CLARKSON-MCCARTHY INTERPOLATED INEQUALITIES IN FINSLER NORMS

CRISTIAN CONDE

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Received:	29 August, 2008
Accepted:	14 January, 2009

Communicated by: R. Bhatia

2000 AMS Sub. Class.: Primary 46B70, 47A30; Secondary 46B20, 47B10.

Key words: p-Schatten class, Complex method, Clarkson-McCarthy inequalities.

Abstract: We apply the complex interpolation method to prove that, given two spaces $B_{p_0,a;s_0}^{(n)}, B_{p_1,b;s_1}^{(n)}$ of *n*-tuples of operators in the *p*-Schatten class of a Hilbert space *H*, endowed with weighted norms associated to positive and invertible operators *a* and *b* of *B*(*H*) then, the curve of interpolation $(B_{p_0,a;s_0}^{(n)}, B_{p_1,b;s_1}^{(n)})_{[t]}$ of the pair is given by the space of *n*-tuples of operators in the *p*-Schatten class of *H*, with the weighted norm associated to the positive invertible element $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$.



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1. Introduction

In [6], J. Clarkson introduced the concept of uniform convexity in Banach spaces and obtained that spaces L_p (or l_p) are uniformly convex for p > 1 throughout the following inequalities

$$2\left(\|f\|_{p}^{p}+\|g\|_{p}^{p}\right) \leq \|f-g\|_{p}^{p}+\|f+g\|_{p}^{p} \leq 2^{p-1}\left(\|f\|_{p}^{p}+\|g\|_{p}^{p}\right),$$

Let $(B(H), \|\cdot\|)$ denote the algebra of bounded operators acting on a complex and separable Hilbert space H, Gl(H) the group of invertible elements of B(H) and $Gl(H)^+$ the set of all positive elements of Gl(H).

If $X \in B(H)$ is compact we denote by $\{s_j(X)\}$ the sequence of singular values of X (decreasingly ordered). For 0 , let

$$\|X\|_p = \left(\sum s_j(X)^p\right)^{\frac{1}{p}}$$

and the linear space

$$B_p(H) = \{ X \in B(H) : ||X||_p < \infty \}.$$

For $1 \le p < \infty$, this space is called the *p*-Schatten class of B(H) (to simplify notation we use B_p) and by convention $||X|| = ||X||_{\infty} = s_1(X)$. A reference for this subject is [9].

C. McCarthy proved in [14], among several other results, the following inequalities for *p*-Schatten norms of Hilbert space operators:

(1.1)
$$2(\|A\|_p^p + \|B\|_p^p) \le \|A - B\|_p^p + \|A + B\|_p^p \le 2^{p-1}(\|A\|_p^p + \|B\|_p^p),$$



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for $2 \leq p < \infty$, and

(1.2)
$$2^{p-1}(\|A\|_{p}^{p} + \|B\|_{p}^{p}) \leq \|A - B\|_{p}^{p} + \|A + B\|_{p}^{p}$$
$$\leq 2(\|A\|_{p}^{p} + \|B\|_{p}^{p}),$$

for $1 \le p \le 2$.

These are non-commutative versions of Clarkson's inequalities. These estimates have been found to be very powerful tools in operator theory (in particular they imply the uniform convexity of B_p for 1) and in mathematical physics (see [16]).

M. Klaus has remarked that there is a simple proof of the Clarkson-McCarthy inequalities which results from mimicking the proof that Boas [4] gave of the Clarkson original inequalities via the complex interpolation method.

In a previous work [7], motivated by [1], we studied the effect of the complex interpolation method on $B_p^{(n)}$ (this set will be defined below) for $p, s \ge 1$ and $n \in \mathbb{N}$ with a Finsler norm associated with $a \in Gl(H)^+$:

$$||X||_{p,a;s} := ||a^{-1/2}Xa^{-1/2}||_p^s$$

From now on, for the sake of simplicity, we denote with lower case letters the elements of $Gl(H)^+$.

As a by-product, we obtain Clarkson type inequalities using the Klaus idea with the linear operator $T_n: B_p^{(n)} \longrightarrow B_p^{(n)}$ given by

$$T_n(\bar{X}) = (T_n(X_1, \dots, X_n)) = \left(\sum_{j=1}^n X_j, \sum_{j=1}^n \theta_j^1 X_j, \dots, \sum_{j=1}^n \theta_j^{n-1} X_j\right),$$

where $\theta_1, \ldots, \theta_n$ are the *n* roots of unity.

Recently, Kissin in [12], motivated by [3], obtained analogues of the Clarkson-McCarthy inequalities for n-tuples of operators from Schatten ideals. In this work



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the author considers H^n , the orthogonal sum of n copies of the Hilbert space H, and each operator $R \in B(H^n)$ can be represented as an $n \times n$ block-matrix operator $R = (R_{jk})$ with $R_{jk} \in B(H)$, and the linear operator $T_R : B_p^{(n)} \to B_p^{(n)}$ is defined by $T_R(\overline{A}) = R\overline{A}$. Finally we remark that the works [3] and [11] are generalizations of [10].

In these notes we obtain inequalities for the linear operator T_R in the Finsler norm $\|\cdot\|_{p,a;s}$ as by-products of the complex interpolation method and Kissin's inequalities.



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2. Geometric Interpolation

We follow the notation used in [2] and we refer the reader to [13] and [5] for details on the complex interpolation method. For completeness, we recall the classical Calderón-Lions theorem.

Theorem 2.1. Let \mathcal{X} and \mathcal{Y} be two compatible couples. Assume that T is a linear operator from \mathcal{X}_j to \mathcal{Y}_j bounded by M_j , j = 0, 1. Then for $t \in [0, 1]$

$$\|T\|_{\mathcal{X}_{[t]}\to\mathcal{Y}_{[t]}} \le M_0^{1-t}M_1^t.$$

Here and subsequently, let $1 \le p < \infty$, $n \in \mathbb{N}$, $s \ge 1$, $a \in Gl(H)^+$ and

$$B_p^{(n)} = \{\overline{A} = (A_1, \dots, A_n)^t : A_i \in B_p\},\$$

(where with t we denote the transpose of the n-tuple) endowed with the norm

$$\|\overline{A}\|_{p,a;s} = (\|A_1\|_{p,a}^s + \dots + \|A_n\|_{p,a}^s)^{1/s}$$

and \mathbb{C}^n endowed with the norm

$$|(a_0,\ldots,a_{n-1})|_s = (|a_0|^s + \cdots + |a_{n-1}|^s)^{1/s}.$$

From now on, we denote with $B_{p,a;s}^{(n)}$ the space $B_p^{(n)}$ endowed with the norm $\|(\cdot, \ldots, \cdot)\|_{p,a;s}$.

From Calderón-Lions interpolation theory we get that for $p_0, p_1, s_0, s_1 \in [1, \infty)$

(2.1)
$$\left(B_{p_0,1;s_0}^{(n)}, B_{p_1,1;s_1}^{(n)} \right)_{[t]} = B_{p_t,1;s_t}^{(n)},$$

where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$$
 and $\frac{1}{s_t} = \frac{1-t}{s_0} + \frac{t}{s_1}$



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Note that for p = 2, (1.1) and (1.2) both reduce to the *parallelogram law*

 $2(||A||_{2}^{2} + ||B||_{2}^{2}) = ||A - B||_{2}^{2} + ||A + B||_{2}^{2},$

while for the cases $p = 1, \infty$ these inequalities follow from the triangle inequality for B_1 and B(H) respectively. Then the inequalities (1.1) and (1.2) can be proved (for n = 2 via Theorem 2.1) by interpolation between the previous elementary cases with the linear operator $T_2: B_{p,1;p}^{(2)} \longrightarrow B_{p,1;p}^{(2)}, T_2(\overline{A}) = (A_1 + A_2, A_1 - A_2)^t$ as observed by Klaus.

In this section, we generalize (2.1) for the Finsler norms $\|(\cdot, \ldots, \cdot)\|_{p,a;s}$. In [7], we have obtained this extension for the particular case when $p_0 = p_1 = p$ and $s_0 = s_1 = s$. For sake of completeness, we recall this result

Theorem 2.2 ([7, Th. 3.1]). Let $a, b \in Gl(H)^+, 1 \leq p, s < \infty, n \in \mathbb{N}$ and $t \in (0, 1)$. Then

$$\left(B_{p,a;s}^{(n)}, B_{p,b;s}^{(n)}\right)_{[t]} = B_{p,\gamma_{a,b}(t);s}^{(n)},$$

where $\gamma_{a,b}(t) = a^{1/2} (a^{-1/2} b a^{-1/2})^t a^{1/2}$.

Remark 1. Note that when a and b commute the curve is given by $\gamma_{a,b}(t) = a^{1-t}b^t$. The previous corollary tells us that the interpolating space, $B_{p,\gamma_{a,b}(t);s}$ can be regarded as a weighted p-Schatten space with weight $a^{1-t}b^t$ (see [2, Th. 5.5.3]).

We observe that the curve $\gamma_{a,b}$ looks formally equal to the geodesic (or shortest curve) between positive definitive matrices ([15]), positive invertible elements of a C^* -algebra ([8]) and positive invertible operators that are perturbations of the *p*-Schatten class by multiples of the identity ([7]).

There is a natural action of Gl(H) on $B_p^{(n)}$, defined by

(2.2)
$$l: Gl(H) \times B_p^{(n)} \longrightarrow B_p^{(n)}, \quad l_g(\overline{A}) = (gA_1g^*, \dots, gA_ng^*)^t.$$



Proposition 2.3 ([7, Prop. 3.1]). The norm in $B_{p,a;s}^{(n)}$ is invariant for the action of the group of invertible elements. By this we mean that for each $\overline{A} \in B_p^{(n)}$, $a \in Gl(H)^+$ and $g \in Gl(H)$, we have

$$\left\|\overline{A}\right\|_{p,a;s} = \left\|l_g(\overline{A})\right\|_{p,gag^*;s}$$

Now, we state the main result of this paper, the general case $1 \le p_0, p_1, s_0, s_1 < \infty$.

Theorem 2.4. Let $a, b \in Gl(H)^+, 1 \le p_0, p_1, s_0, s_1 < \infty, n \in \mathbb{N}$ and $t \in (0, 1)$. Then

$$\left(B_{p_0,a;s_0}^{(n)}, B_{p_1,b;s_1}^{(n)}\right)_{[t]} = B_{p_t,\gamma_{a,b}(t);s_t}^{(n)}$$

where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$$
 and $\frac{1}{s_t} = \frac{1-t}{s_0} + \frac{t}{s_1}$

Proof. For the sake of simplicity, we will only consider the case n = 2 and omit the transpose. The proof below works for *n*-tuples ($n \ge 3$) with obvious modifications.

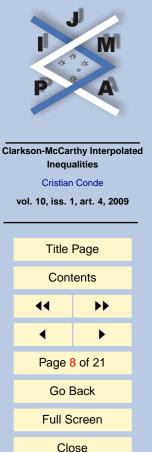
By the previous proposition, $\|(X_1, X_2)\|_{[t]}$ is equal to the norm of $a^{-1/2}(X_1, X_2)a^{-1/2}$ interpolated between the norms $\|(\cdot, \cdot)\|_{p_0,1;s_0}$ and $\|(\cdot, \cdot)\|_{p_1,c;s_1}$. Consequently it is sufficient to prove our statement for these two norms.

Let $t \in (0,1)$ and $(X_1, X_2) \in B_{p_t}^{(2)}$ such that $||(X_1, X_2)||_{p_t, c^t; s_t} = 1$, and define

$$g(z) = \left(U_1 \left| c^{\frac{z}{2}} c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} c^{\frac{z}{2}} \right|^{\lambda(z)}, U_2 \left| c^{\frac{z}{2}} c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} c^{\frac{z}{2}} \right|^{\lambda(z)} \right)$$
$$= (g_1(z), g_2(z)),$$

where

$$\lambda(z) = p_t \left(\frac{1-z}{p_0} + \frac{z}{p_1}\right) s_t \left(\frac{1-z}{s_0} + \frac{z}{s_1}\right)$$



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and $X_i = U_i |X_i|$ is the polar decomposition of X_i for i = 1, 2. Then for each $z \in S$, $g(z) \in B_{p_0}^{(2)} + B_{p_1}^{(2)}$ and

$$\begin{aligned} \|g(iy)\|_{p_{0},1;s_{0}}^{s_{0}} &= \left(\sum_{k=1}^{2} \left\| U_{k} \left| c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_{k} c^{-\frac{t}{2}} c^{\frac{iy}{2}} \right|^{\lambda(iy)} \right\|_{p_{0}}^{s_{0}} \right) \\ &\leq \left(\sum_{k=1}^{2} \left\| c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_{k} c^{-\frac{t}{2}} c^{\frac{iy}{2}} \right\|_{p_{t}}^{p_{t}} \right) \\ &\leq \left(\sum_{k=1}^{2} \| X_{k} \|_{p_{t},c^{t}}^{p_{t}} \right) = 1 \end{aligned}$$

and

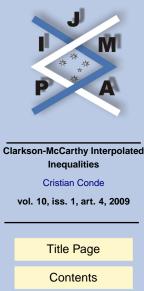
$$\|g(1+iy)\|_{p_1,c;s_1}^{s_1} \le \left(\sum_{k=1}^2 \|X_k\|_{p_t,c^t}^{p_t}\right) = 1.$$

Since $g(t) = (X_1, X_2)$ and $g = (g_1, g_2) \in \mathcal{F}\left(B_{p_0, 1; s_0}^{(2)}, B_{p_1, c; s_1}^{(2)}\right)$, we have $\|(X_1, X_2)\|_{[t]} \leq 1$. Thus we have shown that

$$\|(X_1, X_2)\|_{[t]} \le \|(X_1, X_2)\|_{p_t, c^t; s_t}$$

To prove the converse inequality, let $f = (f_1, f_2) \in \mathcal{F}\left(B_{p_0,1;s_0}^{(2)}, B_{p_1,c;s_1}^{(2)}\right)$; $f(t) = (X_1, X_2)$ and $Y_1, Y_2 \in B_{0,0}(H)$ (the set of finite-rank operators) with $||Y_k||_{q_t} \leq 1$, where q_t is the conjugate exponent for $1 < p_t < \infty$ (or a compact operator and $q = \infty$ if p = 1). For k = 1, 2, let

$$g_k(z) = c^{-\frac{z}{2}} Y_k c^{-\frac{z}{2}}$$



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Consider the function $h: S \to \left(\mathbb{C}^2, \left|(\cdot, \cdot)\right|_{s_t}\right)$,

 $h(z) = (tr(f_1(z)g_1(z)), tr(f_2(z)g_2(z))).$

Since f(z) is analytic in $\overset{\circ}{S}$ and bounded in S, then h is analytic in $\overset{\circ}{S}$ and bounded in S, and

$$h(t) = \left(tr\left(c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} Y_1 \right), tr\left(c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} Y_2 \right) \right) = (h_1(t), h_2(t)).$$

By Hadamard's three line theorem applied to h and the Banach space $(\mathbb{C}^2, |(\cdot, \cdot)|_{s_t})$, we have

$$|h(t)|_{s_t} \le \max\left\{\sup_{y\in\mathbb{R}} |h(iy)|_{s_t}, \sup_{y\in\mathbb{R}} |h(1+iy)|_{s_t}\right\}$$

For j = 0, 1,

$$\begin{split} \sup_{y \in \mathbb{R}} |h(j+iy)|_{s_t} &= \sup_{y \in \mathbb{R}} \left(\sum_{k=1}^2 |tr(f_k(j+iy)g_k(j+iy))|^{s_t} \right)^{\frac{1}{s_t}} \\ &= \sup_{y \in \mathbb{R}} \left(\sum_{k=1}^2 |tr(c^{-j/2}f_k(j+iy)c^{-j/2}g_k(iy))|^{s_t} \right)^{\frac{1}{s_t}} \\ &\leq \sup_{y \in \mathbb{R}} \left(\sum_{k=1}^2 \|f_k(j+iy)\|_{p,c^j}^{s_t} \right)^{\frac{1}{s_t}} \\ &\leq \|f\|_{\mathcal{F}\left(B_{p_0,1;s_0}^{(2)}, B_{p_1,c;s_1}^{(2)}\right)}, \end{split}$$



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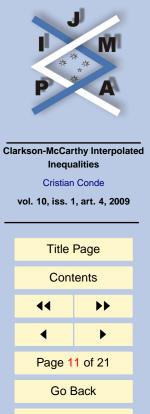
then

$$\begin{split} \|X_1\|_{p_t,c^t}^{s_t} + \|X_2\|_{p_t,c^t}^{s_t} &= \sup_{\substack{\|Y_1\|_{q_t} \le 1, Y_1 \in B_{00}(H) \\ \|Y_2\|_{q_t} \le 1, Y_2 \in B_{00}(H) \\ \end{array}} \{ |h_1(t)|_{s_t}^{s_t} + |h_2(t)|_{s_t}^{s_t} \} \\ &= \sup_{\substack{\|Y_1\|_{q_t} \le 1, Y_1 \in B_{00}(H) \\ \|Y_2\|_{q_t} \le 1, Y_2 \in B_{00}(H) \\ \end{array}} |h(t)|_{s_t}^{s_t} \le \|f\|_{\mathcal{F}\left(B_{p_0,1;s_0}^{(2)}, B_{p_1,c;s_1}^{(2)}\right)}. \end{split}$$

Since the previous inequality is valid for each $f \in \mathcal{F}\left(B_{p_0,1;s_0}^{(2)}, B_{p_1,c;s_1}^{(2)}\right)$ with $f(t) = (X_1, X_2)$, we have

 $\|(X_1, X_2)\|_{p_t, c^t; s_t} \le \|(X_1, X_2)\|_{[t]}.$

In the special case that $p_0 = p_1 = p$ and $s_0 = s_1 = s$ we obtain Theorem 2.2.



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3. Clarkson-Kissin Type Inequalities

Bhatia and Kittaneh [3] proved that if $2 \le p < \infty$, then

$$n^{\frac{2}{p}} \sum_{j=1}^{n} \|A_{j}\|_{p}^{2} \leq \sum_{j=1}^{n} \|B_{j}\|_{p}^{2} \leq n^{2-\frac{2}{p}} \sum_{j=1}^{n} \|A_{j}\|_{p}^{2}.$$
$$n \sum_{j=1}^{n} \|A_{j}\|_{p}^{p} \leq \sum_{j=1}^{n} \|B_{j}\|_{p}^{p} \leq n^{p-1} \sum_{j=1}^{n} \|A_{j}\|_{p}^{p}.$$

(for $0 , these two inequalities are reversed) where <math>B_j = \sum_{k=1}^n \theta_k^j A_k$ with $\theta_1, \ldots, \theta_n$ the *n* roots of unity.

If we interpolate these inequalities we obtain that

$$n^{\frac{1}{p}} \left(\sum_{j=1}^{n} \|A_j\|_p^{s_t} \right)^{\frac{1}{s_t}} \le \left(\sum_{j=1}^{n} \|B_j\|_p^{s_t} \right)^{\frac{1}{s_t}} \le n^{\left(1 - \frac{1}{p}\right)} \left(\sum_{j=1}^{n} \|A_j\|_p^{s_t} \right)^{\frac{1}{s_t}},$$

where

$$\frac{1}{s_t} = \frac{1-t}{2} + \frac{t}{p}.$$

Dividing by n^{s_t} , we obtain

$$n^{\frac{1}{p}} \left(\frac{1}{n} \sum_{j=1}^{n} \|A_{j}\|_{p}^{s_{t}}\right)^{\frac{1}{s_{t}}} \leq \left(\frac{1}{n} \sum_{j=1}^{n} \|B_{j}\|_{p}^{s_{t}}\right)^{\frac{1}{s_{t}}}$$
$$\leq n^{\left(1-\frac{1}{p}\right)} \left(\frac{1}{n} \sum_{j=1}^{n} \|A_{j}\|_{p}^{s_{t}}\right)^{\frac{1}{s_{t}}}$$



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This inequality can be rephrased as follows, if $\mu \in [2, p]$ then

$$n^{\frac{1}{p}} \left(\frac{1}{n} \sum_{j=1}^{n} \|A_{j}\|_{p}^{\mu}\right)^{\frac{1}{\mu}} \leq \left(\frac{1}{n} \sum_{j=1}^{n} \|B_{j}\|_{p}^{\mu}\right)^{\frac{1}{\mu}}$$
$$\leq n^{\left(1-\frac{1}{p}\right)} \left(\frac{1}{n} \sum_{j=1}^{n} \|A_{j}\|_{p}^{\mu}\right)^{\frac{1}{\mu}}$$

In each of the following statements $R \in Gl(H^n)$ and we denote by T_R the linear operator

$$T_R: B_p^{(n)} \longrightarrow B_p^{(n)} \qquad T_R(\overline{A}) = R\overline{A} = (B_1, \dots, B_n)^t,$$

with $B_j = \sum_{k=1}^n R_{jk}A_k$ and $\alpha = ||R^{-1}||, \beta = ||R||$ (we use the same symbol to denote the norm in B(H) and $B(H^n)$).

We observe that if the norm of T_R is at most M when

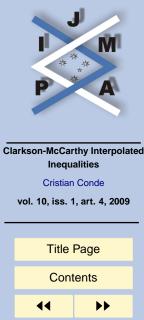
$$T_R: (B_p^{(n)}, \|(\cdot, \ldots, \cdot)\|_{p,1,s}) \to (B_p^{(n)}, \|(\cdot, \ldots, \cdot)\|_{p,1,r}),$$

then if we consider the operator T_R between the spaces

$$T_R: (B_p^{(n)}, \|(\cdot, \ldots, \cdot)\|_{p,a,s}) \to (B_p^{(n)}, \|(\cdot, \ldots, \cdot)\|_{p,b,r}),$$

its norm is at most F(a, b)M with

$$F(a,b) = \begin{cases} \min\{\|b^{-1}\| \|a\|, \|a^{1/2}b^{-1}a^{1/2}\| \|a^{-1}\| \|a\|\} & \text{if } a \neq b, \\ \|a^{-1}\| \|a\| & \text{if } a = b. \end{cases}$$



Remark 2. If $a^{-1/2} \in Gl(H)$ commutes with $R \in B(H^n)$, that is, if $a^{-1/2}$ commutes with R_{jk} for all $1 \leq j, k \leq n$, then F is reduced to

$$F(a,b) = \begin{cases} \min\{\|b^{-1}\| \|a\|, \|a^{1/2}b^{-1}a^{1/2}\|\} = \|a^{1/2}b^{-1}a^{1/2}\| & \text{if } a \neq b, \\ 1 & \text{if } a = b. \end{cases}$$

In [12], Kissin proved the following Clarkson type inequalities for the *n*-tuples $\overline{A} \in B_p^{(n)}$. If $2 \le p < \infty$ and $\lambda, \mu \in [2, p]$, or if $0 and <math>\lambda, \mu \in [p, 2]$, then

(3.1)
$$n^{-f(p)} \alpha^{-1} \left(\frac{1}{n} \sum_{j=1}^{n} \|A_{j}\|_{p}^{\mu} \right)^{\frac{1}{\mu}} \leq \left(\frac{1}{n} \sum_{j=1}^{n} \|B_{j}\|_{p}^{\lambda} \right)^{\frac{1}{\lambda}} \\ \leq n^{f(p)} \beta \left(\frac{1}{n} \sum_{j=1}^{n} \|A_{j}\|_{p}^{\mu} \right)^{\frac{1}{\mu}},$$

where $f(p) = \left| \frac{1}{p} - \frac{1}{2} \right|$.

Remark 3. This result extends the results of Bhatia and Kittaneh proved for $\mu = \lambda = 2$ or p and $R = (R_{jk})$ where

$$R_{jk} = e^{\left(i\frac{2\pi(j-1)(k-1)}{n}\right)}1$$

We use the inequalities (3.1) and the interpolation method to obtain the following inequalities.

Theorem 3.1. Let $a, b \in Gl(H)^+, \overline{A} \in B_p^{(n)}, 1 \le p < \infty$ and $t \in [0, 1]$, then (3.2) $\tilde{k}\left(\sum_{j=1}^n \|A_j\|_{p,a}^{\mu}\right)^{\frac{1}{\mu}} \le \left(\sum_{j=1}^n \|B_j\|_{p,\gamma_{a,b}(t)}^{\lambda}\right)^{\frac{1}{\lambda}} \le \tilde{K}\left(\sum_{j=1}^n \|A_j\|_{p,a}^{\mu}\right)^{\frac{1}{\mu}}$



where

$$\tilde{k} = \tilde{k}(p, a, b, t) = F(a, a)^{t-1} F(b, a)^{-t} n^{\frac{1}{\lambda} - \frac{1}{\mu} - \left|\frac{1}{p} - \frac{1}{2}\right|} \alpha^{-1},$$

and

$$\tilde{K} = \tilde{K}(p, a, b, t) = F(a, a)^{1-t} F(a, b)^t n^{\frac{1}{\lambda} - \frac{1}{\mu} + \left|\frac{1}{p} - \frac{1}{2}\right|} \beta,$$

if $2 \le p$ and $\lambda, \mu \in [2, p]$ or if $1 \le p \le 2$ and $\lambda, \mu \in [p, 2]$.

Proof. We will denote by $\gamma(t) = \gamma_{a,b}(t)$, when no confusion can arise. Consider the space $B_p^{(n)}$ with the norm:

$$\|\overline{A}\|_{p,a;s} = (\|A_1\|_{p,a}^s + \dots + \|A_n\|_{p,a}^s)^{1/s},$$

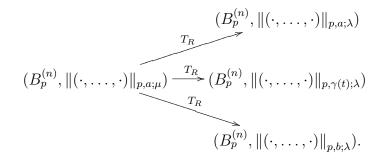
where $a \in Gl(H)^+$.

By (3.1), the norm of T_R is at most $F(a, a)n^{\frac{1}{\lambda} - \frac{1}{\mu} + \left|\frac{1}{p} - \frac{1}{2}\right|} \beta$ when $T_R: \left(B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\mu} \right) \longrightarrow \left(B_p^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\lambda} \right),$

and the norm of T_R is at most $F(a,b)n^{\frac{1}{\lambda}-\frac{1}{\mu}+\left|\frac{1}{p}-\frac{1}{2}\right|}\beta$ when

$$T_R: \left(B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,a;\mu}\right) \longrightarrow \left(B_p^{(n)}, \|(\cdot,\ldots,\cdot)\|_{p,b;\lambda}\right).$$

Therefore, using the complex interpolation, we obtain the following diagram of interpolation for $t \in [0, 1]$





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By Theorem 2.1, T_R satisfies

(3.3)
$$||T_R(\overline{A})||_{p,\gamma(t);\lambda} \le F(a,a)^{1-t}F(a,b)^t n^{\frac{1}{\lambda}-\frac{1}{\mu}+\left|\frac{1}{p}-\frac{1}{2}\right|}\beta ||\overline{A}||_{p,a;\mu}.$$

Now applying the Complex method to

$$(B_{p}^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\lambda}) \xrightarrow{T_{R^{-1}}} (B_{p}^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,\gamma(t);\lambda}) \xrightarrow{T_{R^{-1}}} (B_{p}^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,a;\mu})$$
$$(B_{p}^{(n)}, \|(\cdot, \dots, \cdot)\|_{p,b;\lambda})$$

one obtains

(3.4)
$$\|T_{R^{-1}}(\overline{A})\|_{p,a;\mu} \le F(a,a)^{1-t}F(b,a)^t n^{\frac{1}{\mu}-\frac{1}{\lambda}+\left|\frac{1}{p}-\frac{1}{2}\right|} \alpha \|\overline{A}\|_{p,\gamma(t);\lambda}.$$

Replacing in (3.4) \overline{A} by $R\overline{A}$ we obtain

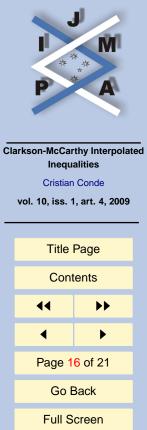
(3.5)
$$\|\overline{A}\|_{p,a;\mu} \leq F(a,a)^{1-t} F(b,a)^t n^{\frac{1}{\mu} - \frac{1}{\lambda} + \left|\frac{1}{p} - \frac{1}{2}\right|} \alpha \|R\overline{A}\|_{p,\gamma(t);\lambda},$$

or equivalently

(3.6)
$$F(a,a)^{t-1}F(b,a)^{-t}n^{\frac{1}{\lambda}-\frac{1}{\mu}-\left|\frac{1}{p}-\frac{1}{2}\right|}\alpha^{-1}\|\overline{A}\|_{p,a;\mu} \le \|T_R(\overline{A})\|_{p,\gamma(t);\lambda}.$$

Finally, the inequalities (3.3) and (3.6) complete the proof.

We remark that the previous statement is a generalization of Th. 4.1 in [7] where $T_n = T_R$ with $R = \left(e^{(i\frac{2\pi(j-1)(k-1)}{n})}1\right)_{1\leq j,k\leq n}$ and $a^{-1/2}$ commutes with R for all $a \in Gl(H)^+$.



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On the other hand, it is well known that if x_1, \ldots, x_n are non-negative numbers, $s \in \mathbb{R}$ and we denote $\mathcal{M}_s(\overline{x}) = \left(\frac{1}{n}\sum_{i=1}^n x_i^s\right)^{1/s}$ then for 0 < s < s', $\mathcal{M}_s(\overline{x}) \leq \mathcal{M}_{s'}(\overline{x})$.

If we denote $\|\overline{B}\| = (\|B_1\|_p, \dots, \|B_n\|_p)$ and we consider $1 , then it holds for <math>t \in [0, 1]$ and $\frac{1}{s_t} = \frac{1-t}{p} + \frac{t}{q}$ that

$$\mathcal{M}_{s_t}(\|\overline{B}\|) \leq \mathcal{M}_q(\|\overline{B}\|) \leq r^{\frac{2}{p}-1} \beta^{\frac{2}{q}} n^{\frac{-1}{q}} \left(\sum_{j=1}^n \|A_j\|_p^p \right)^{\frac{1}{p}},$$

or equivalently

(3.7)
$$\left(\sum_{j=1}^{n} \|B_{j}\|_{p}^{s_{t}}\right)^{\frac{1}{s_{t}}} \leq r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_{t}}-\frac{1}{q}}\left(\sum_{j=1}^{n} \|A_{j}\|_{p}^{p}\right)^{\frac{1}{p}}.$$

Analogously, for $2 \le p < \infty$ we get

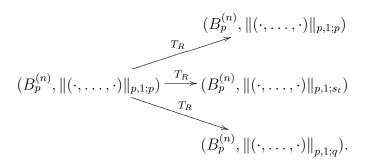
(3.8)
$$\left(\sum_{j=1}^{n} \|A_{j}\|_{p}^{p}\right)^{\frac{1}{p}} \leq \rho^{1-\frac{2}{p}} \alpha^{\frac{2}{p}} n^{\frac{1}{q}-\frac{1}{s_{t}}} \left(\sum_{j=1}^{n} \|B_{j}\|_{p}^{s_{t}}\right)^{\frac{1}{s_{t}}}$$

where $\frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}$.

Now we can use the interpolation method with the inequalities (3.7) and (3.1) (or (3.8) and (3.1)).

If we consider the following diagram of interpolation with $1 and <math>t \in$





By Theorem 2.1 and (3.1), T_R satisfies

(3.9)
$$\|T_R(\overline{A})\|_{p,1;s_t} \le \left(n^{f(p)}\beta\right)^{1-t} \left(r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}\right)^t \|\overline{A}\|_{p,1;p}.$$

Finally, from the inequalities (3.7) and (3.9) we obtain

$$\left(\sum_{j=1}^{n} \|B_{j}\|_{p}^{s_{t}}\right)^{\frac{1}{s_{t}}} \leq \min\left\{r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_{t}}-\frac{1}{q}}, n^{f(p)(1-t)}\beta^{1+t(\frac{2}{q}-1)}r^{(\frac{2}{p}-1)t}\right\} \left(\sum_{j=1}^{n} \|A_{j}\|_{p}^{p}\right)^{\frac{1}{p}}$$

We can summarize the previous facts in the following statement.

Theorem 3.2. Let $\overline{A} \in B_p^{(n)}$ and $B = R\overline{A}$, where $R = (R_{jk})$ is invertible. Let $r = \max ||R_{jk}||$, $\rho = \max ||(R^{-1})_{jk}||$ and q be the conjugate exponent of p. Then, for $t \in [0, 1]$ we get

$$\left(\sum_{j=1}^{n} \|A_{j}\|_{p}^{p}\right)^{\frac{1}{p}} \leq \min\left\{\rho^{1-\frac{2}{p}}\alpha^{\frac{2}{p}}n^{\frac{1}{q}-\frac{1}{s_{t}}}, n^{f(p)t}\alpha^{t+(1-t)\frac{2}{p}}\rho^{(1-\frac{2}{p})(1-t)}\right\}\left(\sum_{j=1}^{n} \|B_{j}\|_{p}^{s_{t}}\right)^{\frac{1}{s_{t}}}$$



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$$\begin{split} & if 2 \le p \text{ and } \frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}, \text{ or} \\ & \left(\sum_{j=1}^n \|B_j\|_p^{s_t}\right)^{\frac{1}{s_t}} \le \min\left\{r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_t}-\frac{1}{q}}, n^{f(p)(1-t)}\beta^{1+t(\frac{2}{q}-1)}r^{(\frac{2}{p}-1)t}\right\} \left(\sum_{j=1}^n \|A_j\|_p^p\right)^{\frac{1}{p}}, \\ & if 1$$

Finally using the Finsler norm $\|(\cdot, \ldots, \cdot)\|_{p,a;s}$, Calderón's method and the previous inequalities we obtain:

Corollary 3.3. Let $a, b \in Gl(H)^+$, $\overline{A} \in B_p^{(n)}$ and $B = R\overline{A}$, where $R = (R_{jk})$ is invertible. Let $r = \max ||R_{jk}||$, $\rho = \max ||(R^{-1})_{jk}||$ and q be the conjugate exponent of p. Then, for $t, u \in [0, 1]$ we get

$$\left(\sum_{j=1}^{n} \|A_j\|_{p,a}^{p}\right)^{\frac{1}{p}} \leq F(a,a)^{1-u} F(b,a)^{u} M_1 \left(\sum_{j=1}^{n} \|B_j\|_{p,\gamma_{a,b}(u)}^{s_t}\right)^{\frac{1}{s_t}},$$

if
$$2 \le p$$
, $\frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}$ and
 $M_1 = M_1(R, p, t) = \min\left\{\rho^{1-\frac{2}{p}}\alpha^{\frac{2}{p}}n^{\frac{1}{q}-\frac{1}{s_t}}, n^{f(p)t}\alpha^{t+(1-t)\frac{2}{p}}\rho^{(1-\frac{2}{p})(1-t)}\right\}$

or

$$\left(\sum_{j=1}^{n} \|B_{j}\|_{p,\gamma_{a,b}(u)}^{s_{t}}\right)^{\frac{1}{s_{t}}} \leq F(a,a)^{1-u}F(a,b)^{u}M_{2}\left(\sum_{j=1}^{n} \|A_{j}\|_{p}^{p}\right)^{\frac{1}{p}},$$

if
$$1 , $\frac{1}{s_t} = \frac{1-t}{p} + \frac{t}{q}$ and
 $M_2 = M_2(R, p, t) = \min\left\{r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_t}-\frac{1}{q}}, n^{f(p)(1-t)}\beta^{1+t(\frac{2}{q}-1)}r^{(\frac{2}{p}-1)t}\right\}.$$$

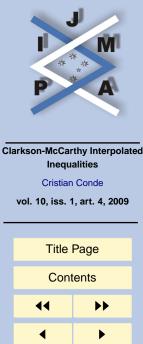


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