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ON CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS INVOLVING INTEGRAL OPERATORS

KHALIDA INAYAT NOOR

Mathematics Department
COMSATS Institute of Information Technology
Islamabad, Pakistan.
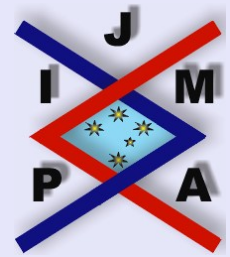
EMail: khalidanoor@hotmail.com

Abstract

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Abstract

We introduce and study some classes of meromorphic functions defined by using a meromorphic analogue of Noor [also Choi-Saigo-Srivastava] operator for analytic functions. Several inclusion results and some other interesting properties of these classes are investigated.

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1. Introduction

Let $\mathcal{M}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in $D = \{z : 0 < |z| < 1\}$.

Let $P_k(\beta)$ be the class of analytic functions $p(z)$ defined in unit disc $E = D \cup \{0\}$, satisfying the properties $p(0) = 1$ and

$$(1.1) \qquad \int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \beta}{1 - \beta} \right| d\theta \leq k\pi,$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \beta < 1$. When $\beta = 0$, we obtain the class P_k defined in [14] and for $\beta = 0$, $k = 2$, we have the class P of functions with positive real part.
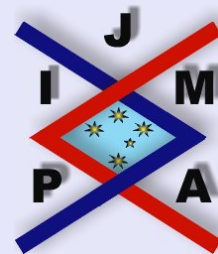
Also, we can write (1.1) as

$$(1.2) \qquad p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\beta)ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$(1.3) \qquad \int_0^{2\pi} d\mu(t) = 2, \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k.$$

From (1.1), we can write, for $p \in P_k(\beta)$,

$$(1.4) \qquad p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z),$$

where $p_1, p_2 \in P_2(\beta) = P(\beta)$, $z \in E$.

We define the function $\lambda(a, b, z)$ by

$$\lambda(a, b, z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(c)_{n+1}} z^n, \quad z \in D,$$

$c \neq 0, -1, -2, \ldots, a > 0$, where $(a)_n$ is the Pochhamer symbol (or the shifted factorial) defined by

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n > 1.$$

We note that

$$\lambda(a, c, z) = \frac{1}{z}{}_2F_1(1, a; c, z),$$

${}_2F_1(1, a; c, z)$ is Gauss hypergeometric function.

Let $f \in \mathcal{M}$. Denote by $\tilde{L}(a, c); \mathcal{M} \longrightarrow \mathcal{M}$, the operator defined by

$$\tilde{L}(a, c)f(z) = \lambda(a, c, z) \star f(z), \quad z \in D,$$

where the symbol $\star$ stands for the Hadamard product (or convolution). The operator $\tilde{L}(a, c)$ was introduced and studied in [5]. This operator is closely related to the Carlson-Shaeffer operator [1] defined for the space of analytic and univalent functions in E, see [11, 13].

We now introduce a function $(\lambda(a, c, z))^{(-1)}$ given by

$$\lambda(a, c, z) \star (\lambda(a, c, z))^{(-1)} = \frac{1}{z(1 - z)^{\mu}}, \quad (\mu > 0), \quad z \in D.$$
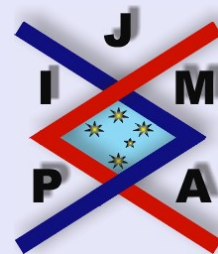
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Analogous to $\tilde{L}(a,c)$, a linear operator $I_\mu(a,c)$ on $\mathcal{M}$ is defined as follows, see [2].

$$(1.5) \qquad I_\mu(a,c)f(z) = (\lambda(a,c,z))^{(-1)} \star f(z),$$
$$(\mu > 0, a > 0, \quad c \neq 0, -1, -2, \ldots, \quad z \in D).$$

We note that

$$I_2(2,1)f(z) = f(z), \quad \text{and} \quad I_2(1,1)f(z) = zf'(z) + 2f(z).$$

It can easily be verified that

$$(1.6) \qquad z\left(I_\mu(a+1,c)f(z)\right)' = aI_\mu(a,c)f(z) - (a+1)I_\mu(a+1,c)f(z),$$
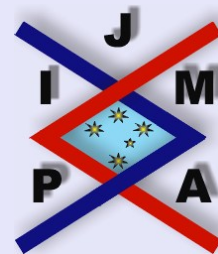$$(1.7) \qquad z\left(I_\mu(a,c)f(z)\right)' = \mu I_{\mu+1}(a,c)f(z) - (\mu+1)I_\mu(a,c)f(z).$$

We note that the operator $I_\mu(a,c)$ is motivated essentially by the operators defined and studied in [2, 11].

Now, using the operator $I_\mu(a,c)$, we define the following classes of meromorphic functions for $\mu > 0$, $0 \leq \eta, \beta < 1$, $\alpha \geq 0$, $z \in D$.

We shall assume, unless stated otherwise, that $a \neq 0, -1, -2, \ldots$, $c \neq 0, -1, -2, \ldots$

Definition 1.1. *A function $f \in \mathcal{M}$ is said to belong to the class $MR_k(\eta)$ for $z \in D, 0 \leq \eta < 1,\ k \geq 2$, if and only if*

$$-\frac{zf'(z)}{f(z)} \in P_k(\eta)$$

and $f \in MV_k(\eta)$, for $z \in D$, $0 \le \eta < 1$, $k \ge 2$, if and only if

$$-\frac{(zf'(z))'}{f'(z)} \in P_k(\eta).$$

We call $f \in MR_k(\eta)$, a meromorphic function with bounded radius rotation of order η and $f \in MV_k$ a meromorphic function with bounded boundary rotation.

Definition 1.2. *Let $f \in \mathcal{M}$, $0 \le \eta < 1$, $k \ge 2$, $z \in D$. Then*

$$f \in MR_k(\mu, \eta, a, c) \quad \text{if and only if} \quad I_\mu(a, c)f \in MR_k(\eta).$$

Also

$$f \in MV_k(\mu, \eta, a, c) \quad \text{if and only if} \quad I_\mu(a, c)f \in MV_k(\eta), \quad z \in D.$$

We note that, for $z \in D$,

$$f \in MV_k(\mu, \eta, a, c) \quad \Longleftrightarrow \quad -zf' \in MR_k(\mu, \eta, a, c).$$

Definition 1.3. *Let $\alpha \ge 0$, $f \in \mathcal{M}$, $0 \le \eta, \beta < 1, \mu > 0$ and $z \in D$. Then $f \in \mathcal{B}_k^\alpha(\mu, \beta, \eta, a, c)$, if and only if there exists a function $g \in MC(\mu, \eta, a, c)$, such that*

$$\left[(1 - \alpha)\frac{(I_\mu(a, c)f(z))'}{(I_\mu(a, c)g(z))'} + \alpha \left\{ -\frac{(z(I_\mu(a, c)f(z))')'}{(I_\mu(a, c)g(z))'} \right\} \right] \in P_k(\beta).$$

In particular, for $\alpha = 0$, $k = a = \mu = 2$, and $c = 1$, we obtain the class of meromorphic close-to-convex functions, see [4]. For $\alpha = 1$, $k = \mu = a = 2$, $c = 1$, we have the class of meromorphic quasi-convex functions defined for $z \in D$. We note that the class $C^\star$ of quasi-convex univalent functions, analytic in E, were first introduced and studied in [7]. See also [9, 12].
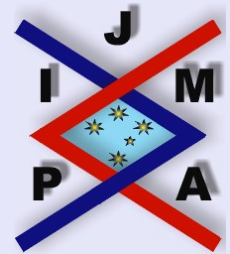
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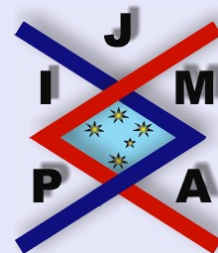*J. Ineq. Pure and Appl. Math.* **7**(4) Art. 138, 2006

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The following lemma will be required in our investigation.

Lemma 1.1 ([6]). *Let* $u = u_1 + iu_2$ *and* $v = v_1 + iv_2$ *and let* $\Phi(u, v)$ *be a complex-valued function satisfying the conditions:*

(i) $\Phi(u, v)$ *is continuous in a domain* $\mathcal{D} \subset \mathcal{C}^2$,

(ii) $(1, 0) \in \mathcal{D}$ *and* $\Phi(1, 0) > 0$.

(iii) $\operatorname{Re} \Phi(iu_2, v_1) \leq 0$ *whenever* $(iu_2, v_1) \in \mathcal{D}$ *and* $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.

If $h(z) = 1 + \sum_{m=1}^{\infty} c_m z^m$ *is a function, analytic in* E, *such that* $(h(z), zh'(z)) \in \mathcal{D}$ *and* $\operatorname{Re}(h(z), zh'(z)) > 0$ *for* $z \in E$, *then* $\operatorname{Re} h(z) > 0$ *in* E.

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2. Main Results

Theorem 2.1.

$$MR_k(\mu + 1, \eta, a, c) \subset MR_k(\mu, \beta, a, c) \subset MR_k(\mu, \gamma, a + 1, c).$$

Proof. We prove the first part of the result and the second part follows by using similar arguments. Let

$$f \in MR_k(\mu + 1, \eta, a, c), \quad z \in D$$

and set

$$H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z)$$

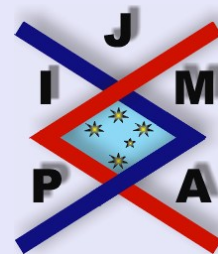$$(2.1) \qquad = - \left[\frac{z(I_\mu(a, c)f(z))'}{I_\mu(a, c)f(z)}\right],$$

where $H(z)$ is analytic in E with $H(0) = 1$.

Simple computation together with (2.1) and (1.7) yields

$$(2.2) \quad - \left[\frac{z\left(I_{\mu+1}(a, c)f(z)\right)'}{I_{\mu+1}(a, c)f(z)}\right]$$

$$= \left[H(z) + \frac{zH'(z)}{-H(z) + \mu + 1}\right] \in P_k(\eta), \qquad z \in E.$$

Let

$$\Phi_\mu(z) = \frac{1}{\mu + 1}\left[\frac{1}{z} + \sum_{k=0}^{\infty} z^k\right] + \frac{\mu}{\mu + 1}\left[\frac{1}{z} + \sum_{k=0}^{\infty} kz^k\right],$$

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then

$$(H(z) \star z\Phi_\mu(z)) = H(z) + \frac{zH'(z)}{-H(z) + \mu + 1}$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)(h_1(z) \star z\Phi_\mu(z)) - \left(\frac{k}{4} - \frac{1}{2}\right)(h_2(z) \star z\Phi_\mu(z))$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\left[h_1(z) + \frac{zh_1'(z)}{-h_1(z) + \mu + 1}\right]$$

(2.3)
$$- \left(\frac{k}{4} - \frac{1}{2}\right)\left[h_2(z) + \frac{zh_2'(z)}{-h_2(z) + \mu + 1}\right].$$

Since $f \in MR_k(\mu + 1, \eta, a, c)$, it follows from (2.2) and (2.3) that

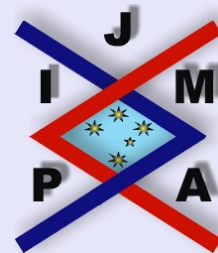$$\left[h_i(z) + \frac{zh_i'(z)}{-h_i(z) + \mu + 1}\right] \in P(\eta), \quad i = 1, 2, \quad z \in E.$$

Let $h_i(z) = (1 - \beta)p_i(z) + \beta$. Then

$$\left\{(1 - \beta)p_i(z) + \left[\frac{(1 - \beta)zp_i'(z)}{-(1 - \beta)p_i(z) - \beta + \mu + 1}\right] + (\beta - \eta)\right\} \in P, \quad z \in E.$$

We shall show that $p_i \in P$, $i = 1, 2$.

We form the functional $\Phi(u, v)$ by taking $u = p_i(z)$, $v = zp_i'(z)$ with $u = u_1 + iu_2$, $v = v_1 + iv_2$. The first two conditions of Lemma 1.1 can easily be verified. We proceed to verify the condition (iii).

$$\Phi(u, v) = (1 - \beta)u + \frac{(1 - \beta)v}{-(1 - \beta)u - \beta + \mu + 1} + (\beta - \eta),$$

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implies that

$$\mathrm{Re}\,\Phi(iu_2, v_1) = (\beta - \eta) + \frac{(1-\beta)(1+\mu-\beta)v_1}{(1+\mu-\beta)^2 + (1-\beta)^2 u_2^2}.$$

By taking $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we have

$$\mathrm{Re}\,\Phi(iu_2, v_1) \leq \frac{A + Bu_2^2}{2C},$$

where

$$A = 2(\beta - \eta)(1 + \mu - \beta)^2 - (1 - \beta)(1 + \mu - \beta),$$
$$B = 2(\beta - \eta)(1 - \beta)^2 - (1 - \beta)(1 + \mu - \beta),$$
$$C = (1 + \mu - \beta)^2 + (1 - \beta)^2 u_2^2 > 0.$$

We note that $\mathrm{Re}\,\Phi(iu_2, v_1) \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we obtain
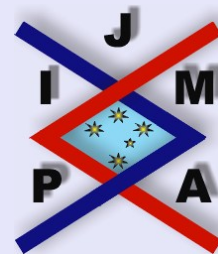
$$(2.4) \qquad \beta = \frac{1}{4}\left[(3 + 2\mu + 2\eta) - \sqrt{(3 + 2\mu + 2\eta)^2 - 8}\right],$$

and $B \leq 0$ gives us $0 \leq \beta < 1$.

Now using Lemma 1.1, we see that $p_i \in P$ for $z \in E$, $i = 1, 2$ and hence $f \in MR_k(\mu, \beta, a, c)$ with β given by (2.4). $\qquad \square$

In particular, we note that

$$\beta = \frac{1}{4}\left[(3 + 2\mu) - \sqrt{4\mu^2 + 12\mu + 1}\right].$$

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Theorem 2.2.

$$MV_k(\mu + 1, \eta, a, c) \subset MV_k(\mu, \beta, a, , c) \subset MV_k(\mu, \gamma, a + 1, c).$$

Proof.

$$
\begin{aligned}
f \in MV_k(\mu + 1, \eta, a, c) \quad &\Longleftrightarrow \quad -zf' \in MR_k(\mu + 1, \eta, a, c) \\
&\Rightarrow -zf' \in MR_k(\mu, \beta, a, c) \\
&\Longleftrightarrow f \in MV_k(\mu, \beta, a, c),
\end{aligned}
$$

where β is given by (2.4).

The second part can be proved with similar arguments. $\qquad\square$

Theorem 2.3.

$$\mathcal{B}_k^\alpha(\mu + 1, \beta_1, \eta_1, a, c) \subset \mathcal{B}_k^\alpha(\mu, \beta_2, \eta_2, a, c) \subset \mathcal{B}_k^\alpha(\mu, \beta_3, \eta_3, a + 1, c),$$
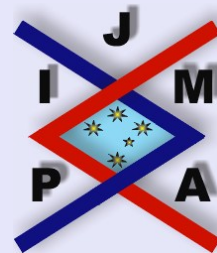
where $\eta_i = \eta_i(\beta_i, \mu)$, $i = 1, 2, 3$ are given in the proof.

Proof. We prove the first inclusion of this result and other part follows along similar lines. Let $f \in \mathcal{B}_k^\alpha(\mu + 1, \beta_1, \eta_1, a, c)$. Then, by Definition 1.3, there exists a function $g \in MV_2(\mu + 1, \eta_1, a, c)$ such that

$$(2.5) \quad (1 - \alpha)\left[\frac{(I_{\mu+1}(a, c)f(z))'}{(I_{\mu+1}(a, c)g(z))'}\right] + \alpha\left[-\frac{(z(I_{\mu+1}(a, c)f(z))')'}{(I_{\mu+1}(a, c)g(z))'}\right] \in P_k(\beta_1).$$

Set

$$(2.6) \quad p(z) = (1 - \alpha)\left[\frac{(I_\mu(a, c)f(z))'}{(I_\mu(a, c)g(z))'}\right] + \alpha\left[-\frac{(z(I_\mu(a, c)f(z))')'}{(I_\mu(a, c)g(z))'}\right],$$

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where p is an analytic function in E with $p(0) = 1$.

Now, $g \in MV_2(\mu + 1, \eta_1, a, c) \subset MV_2(\mu, \eta_2, a, c)$, where η_2 is given by the equation

$$(2.7) \qquad 2\eta_2^2 + (3 + 2\mu - 2\eta_1)\eta_2 - [2\eta_1(1 + \mu) + 1] = 0.$$

Therefore,

$$q(z) = \left(-\frac{(z(I_\mu(a,c)g(z))')'}{(I_\mu(a,c)g(z))'} \right) \in P(\eta_2), \quad z \in E.$$

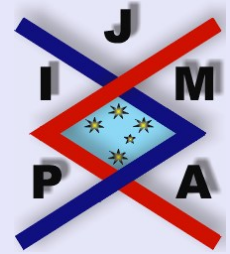By using (1.7), (2.5), (2.6) and (2.7), we have

$$(2.8) \qquad \left[p(z) + \alpha \frac{zp'(z)}{-q(z) + \mu + 1} \right] \in P_k(\beta_1), \quad q \in P(\eta_2), \quad z \in E.$$

With

$$p(z) = \left(\frac{k}{4} + \frac{1}{2} \right) [(1 - \beta_2)p_1(z) + \beta_2] - \left(\frac{k}{4} - \frac{1}{2} \right) [(1 - \beta_2)p_2(z) + \beta_2],$$

(2.8) can be written as

$$\left(\frac{k}{4} + \frac{1}{2} \right) \left[(1 - \beta_2)p_1(z) + \alpha \frac{(1 - \beta_2)zp_1'(z)}{-q(z) + \mu + 1} + \beta_2 \right]$$
$$- \left(\frac{k}{4} - \frac{1}{2} \right) \left[(1 - \beta_2)p_2(z) + \alpha \frac{(1 - \beta_2)zp_2'(z)}{-q(z) + \mu + 1} + \beta_2 \right],$$

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where

$$\left[(1 - \beta_2)p_i(z) + \alpha\frac{(1 - \beta_2)zp_i'(z)}{-q(z) + \mu + 1} + \beta_2\right] \in P(\beta_1), \quad z \in E, i = 1, 2.$$
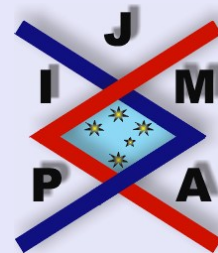
That is

$$\left[(1 - \beta_2)p_i(z) + \alpha\frac{(1 - \beta_2)zp_i'(z)}{-q(z) + \mu + 1} + (\beta_2 - \beta_1)\right] \in P, \quad z \in E, \quad i = 1, 2.$$

We form the functional $\Psi(u, v)$ by taking $u = u_1 + iu_2 = p_i$, $v = v_1 + iv_2 = zp_i'$, and

$$\Psi(u, v) = (1 - \beta_2)u + \alpha\frac{(1 - \beta_2)v}{(-q_1 + iq_2) + \mu + 1} + (\beta_2 - \beta_1), \quad (q = q_1 + iq_2).$$

The first two conditions of Lemma 1.1 are clearly satisfied. We verify (iii), with $v_1 \leq -\frac{1}{2}(1 + u_2^2)$ as follows

$$\begin{aligned}
&\mathrm{Re}\,\Psi(iu_2, v_1) \\
&= (\beta_2 - \beta_1) + \mathrm{Re}\left[\frac{\alpha(1 - \beta_2)v_1\{(-q_1 + \mu + 1) + iq_2\}}{(-q + \mu + 1)^2 + q_2^2}\right] \\
&\leq \frac{2(\beta - 2 - \beta_1)| - q + \mu + 1|^2 - \alpha(1 - \beta_2)(-q_1 + \mu + 1)(1 + u_2^2)}{2| - q + \mu + 1|^2} \\
&= \frac{A + Bu_2^2}{2C}, \quad C = | - q + \mu + 1|^2 > 0 \\
&\leq 0, \quad \text{if} \quad A \leq 0 \quad \text{and} \quad B \leq 0,
\end{aligned}$$

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where

$$A = 2(\beta_2 - \beta_1)|-q+\mu+1|^2 - \alpha(1-\beta_2)(-q_1+\mu+1),$$
$$B = -\alpha(1-\beta_2)(-q_1+\mu+1) \leq 0.$$

From $A \leq 0$, we get

(2.9) $$\beta_2 = \frac{2\beta_1|-q+\mu+1|^2 + \alpha \operatorname{Re}(-q(z)+\mu+1)}{2|-q+\mu+1|^2 + \alpha \operatorname{Re}(-q(z)+\mu+1)}.$$

Hence, using Lemma 1.1, it follows that $p(z)$, defined by (2.6), belongs to $P_k(\beta_2)$ and thus $f \in \mathcal{B}_k^\alpha(\mu, \beta_2, \eta_2, a, c)$, $z \in D$. This completes the proof of the first part. The second part of this result can be obtained by using similar arguments and the relation (1.6). □

Theorem 2.4.

(i) $\qquad \mathcal{B}_k^\alpha(\mu, \beta, \eta, a, c) \subset \mathcal{B}_k^0(\mu, \gamma, \eta, a, c)$

(ii) $\qquad \mathcal{B}_k^{\alpha_1}(\mu, \beta, \eta, a, c) \subset \mathcal{B}_k^{\alpha_2}(\mu, \beta, \eta, a, c), \quad for \quad 0 \leq \alpha_2 < \alpha_1.$

Proof. (i). Let

$$h(z) = \frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'},$$

$h(z)$ is analytic in E and $h(0) = 1$. Then

(2.10) $\quad (1-\alpha)\left[\dfrac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'}\right] + \alpha\left[-\dfrac{(z(I_\mu(a,c)f(z))')'}{(I_\mu(a,c)g(z))'}\right]$

$$= h(z) + \alpha\frac{zh'(z)}{-h_0(z)},$$

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where
$$h_0(z) = -\frac{(z(I_\mu(a,c)g(z))')'}{(I_\mu(a,c)g(z))'} \in P(\eta).$$

Since $f \in \mathcal{B}_k^\alpha(\mu, \beta, \eta, a, c)$, it follows that

$$\left[h(z) + \alpha \frac{zh'(z)}{-h_0(z)}\right] \in P_k(\beta), \quad h_0 \in P(\eta), \quad \text{for} \quad z \in E.$$

Let

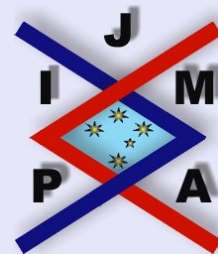$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z).$$

Then (2.10) implies that

$$\left[h_i(z) + \alpha \frac{zh_i'(z)}{-h_0(z)}\right] \in P(\beta), \quad z \in E, \quad i = 1, 2,$$

and from use of similar arguments, together with Lemma 1.1, it follows that $h_i \in P(\gamma)$, $i = 1, 2$, where

$$\gamma = \frac{2\beta|h_0|^2 + \alpha \operatorname{Re} h_0}{2|h_0|^2 + \alpha \operatorname{Re} h_0}.$$

Therefore $h \in P_k(\gamma)$, and $f \in \mathcal{B}_k^0(\mu, \gamma, \eta, a, c)$, $z \in D$. In particular, it can be shown that $h_i \in P(\beta)$, $i = 1, 2$. Consequently $h \in P_k(\beta)$ and $f \in \mathcal{B}_k^0(\mu, \beta, \eta, a, c)$ in D.

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For $\alpha_2 = 0$, we have (i). Therefore, we let $\alpha_2 > 0$ and $f \in \mathcal{B}_k^{\alpha_1}(\mu, \beta, \eta, a, c)$. There exist two functions $H_1, H_2 \in P_k(\beta)$ such that

$$H_1(z) = (1 - \alpha_1) \left[\frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'} \right] + \alpha_1 \left[-\frac{(z(I_\mu(a,c)f(z))')'}{(I_\mu(a,c)g(z))'} \right]$$

$$H_2(z) = \frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'}, \quad g \in MV_2(\mu, \eta, a, c).$$

Now

$$(2.11) \quad (1 - \alpha_2) \left[\frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'} \right] + \alpha_2 \left[-\frac{(z(I_\mu(a,c)f(z))')'}{(I_\mu(a,c)g(z))'} \right]$$

$$= \frac{\alpha_2}{\alpha_1} H_1(z) + \left(1 - \frac{\alpha_2}{\alpha_1} \right) H_2(z).$$

Since the class $P_k(\beta)$ is a convex set [10], it follows that the right hand side of (2.11) belongs to $P_k(\beta)$ and this shows that $f \in \mathcal{B}_k^{\alpha_2}(\mu, \beta, \eta, a, c)$ for $z \in D$. This completes the proof. $\qquad \square$
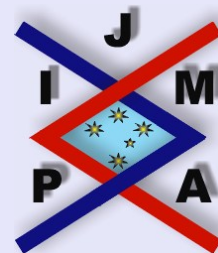
Let $f \in \mathcal{M}$, $b > 0$ and let the integral operator F_b be defined by

$$(2.12) \qquad F_b(f) = F_b(f)(z) = \frac{b}{z^{b+1}} \int_0^z t^b f(t) dt.$$

From (2.12), we note that

$$(2.13) \qquad z \left(I_\mu(a,c) F_b(f)(z) \right)' = b I_\mu(a,c) f(z) - (b+1) I_\mu(a,c) F_b(f)(z).$$

Using (2.12), (2.13) with similar techniques used earlier, we can prove the following:

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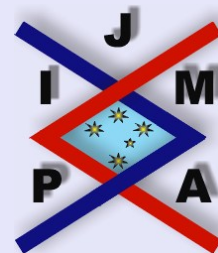
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Theorem 2.5. *Let* $f \in MR_k(\mu, \beta, a, c)$, *or* $MV_k(\mu, \beta, a, c)$, *or* $\mathcal{B}_k^\alpha(\mu, \beta, \eta, a, c)$, *for* $z \in D$. *Then* $F_b(f)$ *defined by (2.12) is also in the same class for* $z \in D$.

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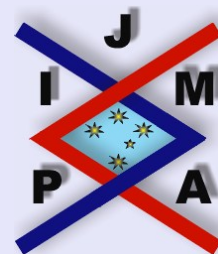
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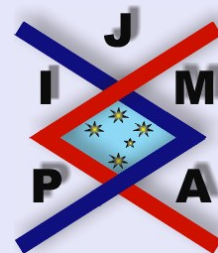
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