journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 9 (2008), Issue 2, Article 48, 6 pp.

AN INEQUALITY ABOUT ${}_3\phi_2$ AND ITS APPLICATIONS

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Received 17 August, 2007; accepted 18 May, 2008
Communicated by S.S. Dragomir

ABSTRACT. In this paper, we use the terminating case of the q-binomial formula, the q-Chu-Vandermonde formula and the Grüss inequality to drive an inequality about ${}_3\phi_2$. As applications of the inequality, we discuss the convergence of some q-series involving ${}_3\phi_2$.

Key words and phrases: Basic hypergeometric function ${}_3\phi_2$; q-binomial theorem; q-Chu-Vandermonde formula; Grüss inequality.

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 33D15.

1. STATEMENT OF MAIN RESULTS

q-series, which are also called basic hypergeometric series, play a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics, etc. Inequalities techniques provide useful tools in the study of special functions (see [1, 6, 7, 8, 9, 10]). For example, Ito used inequalities techniques to give a sufficient condition for convergence of a special q-series called the Jackson integral [6]. In this paper, we derive the following new inequality about q-series involving ${}_3\phi_2$.

Theorem 1.1. *Let a_1, a_2, b_1, b_2 be some real numbers such that $b_i < 1$ for $i = 1, 2$. Then for all positive integers n, we have:*

$$\left| {}_3\phi_2\left(\begin{matrix} b_1/a_1, b_2/a_2, q^{-n} \\ b_1, b_2 \end{matrix}; q, -a_1a_2q^n\right) - \frac{(a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \right| \le \lambda\mu(-1; q)_n, \tag{1.1}$$

where

$$\lambda = \max\{1, M^n\},\ M = \max\left\{|a_1|, \frac{|a_1 - b_1|}{1 - b_1}\right\},\ \mu = \max\{1, N^n\},\ N = \max\left\{|a_2|, \frac{|a_2 - b_2|}{1 - b_2}\right\}.$$

Applications of this inequality are also given.

The author would like to express deep appreciation to the referee for the helpful suggestions. In particular, the author thanks the referee for helping to improve the presentation of the paper.

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2. NOTATIONS AND KNOWN RESULTS

We recall some definitions, notations and known results which will be used in the proof. Throughout this paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{2.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n, \tag{2.2}$$

where n is an integer or ∞.

The q-binomial theorem (see [2, 3, 4]) is given by

$$\sum_{k=0}^{\infty} \frac{(a;q)_k z^k}{(q;q)_k} = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z| < 1. \tag{2.3}$$

When $a = q^{-n}$, where n denotes a nonnegative integer, we have

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k z^k}{(q;q)_k} = (zq^{-n};q)_n. \tag{2.4}$$

Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series, which is defined by

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n x^n}{(q, b_1, b_2, \ldots, b_r; q)_n}. \tag{2.5}$$

The q-Chu-Vandermonde sums (see [2, 3, 4]) are

$${}_2\phi_1 \left(\begin{matrix} a, q^{-n} \\ c \end{matrix} ; q, q \right) = \frac{a^n (c/a;q)_n}{(c;q)_n} \tag{2.6}$$

and, reversing the order of summation, we have

$${}_2\phi_1 \left(\begin{matrix} a, q^{-n} \\ c \end{matrix} ; q, cq^n/a \right) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{2.7}$$

At the end of this section, we recall the Grüss inequality (see [5]):

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \right| \le \frac{(M-m)(N-n)}{4}, \tag{2.8}$$

provided that $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and $m \le f(x) \le M$, $n \le g(x) \le N$ for all $x \in [a,b]$, where m, M, n, N are given constants.

By simple calculus, one can prove that the discrete version of the Grüss inequality can be stated as:

if $a \le \lambda_i \le A$ and $b \le \mu_i \le B$, $i = 1, 2, \ldots, n$, then for all sequences $(p_i)_{0 \le i \le n}$ of nonnegative real numbers satisfying $\sum_{i=1}^n p_i = 1$, we have

$$\left| \sum_{i=1}^{n} \lambda_i \mu_i p_i - \left(\sum_{i=1}^{n} \lambda_i p_i\right) \cdot \left(\sum_{i=1}^{n} \mu_i p_i\right) \right| \le \frac{(A-a)(B-b)}{4}, \tag{2.9}$$

where a, A, b, B are some given real constants.

3. PROOF OF THE THEOREM

In this section, we use the terminating case of the q-binomial formula, the q-Chu-Vandermonde formula and the Grüss inequality to prove (1.1). For this purpose, we need the following lemma.

Lemma 3.1. *Let a and b be two real numbers such that $b < 1$, and let $0 \le t \le 1$. Then,*

$$\left|\frac{a-bt}{1-bt}\right| \le \max\left\{|a|, \frac{|a-b|}{1-b}\right\}. \tag{3.1}$$

Proof. Let

$$f(t) = \frac{a-bt}{1-bt}, \qquad 0 \le t \le 1,$$

then

$$f'(t) = \frac{b(a-1)}{(1-bt)^2}, \qquad 0 \le t \le 1.$$

So $f(t)$ is a monotonic function with respect to $0 \le t \le 1$. Since $f(0) = a$ and $f(1) = \frac{a-b}{1-b}$, (3.1) holds. □

Now, we are in a position to prove the inequality (1.1).

Proof. Put

$$p_k = \frac{(q^{-n};q)_k(-q^n)^k}{(q;q)_k(-1;q)_n}, \qquad k = 0, 1, \dots, n. \tag{3.2}$$

It is obvious that $p_k \ge 0$.

On the other hand, using (2.4), we obtain

$$\sum_{k=0}^{n} p_k = \frac{1}{(-1;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(-q^n)^k}{(q;q)_k} = 1.$$

Let

$$\lambda_k = \frac{(-a_1)^k(b_1/a_1;q)_k}{(b_1;q)_k}, \tag{3.3}$$

and

$$\mu_k = \frac{(-a_2)^k(b_2/a_2;q)_k}{(b_2;q)_k}, \tag{3.4}$$

where $k = 0, 1, \dots, n$.

According to the definitions of M, N, λ and μ, it is easy to see that

$$M^k \le \lambda \text{ and } N^k \le \mu, \ \ 0 \le k \le n.$$

Using the lemma, one can get for all $0 \le k \le n$,

$$|\lambda_k| = \left|\frac{a_1-b_1}{1-b_1}\cdot\frac{a_1-b_1q}{1-b_1q}\cdot\dots\cdot\frac{a_1-b_1q^{k-1}}{1-b_1q^{k-1}}\right| \le M^k \le \lambda \tag{3.5}$$

and

$$|\mu_k| = \left|\frac{a_2-b_2}{1-b_2}\cdot\frac{a_2-b_2q}{1-b_2q}\cdot\dots\cdot\frac{a_2-b_2q^{k-1}}{1-b_2q^{k-1}}\right| \le N^k \le \mu. \tag{3.6}$$

Substitution of (3.2), (3.3), (3.4), (3.5) and (3.6) into (2.9), gives

$$\begin{aligned}(3.7)\quad &\left|\sum_{k=0}^{n}\frac{(q^{-n};q)_k(-q^n)^k}{(q;q)_k(-1;q)_n}\cdot\frac{(-a_1)^k(b_1/a_1;q)_k}{(b_1;q)_k}\cdot\frac{(-a_2)^k(b_2/a_2;q)_k}{(b_2;q)_k}\right.\\ &\qquad -\sum_{k=0}^{n}\frac{(q^{-n};q)_k(-q^n)^k}{(q;q)_k(-1;q)_n}\cdot\frac{(-a_1)^k(b_1/a_1;q)_k}{(b_1;q)_k}\\ &\qquad\quad\left.\times\sum_{k=0}^{n}\frac{(q^{-n};q)_k(-q^n)^k}{(q;q)_k(-1;q)_n}\cdot\frac{(-a_2)^k(b_2/a_2;q)_k}{(b_2;q)_k}\right|\le\lambda\mu.\end{aligned}$$

Using (2.5) and (2.7), we get

$$\begin{aligned}(3.8)\quad &\sum_{k=0}^{n}\frac{(q^{-n};q)_k(-q^n)^k}{(q;q)_k(-1;q)_n}\cdot\frac{(-a_1)^k(b_1/a_1;q)_k}{(b_1;q)_k}\cdot\frac{(-a_2)^k(b_2/a_2;q)_k}{(b_2;q)_k}\\ &=\frac{1}{(-1;q)_n}\sum_{k=0}^{n}\frac{(q^{-n},b_1/a_1,b_2/a_2;q)_k}{(q,b_1,b_2;q)_k}(-a_1a_2q^n)^k\\ &=\frac{1}{(-1;q)_n}{}_3\phi_2\left(\begin{matrix}b_1/a_1,b_2/a_2,q^{-n}\\ b_1,b_2\end{matrix};q,-a_1a_2q^n\right),\end{aligned}$$

$$(3.9)\quad \sum_{k=0}^{n}\frac{(q^{-n};q)_k(-q^n)^k}{(q;q)_k(-1;q)_n}\cdot\frac{(-a_1)^k(b_1/a_1;q)_k}{(b_1;q)_k}=\frac{(a_1;q)_n}{(-1,b_1;q)_n}$$

and

$$(3.10)\quad \sum_{k=0}^{n}\frac{(q^{-n};q)_k(-q^n)^k}{(q;q)_k(-1;q)_n}\cdot\frac{(-a_2)^k(b_2/a_2;q)_k}{(b_2;q)_k}=\frac{(a_2;q)_n}{(-1,b_2;q)_n}.$$

Substituting (3.8), (3.9) and (3.10) into (3.7), we obtain (1.1). □

Taking $a_2=1$ in (1.1), we get the following corollary.

Corollary 3.2. *We have*

$$(3.11)\quad \left|{}_2\phi_1\left(\begin{matrix}b_1/a_1,q^{-n}\\ b_1\end{matrix};q,-a_1q^n\right)\right|\le\lambda(-1;q)_n.$$

4. SOME APPLICATIONS OF THE INEQUALITY

Convergence of q-series is an important problem which is at times very difficult. As applications of the inequality derived in this paper, we obtain some results about the convergence of the q-series involving ${}_3\phi_2$. In this section, we mainly discuss the convergence of the following q-series:

$$(4.1)\quad \sum_{n=0}^{\infty}c_n{}_3\phi_2\left(\begin{matrix}b_1/a_1,b_2/a_2,q^{-n}\\ b_1,b_2\end{matrix};q,-a_1a_2q^n\right).$$

Theorem 4.1. *Suppose $|a_i|\le 1$ and $b_i<\frac{a_i+1}{2}$ for $i=1,2$. Let $\{c_n\}$ be a real sequence satisfying*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=p<1.$$

Then the series (4.1) is absolutely convergent.

Proof. It is obvious that $b_i < 1$ for $i = 1, 2$. Combining the following inequality

$$\left| {}_3\phi_2 \left(\begin{matrix} b_1/a_1, b_2/a_2, q^{-n} \\ b_1, b_2 \end{matrix} ; q, -a_1 a_2 q^n \right) \right| - \left| \frac{(a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \right| \leq \left| {}_3\phi_2 \left(\begin{matrix} b_1/a_1, b_2/a_2, q^{-n} \\ b_1, b_2 \end{matrix} ; q, -a_1 a_2 q^n \right) - \frac{(a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \right| \tag{4.2}$$

with (2.1), shows that

$$\left| {}_3\phi_2 \left(\begin{matrix} b_1/a_1, b_2/a_2, q^{-n} \\ b_1, b_2 \end{matrix} ; q, -a_1 a_2 q^n \right) \right| \leq \left| \frac{(a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \right| + \lambda\mu(-1; q)_n. \tag{4.3}$$

Since

$$|a_i| \leq 1, \quad b_i < \frac{a_i + 1}{2}, \qquad i = 1, 2,$$

which is equivalent to

$$|a_i| \leq 1, \quad \frac{|a_i - b_i|}{1 - b_i} < 1, \qquad i = 1, 2,$$

then

$$\lambda = \mu = 1. \tag{4.4}$$

Substituting (4.4) into (4.3), we obtain

$$\left| {}_3\phi_2 \left(\begin{matrix} b_1/a_1, b_2/a_2, q^{-n} \\ b_1, b_2 \end{matrix} ; q, -a_1 a_2 q^n \right) \right| \leq \left| \frac{(a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \right| + (-1; q)_n. \tag{4.5}$$

Multiplication of the two sides of (4.5) by $|c_n|$ gives

$$\left| c_n {}_3\phi_2 \left(\begin{matrix} b_1/a_1, b_2/a_2, q^{-n} \\ b_1, b_2 \end{matrix} ; q, -a_1 a_2 q^n \right) \right| \leq \left| \frac{c_n (a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \right| + |c_n (-1; q)_n| . \tag{4.6}$$

The ratio test shows that both

$$\sum_{n=0}^{\infty} \frac{c_n (a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} c_n (-1; q)_n$$

are absolutely convergent. From (4.6), we get that (4.1) is absolutely convergent. □

Theorem 4.2. *Suppose $|a_1| > 1$ or $a_1 < 2b_1 - 1$, $b_1 < 1$, $|a_2| \leq 1$ and $b_2 \leq \frac{a_2+1}{2}$. Let $\{c_n\}$ be a real sequence satisfying*

$$\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = p < \frac{1}{M},$$

where $M = \max\left\{ |a_1|, \frac{|a_1 - b_1|}{1 - b_1} \right\}$. Then the series (4.1) is absolutely convergent.

Proof. First we point out that $a_1 < 2b_1 - 1$ implies

$$\frac{|a_1 - b_1|}{1 - b_1} > 1.$$

So, under the conditions of the theorem, we know

$$\lambda = M^n \quad \text{and} \quad \mu = 1.$$

Multiplying both sides of (4.3) by $|c_n|$, one gets

$$\left| c_n {}_3\phi_2 \left(\begin{matrix} b_1/a_1, b_2/a_2, q^{-n} \\ b_1, b_2 \end{matrix} ; q, -a_1 a_2 q^n \right) \right| \leq \left| \frac{c_n (a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \right| + |c_n M^n (-1; q)_n| . \tag{4.7}$$

The ratio test shows that both

$$\sum_{n=0}^{\infty} \frac{c_n(a_1, a_2; q)_n}{(-1, b_1, b_2; q)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} c_n M^n(-1; q)_n$$

are absolutely convergent. From (4.7), we get that (4.1) is absolutely convergent. $\square$

Similarly, we have

Theorem 4.3. *Suppose* $|a_i| > 1$ *or* $a_i < 2b_i - 1$, $b_i < 1$ *with* $i = 1, 2$. *Let* $\{c_n\}$ *be a real sequence satisfying*

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = p < \frac{1}{MN},$$

where $M = \max\left\{|a_1|, \frac{|a_1-b_1|}{1-b_1}\right\}$ *and* $N = \max\left\{|a_2|, \frac{|a_2-b_2|}{1-b_2}\right\}$. *Then the series (4.1) is absolutely convergent.*

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