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SOME APPLICATIONS OF THE GENERALIZED BERNARDI–LIBERA–LIVINGSTON INTEGRAL OPERATOR ON UNIVALENT FUNCTIONS

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ABSTRACT. In this paper by making use of the generalized Bernardi–Libera–Livingston integral operator we introduce and study some new subclasses of univalent functions. Also we investigate the relations between those classes and the classes which are studied by Jin–Lin Liu.

Key words and phrases: Analytic function, Integral operator, Univalent function.

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1. INTRODUCTION

Let A be the class of functions of the form, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $U = \{z : |z| < 1\}$. Also, let S denote the subclass of A consisting of all univalent functions in U. Suppose λ is a real number with $0 \leq \lambda < 1$. A function $f \in S$ is said to be starlike of order λ if and only if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \lambda$, $z \in U$. Also, $f \in S$ is said to be convex of order λ if and only if $\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \lambda$, $z \in U$. We denote by $S^*(\lambda), C(\lambda)$ the classes of starlike and convex functions of order λ respectively. It is well known that $f \in C(\lambda)$ if and only

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if $zf'^*(\lambda)$. If $f \in A$, then $f \in K(\beta,\lambda)$ if and only if there exists a function $g \in S^*(\lambda)$ such that $\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \beta$, $z \in U$, where $0 \leq \beta < 1$. These functions are called close-to-convex functions of order β type λ. A function $f \in A$ is called quasi-convex of order β type λ if there exists a function $g \in C(\lambda)$ such that $\mathrm{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} > \beta$. We denote this class by $K^*(\beta,\lambda)$ [10]. It is easy to see that $f \in K^*(\beta,\gamma)$ if and only if $zf' \in K(\beta,\gamma)$ [9]. For $f \in A$ if for some $\lambda(0 \leq \lambda < 1)$ and $\eta(0 < \eta \leq 1)$ we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \lambda\right)\right| < \frac{\pi}{2}\eta, \quad (z \in U), \tag{1.1}$$

then $f(z)$ is said to be strongly starlike of order η and type λ in U and we denote this class by $S^*(\eta,\lambda)$. If $f \in A$ satisfies the condition

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \lambda\right)\right| < \frac{\pi}{2}\eta, \quad (z \in U) \tag{1.2}$$

for some λ and η as above, then we say that $f(z)$ is strongly convex of order η and type λ in U and we denote this class by $C(\eta,\lambda)$. Clearly $f \in C(\eta,\lambda)$ if and only if $zf'^*(\eta,\lambda)$, and in particular, we have $S^*(1,\lambda) = S^*(\lambda)$ and $C(1,\lambda) = C(\lambda)$.

For $c > -1$ and $f \in A$ the generalized Bernardi–Libera–Livingston integral operator $L_c f$ is defined as follows

$$L_c f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt. \tag{1.3}$$

This operator for $c \in \mathbb{N} = \{1,2,3,\dots\}$ was studied by Bernardi [1] and for $c = 1$ by Libera [4] (see also [8]). The classes $ST_c(\eta,\lambda)$ and $CV_c(\eta,\lambda)$ were introduced by Liu [7], where

$$ST_c(\eta,\lambda) = \left\{f \in A : L_c f \in S^*(\eta,\lambda), \frac{z(L_c f(z))'}{L_c f(z)} \neq \lambda, z \in U\right\},$$
$$CV_c(\eta,\lambda) = \left\{f \in A : L_c f \in C(\eta,\lambda), \frac{(z(L_c f(z))')'}{(L_c f(z))'} \neq \lambda, z \in U\right\}.$$

Now by making use of the operator given by (1.3) we introduce the following classes.

$$S_c^*(\lambda) = \{f \in A : L_c f \in S^*(\lambda)\},$$
$$C_c(\lambda) = \{f \in A : L_c f \in C(\lambda)\}.$$

Obviously $f \in CV_c(\eta,\lambda)$ if and only if $zf' \in ST_c(\eta,\lambda)$. J. L. Liu [5] and [6] introduced and similarly investigated the classes $S_\sigma^*(\lambda)$, $C_\sigma(\lambda)$, $K_\sigma(\beta,\lambda)$, $K_\sigma^*(\beta,\lambda)$, $ST_\sigma(\eta,\lambda)$, $CV_\sigma(\eta,\lambda)$ by making use of the integral operator $I^\sigma f$ given by

$$I^\sigma f(z) = \frac{2^\sigma}{z\Gamma(\sigma)}\int_0^z \left(\log\frac{z}{t}\right)^{\sigma-1} f(t)dt, \quad \sigma > 0, f \in A. \tag{1.4}$$

The operator I^σ was introduced by Jung, Kim and Srivastava [2] and then investigated by Uralegaddi and Somanatha [13], Li [3] and Liu [5]. For the integral operators given by (1.3) and (1.4) we have verified following relationships.

$$I^\sigma f(z) = z + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^\sigma a_n z^n, \tag{1.5}$$

$$L_c f(z) = z + \sum_{n=2}^{\infty}\frac{c+1}{n+c} a_n z^n, \tag{1.6}$$

$$z(I^\sigma L_c f(z))'^\sigma f(z) - cI^\sigma L_c f(z), \tag{1.7}$$

$$z(L_c I^\sigma f(z))'^\sigma f(z) - cL_c I^\sigma f(z). \tag{1.8}$$

It follows from (1.5) that one can define the operator I^σ for any real number σ. In this paper we investigate the properties of the classes $S_c^*(\lambda)$, $C_c(\lambda)$, $K_c(\beta,\lambda)$, $K_c^*(\beta,\lambda)$, $ST_c(\eta,\lambda)$ and $CV_c(\eta,\lambda)$. We also study the relations between these classes by the classes which are introduced by Liu in [5] and [6]. For our purposes we need the following lemmas.

Lemma 1.1 ([9]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $\psi(u,v)$ be a complex function $\psi : D \subset \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. Suppose that ψ satisfies the following conditions*

(i) $\psi(u,v)$ *is continuous in* D;

(ii) $(1,0) \in D$ *and* $\operatorname{Re}\{\psi(1,0)\} > 0$;

(iii) $\operatorname{Re}\{\psi(iu_2,v_1)\} \le 0$ *for all* $(iu_2,v_1) \in D$ *with* $v_1 \le -\frac{1+u_2^2}{2}$.
Let $p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$ *be analytic in* U *so that* $(p(z), zp'(z)) \in D$ *for all* $z \in U$. *If* $\operatorname{Re}\{\psi(p(z), zp'(z))\} > 0$, $z \in U$ *then* $\operatorname{Re}\{p(z)\} > 0$, $z \in U$.

Lemma 1.2 ([11]). *Let the function $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in U and $p(z) \neq 0$, $z \in U$. If there exists a point $z_0 \in U$ such that $|\arg(p(z))| < \frac{\pi}{2}\eta$ for $|z| < |z_0|$ and $\arg p(z_0)| = \frac{\pi}{2}\eta$ where $0 < \eta \le 1$, then $\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta$ and $k \ge \frac{1}{2}(r + \frac{1}{r})$ when $\arg p(z_0) = \frac{\pi}{2}\eta$, Also, $k \le \frac{-1}{2}(r + \frac{1}{r})$ when $\arg p(z_0) = \frac{-\pi}{2}\eta$, and $p(z_0)^{1/\eta} = \pm ir (r > 0)$.*

2. MAIN RESULTS

In this section we obtain some inclusion theorems by following the method of proof adopted in [12].

Theorem 2.1.

(i) *For $f \in A$ if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{z(L_c f(z))'}{L_c f(z)}\right\} > 0$ and $\frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)}$ is an analytic function, then $S_c^*(\lambda) \subset S_{c+1}^*(\lambda)$.*

(ii) *Let $c > -\lambda$. For $f \in A$ if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)}\right\} > 0$ and $\frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)}$ is an analytic function, then $S_{c+1}^*(\lambda) \subset S_c^*(\lambda)$.*

Proof. (i) Suppose that $f \in S_c^*(\lambda)$ and set

$$\frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)} - \lambda = (1-\lambda)p(z), \tag{2.1}$$

where $p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$. An easy calculation shows that

$$\frac{\frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)}\left[2 + c + \frac{z(L_{c+1} f(z))''}{(L_{c+1} f(z))'}\right]}{\frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)} + c + 1} = \frac{zf'(z)}{f(z)}. \tag{2.2}$$

By setting $H(z) = \frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)}$ we have

$$1 + \frac{z(L_{c+1} f(z))''}{(L_{c+1} f(z))'} = H(z) + \frac{zH'(z)}{H(z)}. \tag{2.3}$$

By making use of (2.3) in (2.2) , since $H(z)=\lambda+(1-\lambda)p(z)$, we obtain

$$(1-\lambda)p(z)+\frac{(1-\lambda)zp'(z)}{\lambda+c+1+(1-\lambda)p(z)}=\frac{zf'(z)}{f(z)}-\lambda. \tag{2.4}$$

If we consider

$$\psi(u,v)=(1-\lambda)u+\frac{(1-\lambda)v}{\lambda+c+1+(1-\lambda)u},$$

then $\psi(u,v)$ is a continuous function in $D=\left\{\mathbb{C}-\frac{\lambda+c+1}{\lambda-1}\right\}\times\mathbb{C}$ and $(1,0)\in D$. Also, $\psi(1,0)>0$ and for all $(iu_2,v_1)\in D$ with $v_1\le-\frac{1+u_2^2}{2}$ we have

$$\begin{aligned}\operatorname{Re}\psi(iu_2,v_1)&=\frac{(1-\lambda)(\lambda+c+1)v_1}{(1-\lambda)^2u_2^2+(\lambda+c+1)^2}\\&\le\frac{-(1-\lambda)(\lambda+c+1)(1+u_2^2)}{2[(1-\lambda)^2u_2^2+(\lambda+c+1)^2]}<0.\end{aligned}$$

Therefore the function $\psi(u,v)$ satisfies the conditions of Lemma 1.1 and since in view of the assumption, by considering (2.4) , we have $\operatorname{Re}\{\psi(p(z),zp'(z))\}>0$, Lemma 1.1 implies that $\operatorname{Re}p(z)>0, z\in U$ and this completes the proof of (i).

(ii) For proving this part of the theorem, we use the same method and a easily verified formula similar to (2.2). By replacing $c+1$ with c we get the desired result. □

Theorem 2.2.

(i) *For* $f\in A$ *if* $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\frac{z(L_cf(z))'}{L_cf(z)}\right\}>0$ *and* $\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}$ *is an analytic function, then* $C_c(\lambda)\subset C_{c+1}(\lambda)$.

(ii) *Let* $c>-\lambda$. *For* $f\in A$ *if* $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}\right\}>0$ *and* $\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}$ *is an analytic function, then* $C_{c+1}(\lambda)\subset C_c(\lambda)$.

Proof. (i) In view of part (i) of Theorem 2.1 we can write

$$\begin{aligned}f\in C_c(\lambda)&\Leftrightarrow L_cf\in C(\lambda)\Leftrightarrow z(L_cf)'^*(\lambda)\Leftrightarrow L_czf'^*(\lambda)\Leftrightarrow zf_c'^*(\lambda)\Rightarrow zf_{c+1}'^*(\lambda)\\&\Leftrightarrow L_{c+1}zf'^*(\lambda)\Leftrightarrow z(L_{c+1}f)'^*(\lambda)\Leftrightarrow L_{c+1}f\in C(\lambda)\Leftrightarrow f\in C_{c+1}(\lambda).\end{aligned}$$

Part (ii) of the theorem can be proved in a similar manner. □

Theorem 2.3. *If* $c\ge-\lambda$ *and* $\frac{zf'(z)}{f(z)}$ *is an analytic function, then* $f\in S^*(\lambda)$ *implies* $f\in S_c^*(\lambda)$.

Proof. By differentiating logarithmically both sides of (1.3) with respect to z we obtain

$$\frac{z(L_cf(z))'}{L_cf(z)}+c=\frac{(c+1)f(z)}{L_cf(z)}. \tag{2.5}$$

Again differentiating logarithmically both sides of (2.5) we have

$$p(z)+\frac{zp'(z)}{c+\lambda+p(z)}=\frac{zf'(z)}{f(z)}-\lambda, \tag{2.6}$$

where $p(z)=\frac{z(L_cf(z))'}{L_cf(z)}-\lambda$. Let us consider $\psi(u,v)=u+\frac{v}{u+c+\lambda}$. Then ψ is a continuous function in $D=\{\mathbb{C}-(-c-\lambda)\}\times\mathbb{C}$, $(1,0)\in D$ and $\operatorname{Re}\ \psi(1,0)>0$. If $(iu_2,v_1)\in D$ with $v_1\le-\frac{1+u_2^2}{2}$, then

$$\operatorname{Re}\psi(iu_2,v_1)=\frac{v_1(c+\lambda)}{u_2^2+(c+\lambda)^2}\le 0.$$

Since $f \in S^*(\lambda)$, then (2.6) gives

$$\mathrm{Re}(\psi(p(z), zp'(z))) = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \lambda\right\} > 0.$$

Therefore Lemma 1.1 concludes that $\mathrm{Re}\{p(z)\} > 0$ and this completes the proof. $\square$

Corollary 2.4. *If $c \geq \lambda$ and $\frac{zf'(z)}{f(z)}$ is an analytic function, then $f \in C(\lambda)$ implies $f \in C_c(\lambda)$.*

Proof. We have

$$\begin{aligned} f \in C(\lambda) &\Leftrightarrow zf'^*(\lambda)\Lambda z f_c'^*(\lambda) \Leftrightarrow L_c zf' \in S^*(\lambda) \\ &\Leftrightarrow z(L_c f)'^*(\lambda) \Leftrightarrow L_c f \in C(\lambda) \Leftrightarrow f \in C_c(\lambda). \end{aligned}$$

$\square$

REFERENCES

[1] S.D. BERNARDI, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.,* **135** (1969), 429–446.

[2] I.B. JUNG, Y.C. KIM AND H.M. SRIVASTAVA, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.,* **176** (1993), 138–147.

[3] J.L. LI, Some properties of two integral operators, *Soochow. J. Math.,* **25** (1999), 91–96.

[4] R.J. LIBERA, Some classes of regular functions, *Proc. Amer. Math. Soc.,***16** (1965), 755–758.

[5] J.L. LIU, A linear operator and strongly starlike functions, *J. Math. Soc. Japan,* **54**(4) (2002), 975–981.

[6] J.L. LIU, Some applications of certain integral operator, *Kyungpook Math. J.,* **43**(2003), 21–219.

[7] J.L. LIU, Certain integral operator and strongly starlike functions, *Int. J. Math. Math. Sci.,* **30**(9) (2002), 569–574.

[8] A.E. LIVINGSTON, On the radius of univalence of certain analytic functions, *Proc. Amer. Math. Soc.,* **17** (1996), 352–357.

[9] S.S. MILLER and P.T. MOCANU, Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.,* **65** (1978), 289–305.

[10] K.I. NOOR, On quasi-convex functions and related topics, *Internat. J. Math. Math. Sci.,***10** (1987), 241–258.

[11] M. NUNOKAWA, S. OWA, H. SAITOH, A. IKEDA AND N. KOIKE, Some results for strongly starlike functions, *J. Math. Anal. Appl.,* **212** (1997), 98–106.

[12] J. SOKOL, A linear operator and associated class of multivalent analytic functions, *Demonstratio Math.,* **40**(3) (2007), 559–566.

[13] B.A. URALEGADDI AND C. SOMANATHA, Certain integral operators for starlike functions, *J. Math. Res. Expo.,* **15** (1995), 14–16.