/q} x^n h_v(1, v)}{(1 + x^n h(1, v))^\lambda} dv \\ &= \frac{1}{n^{p/q}} B\left(1 + \mu - \frac{p}{q}, \lambda - 1 - \mu + \frac{p}{q}\right) \int_0^\infty x^{p/q-1} f^p(x) dx. \end{aligned}$$

Similarly,

$$N = \frac{1}{n^{q/p}} B\left(1 + \mu - \frac{q}{p}, \lambda - 1 - \mu + \frac{q}{p}\right) \int_0^\infty y^{q/p-1} g^q(y) dy.$$

This implies

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)h^\mu(x, y)}{(1 + h(x, y))^\lambda} dx dy$$



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 10 of 18

$$\leq \frac{1}{n} B^{\frac{1}{p}} \left(1 + \mu - \frac{p}{q}, \lambda - 1 - \mu + \frac{p}{q} \right) B^{\frac{1}{q}} \left(1 + \mu - \frac{q}{p}, \lambda - 1 - \mu + \frac{q}{p} \right) \\ \times \left(\int_0^\infty t^{p/q-1} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_0^\infty t^{q/p-1} g^q(t) dt \right)^{\frac{1}{q}}.$$

□

The following lemma is needed for the coming result.

Lemma 2.3. *Let $s \geq 1$, $0 < 1 + \mu \leq \min \{ \alpha, \lambda \}$ and define*

$$f(s) = s^{-\alpha} \int_0^s \frac{t^\mu}{(1+t)^\lambda} dt,$$

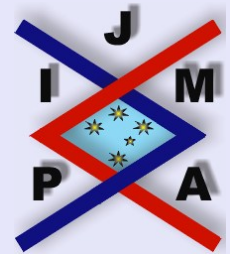
then $f(s) \leq f(1)$.

Proof. We have

$$f'(s) = s^{-\alpha} \frac{s^\mu}{(1+s)^\lambda} + \int_0^s \frac{t^\mu}{(1+t)^\lambda} dt (-\alpha) s^{-\alpha-1} \\ \leq \frac{s^{\mu-\alpha}}{(1+s)^\lambda} - \frac{\alpha s^{-\alpha-1}}{(1+s)^\lambda} \int_0^s t^\mu dt \\ = \frac{s^{\mu-\alpha}}{(1+s)^\lambda} \left(1 - \frac{\alpha}{1+\mu} \right) \leq 0.$$

This shows that f is nonincreasing and hence $f(s) \leq f(1)$.

□



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 11 of 18

Theorem 2.4. Let f, g, F, G be nonnegative functions such that $F(s) = \int_0^s f(t)dt$, $G(s) = \int_0^s g(t)dt$, let, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $(1 - \lambda/2)r \leq \lambda/2 + \alpha \leq 2\alpha$, $r \in \left\{ \frac{p}{q}, \frac{q}{p} \right\}$,

$$0 < \int_0^x (x-t)t^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^{p-1}(t)f(t)dt < \infty,$$

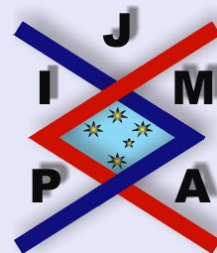
$$0 < \int_0^x (x-t)t^{(1-\lambda/2)q/p-\lambda/2-\alpha} G^{q-1}(t)g(t)dt < \infty,$$

then

$$(2.3) \quad \int_0^x \int_0^x \frac{F(s)G(t)}{(s+t)^\lambda} dsdt \leq \frac{\sqrt[p]{p}\sqrt[q]{q}}{2} x^\alpha B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \\ \times \left(\int_0^x (x-t)t^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^{p-1}(t)f(t)dt \right)^{\frac{1}{p}} \\ \times \left(\int_0^x (x-t)t^{(1-\lambda/2)q/p-\lambda/2-\alpha} G^{q-1}(t)g(t)dt \right)^{\frac{1}{q}}.$$

Proof. Observe that

$$\int_0^x \int_0^x \frac{F(s)G(t)}{(s+t)^\lambda} dsdt \\ = \int_0^x \int_0^x \frac{F(s) \left(\frac{t^{1/p}}{s^{1/q}} \right)^{\lambda/2-1}}{(s+t)^{\lambda/p}} \cdot \frac{G(t) \left(\frac{s^{1/q}}{t^{1/p}} \right)^{\lambda/2-1}}{(s+t)^{\lambda/q}} dsdt$$



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 12 of 18

$$\leq \left(\int_0^x \int_0^x \frac{F^p(s)t^{\lambda/2-1}}{(s+t)^\lambda s^{(\lambda/2-1)p/q}} dsdt \right)^{\frac{1}{p}} \left(\int_0^x \int_0^x \frac{G^q(t)s^{\lambda/2-1}}{(s+t)^\lambda t^{(\lambda/2-1)q/p}} dsdt \right)^{\frac{1}{q}}$$

$$= M^{\frac{1}{p}} N^{\frac{1}{q}}, \quad \text{say.}$$

Then

$$M = \int_0^x s^{(1-\lambda/2)p/q-\lambda/2} F^p(s) ds \int_0^x \frac{\left(\frac{t}{s}\right)^{\lambda/2-1} \frac{1}{s}}{\left(1+\frac{t}{s}\right)^\lambda} dt$$

$$= \int_0^x s^{(1-\lambda/2)p/q-\lambda/2} F^p(s) ds \left(\frac{x}{s}\right)^\alpha \left(\frac{x}{s}\right)^{-\alpha} \int_0^{x/s} \frac{u^{\lambda/2-1}}{(1+u)^\lambda} du$$

$$\leq \int_0^x s^{(1-\lambda/2)p/q-\lambda/2} F^p(s) ds \left(\frac{x}{s}\right)^\alpha \int_0^1 \frac{u^{\lambda/2-1}}{(1+u)^\lambda} du$$

$$= \frac{x^\alpha}{2} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x s^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^p(s) ds,$$

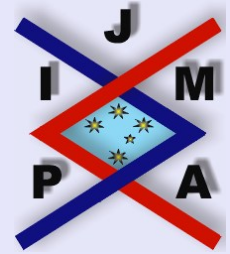
by virtue of the lemma.

As

$$F^p(s) = p \int_0^s F^{p-1}(u) f(u) du,$$

then

$$M = \frac{p}{2} x^\alpha B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x s^{(1-\lambda/2)p/q-\lambda/2-\alpha} ds \int_0^s F^{p-1}(u) f(u) du$$



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 13 of 18

$$\begin{aligned}
&= \frac{p}{2} x^\alpha B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x \int_0^s u^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^{p-1}(u) f(u) ds du \\
&= \frac{p}{2} x^\alpha B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x (x-s) s^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^{p-1}(s) f(s) ds.
\end{aligned}$$

Similarly,

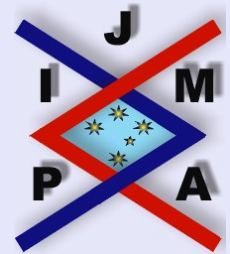
$$N = \frac{q}{2} x^\alpha B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x (x-t) t^{(1-\lambda/2)q/p-\lambda/2-\alpha} G^{q-1}(t) g(t) dt.$$

Therefore, we have

$$\begin{aligned}
&\int_0^x \int_0^x \frac{F(s)G(t)}{(s+t)^\lambda} ds dt \\
&\leq \frac{\sqrt[p]{p}\sqrt[q]{q}}{2} x^\alpha B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^x (x-t) t^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^{p-1}(t) f(t) dt\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^x (x-t) t^{(1-\lambda/2)q/p-\lambda/2-\alpha} G^{q-1}(t) g(t) dt\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Theorem 2.5. Let f, g be nonnegative functions, f is submultiplicative and g is concave nonincreasing, $f'(x), g'(y) \geq 0$, $f(0) = g(0) = 0$, $f(\infty) = g(\infty) = \infty$



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 14 of 18

$$\infty, p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < a + 1 < \lambda, 0 < b + 1 < \lambda,$$

$$0 < \int_0^\infty \frac{[f(x)]^{\mu p - bp/q} [g(x)]^{1+a-\lambda}}{[f'(x)]^{\frac{p}{q}}} dx < \infty,$$

$$0 < \int_0^\infty \frac{[g(y)]^{\mu q - aq/p} [f(y)]^{1+b-\lambda}}{[g'(y)]^{\frac{q}{p}}} dy < \infty,$$

then

$$(2.4) \quad \int_0^\infty \int_0^\infty \frac{f^\mu(xy)}{g^{\lambda/2}(xy)} dx dy$$

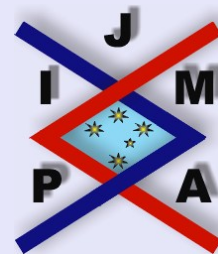
$$\leq 2^{\lambda/2} B^{\frac{1}{p}}(a+1, \lambda-a-1) B^{\frac{1}{q}}(b+1, \lambda-b-1)$$

$$\times \left(\int_0^\infty \frac{[f(t)]^{\mu p} [g(t)]^{1+a-\lambda-bp/q}}{[g'(t)]^{\frac{p}{q}}} dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty \frac{[f(t)]^{\mu q} [g(t)]^{1+b-\lambda-aq/p}}{[g'(t)]^{\frac{q}{p}}} dt \right)^{\frac{1}{q}}.$$

Proof. Since $\sqrt{xy} \leq \frac{x+y}{2}$, then

$$g(xy) = g\left((\sqrt{xy})^2\right) \geq (g(\sqrt{xy}))^2 \geq \left(g\left(\frac{x+y}{2}\right)\right)^2 \geq \left(\frac{g(x)+g(y)}{2}\right)^2,$$



Four Inequalities Similar To Hardy-Hilbert's Integral Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

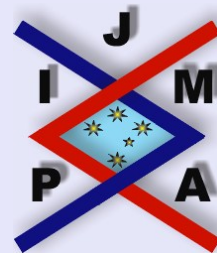
Page 15 of 18

and hence

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty \frac{f^\mu(xy)}{g^{\lambda/2}(xy)} dx dy \\
 & \leq 2^{\lambda/2} \int_0^\infty \int_0^\infty \frac{f^\mu(x) f^\mu(y)}{(g(x) + g(y))^\lambda} dx dy \\
 & = 2^{\lambda/2} \int_0^\infty \int_0^\infty \frac{f^\mu(x) \frac{[g(y)]^{a/p} [g'(y)]^{1/p}}{[g(x)]^{b/q} [g'(x)]^{1/q}}}{(g(x) + g(y))^{\frac{\lambda}{p}}} \cdot \frac{f^\mu(y) \frac{[g(x)]^{b/q}}{[g(y)]^{a/p}} \cdot \frac{[g'(x)]^{1/q}}{[g'(y)]^{1/p}}}{(g(x) + g(y))^{\frac{\lambda}{q}}} \\
 & \leq 2^{\lambda/2} \left(\int_0^\infty \int_0^\infty \frac{f^{\mu p}(x) g^a(y) g'(y)}{[g(x)]^{bp/q} [g'(x)]^{\frac{p}{q}} (g(x) + g(y))^\lambda} dx dy \right)^{\frac{1}{p}} \\
 & \quad \times \left(\int_0^\infty \int_0^\infty \frac{f^{\mu q}(y) g^b(x) g'(x)}{[g(y)]^{aq/p} [g'(y)]^{\frac{q}{p}} (g(x) + g(y))^\lambda} dx dy \right)^{\frac{1}{q}} \\
 & = 2^{\lambda/2} M^{\frac{1}{p}} N^{\frac{1}{q}}, \quad \text{say.}
 \end{aligned}$$

Then

$$\begin{aligned}
 M & = \int_0^\infty \frac{[f(x)]^{\mu p} [g(x)]^{1+a-\lambda-bp/q}}{[g'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{\left(\frac{g(y)}{g(x)}\right)^a \frac{g'(y)}{g(x)}}{\left(1 + \frac{g(y)}{g(x)}\right)^\lambda} dy \\
 & = B(a+1, \lambda-a-1) \int_0^\infty \frac{[f(x)]^{\mu p} [g(x)]^{1+a-\lambda-bp/q}}{[g'(x)]^{\frac{p}{q}}} dx.
 \end{aligned}$$



**Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality**

W.T. Sulaiman

Title Page

Contents



Go Back

Close

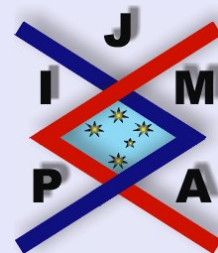
Quit

Page 16 of 18

Similarly, we can show that

$$N = B(b+1, \lambda - b - 1) \int_0^\infty \frac{[f(y)]^{\mu q} [g(y)]^{1+b-\lambda-aq/p}}{[g'(y)]^{\frac{q}{p}}} dy.$$

The result follows. □



**Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality**

W.T. Sulaiman

Title Page

Contents



Go Back

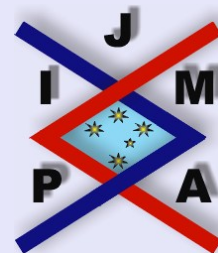
Close

Quit

Page 17 of 18

References

- [1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press. Cambridge, 1952.
- [2] B. YANG AND L. DEBNATH, On the extended Hardy-Hilbert's inequality, *J. Math. Anal. Appl.*, **272** (2002), 187–199.
- [3] M. GAO, TAN LI AND L. DEBNATH, Some improvements on Hilbert's integral inequality, *J. Math. Anal. Appl.*, **229** (1999), 682–689.
- [4] M. GAO, A note on Hilbert double series theorem, *Hunan Ann. Math.*, **12**(1-2) (1992), 142–147.
- [5] HU KE, On Hilbert's inequality, *Chinese Ann. of Math.*, **13B**(1) (1992), 35–39.
- [6] W. RUDIN, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill Book Co. 1976.



Four Inequalities Similar To
Hardy-Hilbert's Integral
Inequality

W.T. Sulaiman

Title Page

Contents



Go Back

Close

Quit

Page 18 of 18