Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 4, Article 99, 2005

L'HOSPITAL-TYPE RULES FOR MONOTONICITY, AND THE LAMBERT AND SACCHERI QUADRILATERALS IN HYPERBOLIC GEOMETRY

IOSIF PINELIS

DEPARTMENT OF MATHEMATICAL SCIENCES
MICHIGAN TECHNOLOGICAL UNIVERSITY
HOUGHTON, MICHIGAN 49931
ipinelis@mtu.edu

Received 10 August, 2005; accepted 24 August, 2005
Communicated by M. Vuorinen

ABSTRACT. Elsewhere we developed rules for the monotonicity pattern of the ratio f/g of two functions on an interval of the real line based on the monotonicity pattern of the ratio f'/g' of the derivatives. These rules are applicable even more broadly than the l'Hospital rules for limits, since we do not require that both f and g, or either of them, tend to 0 or ∞ at an endpoint of the interval.

Here these rules are used to obtain monotonicity patterns of the ratios of the pairwise distances between the vertices of the Lambert and Saccheri quadrilaterals in the Poincaré model of hyperbolic geometry. Some of the results may seem surprising. Apparently, the methods will work for other ratios of distances in hyperbolic geometry and other Riemann geometries.

The presentation is mainly self-contained.

Key words and phrases: L'Hospital type rules for monotonicity, Hyperbolic geometry, Poincaré model, Lambert quadrilaterals, Saccheri quadrilaterals, Riemann geometry, Differential geometry.

2000 *Mathematics Subject Classification.* Primary 53A35, 26A48; Secondary 51M25, 51F20, 51M15, 26A24.

1. L'HOSPITAL-TYPE RULES FOR MONOTONICITY

Let $-\infty \le a < b \le \infty$. Let f and g be differentiable functions defined on the interval (a, b), and let $r := f/g$. It is assumed throughout that g and g' do not take on the zero value and do not change their respective signs on (a, b). In [16], general "rules" for monotonicity patterns, resembling the usual l'Hospital rules for limits, were given. In particular, according to [16, Proposition 1.9], the dependence of the monotonicity pattern of r (on (a, b)) on that of $\rho := f'/g'$ (and also on the sign of gg') is given by Table 1.1, where, for instance, $r \searrow\nearrow$ means that there is some $c \in (a, b)$ such that $r \searrow$ (that is, r is decreasing) on (a, c) and $r \nearrow$ on (c, b). Now suppose that one also knows whether $r \nearrow$ or $r \searrow$ in a right neighborhood of a and in a left neighborhood of b; then Table 1.1 uniquely determines the monotonicity pattern of r.

ISSN (electronic): 1443-5756

239-05

ρ	gg'	r
$\nearrow$	> 0	$\nearrow$ or $\searrow$ or $\searrow\nearrow$
$\searrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$
$\nearrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$
$\searrow$	< 0	$\nearrow$ or $\searrow$ or $\searrow\nearrow$

Table 1.1: Basic rules for monotonicity

Clearly, these l'Hospital-type rules for monotonicity patterns are helpful wherever the l'Hospital rules for limits are so, and even beyond that, because the monotonicity rules do not require that both f and g (or either of them) tend to 0 or ∞ at any point.

The proof of these rules is very easy if one additionally assumes that the derivatives f' and g' are continuous and r' has only finitely many roots in (a, b) (which will be the case if, for instance, r is not a constant and f and g are real-analytic functions on $[a, b]$): Indeed, suppose that the assumptions $\rho \nearrow$ and $gg' > 0$ of the first line of Table 1.1 hold. Then it suffices to show that $r'(x)$ may change sign only from $-$ to $+$ as x increases from a to b. To obtain a contradiction, suppose the contrary, so that there is some root u of r' in (a, b) such that in some right neighborhood (u, t) of the root u one has $r' < 0$ and hence $r < r(u)$. Consider now the key identity

$$g^2\, r' = (\rho - r)\, g\, g', \tag{1.1}$$

which is easy to check. Then the conditions $r'(u) = 0$ and $r' < 0$ on (u, t) imply, respectively, that $\rho(u) = r(u)$ and $\rho < r$ on (u, t). It follows that $\rho < r < r(u) = \rho(u)$ on (u, t), which contradicts the condition $\rho \nearrow$. The other three lines of Table 1.1 can be treated similarly. A proof without using the additional conditions (that the derivatives f' and g' are continuous and r' has only finitely many roots) was given in [16].

Based on Table 1.1, one can generally infer the monotonicity pattern of r given that of ρ, however complicated the latter is. In particular, one has Table 1.2.

ρ	gg'	r
$\nearrow\searrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\searrow\nearrow\searrow$
$\searrow\nearrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\nearrow\searrow\nearrow$
$\nearrow\searrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\nearrow\searrow\nearrow$
$\searrow\nearrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\searrow\nearrow\searrow$

Table 1.2: Derived rules for monotonicity

In the special case when both f and g vanish at an endpoint of the interval (a, b), l'Hospital-type rules for monotonicity and their applications can be found, in different forms and with different proofs, in [9, 11, 14, 8, 2, 3, 1, 4, 5, 15, 16, 17, 18].

The *special-case* rule can be stated as follows: Suppose that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$; suppose also that ρ is increasing or decreasing on the entire interval (a, b); then, respectively, r is increasing or decreasing on (a, b). When the condition $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$ does hold, the special-case rule may be more convenient, because then one does not have to investigate the monotonicity pattern of ratio r near the endpoints of the interval (a, b).

The special-case rule is easy to prove. For instance, suppose that $f(a+) = g(a+) = 0$. Then g and g' must have the same sign on (a, b). By the mean-value theorem, for every $x \in (a, b)$ there is some $\xi \in (a, x)$ such that $r(x) = \rho(\xi)$. Now the rule follows by identity (1.1).

This latter proof is essentially borrowed from [2, Lemma 2.2]. Another very simple proof of the special-case rule was given in [15]; that proof remains valid under somewhat more general conditions on f and g. A unified treatment of the monotonicity rules, applicable whether or not f and g vanish at an endpoint of (a, b), can be found in [16].

(L'Hospital's rule for the limit $r(b-)$ (say) when $g(b-) = \infty$ does not have a "special-case" analogue for monotonicity, even if one also has $f(b-) = \infty$. For example, consider $f(x) = x - 1 - e^{-x}$ and $g(x) = x$ for $x > 0$. Then $r \nearrow$ on $(0, \infty)$, even though $\rho \searrow$ on $(0, \infty)$ and $f(\infty-) = g(\infty-) = \infty$.)

In view of what has been said here, it should not be surprising that a very wide variety of applications of these l'Hospital-type rules for monotonicity patterns were given: in areas of analytic inequalities [15, 16, 19, 5], approximation theory [17], differential geometry [8, 9, 11], information theory [15, 16], (quasi)conformal mappings [1, 2, 3, 4], statistics and probability [14, 16, 17, 18], etc.

Clearly, the stated rules for monotonicity could be helpful when f' or g' can be expressed simpler than f or g, respectively. Such functions f and g are essentially the same as the functions that could be taken to play the role of u in the integration-by-parts formula $\int u\,dv = uv - \int v\,du$; this class of functions includes polynomial, logarithmic, inverse trigonometric and inverse hyperbolic functions, and as well as non-elementary "anti-derivative" functions of the form $x \mapsto \int_a^x h(u)\,du$ or $x \mapsto \int_x^b h(u)\,du$.

("Discrete" analogues, for f and g defined on $\mathbb{Z}$, of the l'Hospital-type rules for monotonicity, are available as well [20].)

In the present paper, we use the stated rules for monotonicity to obtain monotonicity properties of the Lambert and Saccheri quadrilaterals in hyperbolic geometry. This case represents a perfect match between the two areas. Indeed, the distances in hyperbolic geometry are expressed in terms of inverse hyperbolic functions, whose derivatives are algebraic. One can expect these rules to work for other Riemann geometries as well, since the geodesic distances there are line integrals, too.

2. MONOTONICITY PROPERTIES OF THE LAMBERT AND SACCHERI QUADRILATERALS

2.1. Background.

2.1.1. *Hyperbolic plane.* The Lambert and Saccheri quadrilaterals are quadrilaterals in the Poincaré hyperbolic plane H^2.

The significance of the Poincaré model is that, by the Riemann mapping theorem, any simply connected analytic Riemann surface is conformally equivalent to H^2, $\mathbb{C}$, or $\mathbb{C} \cup \{\infty\}$ [7, Theorem 9.1]. Moreover, any analytic Riemann surface is conformally equivalent to the quotient surface $\tilde{R}/G$, where $\tilde{R}$ is H^2, $\mathbb{C}$, or $\mathbb{C} \cup \{\infty\}$, and G is a group of Möbius transformations acting discontinuously on (the covering surface) $\tilde{R}$ [7, Proposition 9.2.3]. However, this comment will not be used further in this paper.

To make this section mainly self-contained, let us fix the terminology and basic facts concerning the Poincaré model of hyperbolic plane geometry. The set of points in this model is the upper half-plane

$$H^2 := \{z \in \mathbb{C} \colon \ \operatorname{Im} z > 0\}.$$

This set is endowed with the differential metric element

$$\mathrm{d}s := \frac{|\mathrm{d}z|}{\operatorname{Im} z},$$

so that the length of any rectifiable curve in H^2 is obtained as the line integral of $\mathrm{d}s$. For $x \in \mathbb{R}$ and $r \in \mathbb{R} \setminus \{0\}$, let us refer to the semicircles

$$[\![x-r, x+r]\!] := \{z \in H^2 \colon |z-x| = |r|\},$$

centered at point x and of radius $|r|$, and the vertical rays

$$[\![x, \infty]\!] := \{z \in H^2 \colon \operatorname{Re} z = x\}$$

as the *"lines"*. It will be seen in a moment that these "lines" are precisely the geodesics in this geometry, so that the geodesics are orthogonal to the real axis.

For $x \in \mathbb{R}$ and $r \in \mathbb{R}\setminus\{0\}$, let $\iota_{x,r}$ denote the reflection of H^2 in the semicircle $[\![x-r, x+r]\!]$, so that, for $z \in H^2$,

$$\iota_{x,r}(z) := x + \frac{r^2}{\overline{z} - x}.$$

It is easy to see that this transformation is inverse to itself and preserves H^2 as well as the metric element $\mathrm{d}s$, and hence also the (absolute value of the) angles. Indeed, if $w := \iota_{x,r}(z)$ for $z \in H^2$, then $\operatorname{Im} w = r^2 \operatorname{Im} z/|\overline{z}-x|^2$ and $\mathrm{d}w = -r^2\,\mathrm{d}\overline{z}/(\overline{z}-x)^2$, so that $\operatorname{Im} w > 0$ and $|\mathrm{d}w|/\operatorname{Im} w = |\mathrm{d}z|/\operatorname{Im} z$.

Let G be the group of transformations of H^2 generated by all such reflections. Then G preserves the metric element $\mathrm{d}s$. Note that G contains all the homotheties $z \mapsto \eta_{x,\lambda}(z) := x + \lambda(z-x)$, horizontal parallel translations $z \mapsto \sigma_x(z) := z + x$, and reflections $z \mapsto \iota_{x,\infty}(z) := 2x - \overline{z}$ in the vertical rays $[\![x, \infty]\!]$, where $x \in \mathbb{R}$ and $\lambda > 0$; indeed, $\eta_{x,\lambda} = \iota_{x,\sqrt{\lambda}} \circ \iota_{x,1}$, $\iota_{x,\infty} = \iota_{x+r,2r} \circ \iota_{x-r,2r} \circ \iota_{x+r,2r}$, and $\sigma_x = \iota_{x/2,\infty} \circ \iota_{0,\infty}$.

It is easy to see that the geodesic connecting two points z_1 and z_2 on the same vertical ray $[\![x, \infty]\!]$ ($x \in \mathbb{R}$) is the segment of that ray with the endpoints z_1 and z_2, so that the geodesic distance $d(z_1, z_2)$ between such z_1 and z_2 is $|\ln(y_1/y_2)|$, where $y_j := \operatorname{Im} z_j$, $j = 1, 2$. Now it is seen that group G acts transitively on the set of all ordered pairs (z_1, z_2) of points on the vertical ray $[\![x, \infty]\!]$ with a fixed value of the distance $d(z_1, z_2)$ — in the sense that, for any two pairs (z_1, z_2) and (w_1, w_2) of points on $[\![x, \infty]\!]$ with $d(z_1, z_2) = d(w_1, w_2)$, there is some transformation g in G such that $g(z_j) = w_j$, $j = 1, 2$; indeed, it suffices to take g to be a single reflection $\iota_{x,r}$ or a single homothety $\eta_{x,\lambda}$, for some $r > 0$ or $\lambda > 0$.

Next, the reflection $\iota_{x+r,2r}$ maps the semicircle $[\![x-r, x+r]\!]$ onto the vertical ray $[\![x-r, \infty]\!]$, and hence vice versa, for all $x \in \mathbb{R}$ and $r \in \mathbb{R} \setminus \{0\}$. Moreover, any two distinct points in H^2 lie on exactly one "line".

It follows now that indeed the "lines" are precisely the geodesics, and group G acts transitively on the set of all ordered pairs (z_1, z_2) of points in H^2 with any fixed value of the geodesic distance $d(z_1, z_2)$. Another corollary here is the formula for the geodesic distance between any two points z_1 and z_2 of H^2:

$$d(z_1, z_2) = \operatorname{arcch}\left(1 + \frac{|z_1 - z_2|^2}{2 \operatorname{Im} z_1 \, \operatorname{Im} z_2}\right), \tag{2.1}$$

where $\operatorname{arcch} x := \ln\left(x + \sqrt{x^2-1}\right)$ for $x \geqslant 1$; cf. [6, Theorem 7.2.1(ii)]. One can now also easily derive Pythagoras' theorem,

$$\operatorname{ch} c = \operatorname{ch} a \, \operatorname{ch} b, \tag{2.2}$$

for a right-angled (geodesic) triangle ABC with side c opposite to the right-angle vertex C and two other sides a and b; indeed, such a triangle is G-congruent, for some $k \in (0, 1)$ and

$\theta \in (0, \pi/2)$, to the triangle with vertices $C_* = i$, $A_* = k\, i$, and $B_* = e^{i\theta}$; cf. [6, Theorem 7.11.1]. (Yet another corollary, not to be used in this paper, is that G is the group of all isometries of H^2.)

2.1.2. *Lambert's and Saccheri's quadrilaterals.* A Lambert quadrilateral is a quadrilateral in the Poincaré hyperbolic plane with angles $\pi/2$, $\pi/2$, $\pi/2$, and φ, for some φ; a Saccheri quadrilateral is a quadrilateral (also in the hyperbolic plane) with angles $\pi/2$, $\pi/2$, ψ and ψ, for some ψ [6, Section 7.17]. See Figure 2.1.

For a Saccheri quadrilateral, let us refer to (the length of) its side adjacent to the right angles as the *base*, its opposite side as the *top*, and to either of the other two (congruent to each other) sides simply as the *side*.

A Lambert quadrilateral has two sides each adjacent to two of the three right angles. Let us arbitrarily choose one of these two sides and refer to it as the *base*, and to the other one of the two as the *(short) side*. The side opposite to the base will again be referred to as the *top*, and the fourth side as the *long side*. It will be seen in the next subsection that indeed the long side is always longer than the short one.

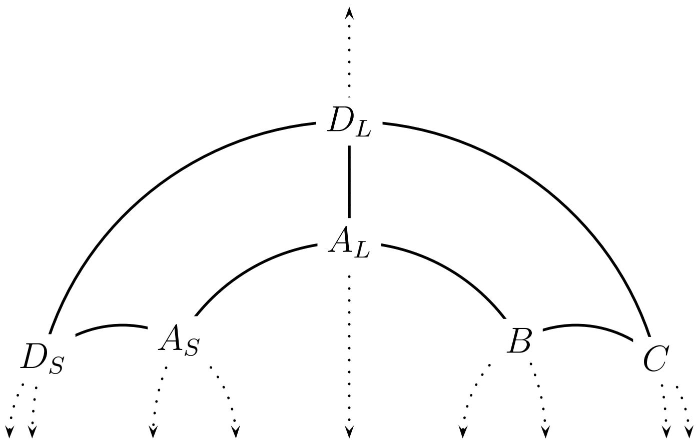

Figure 2.1: Lambert's (A_LBCD_L) and Saccheri's (A_SBCD_S) quadrilaterals; A_LB, A_LD_L, BC, and CD_L are respectively the base, short side, long side, and top of the Lambert quadrilateral; A_SB, $A_SD_S = BC$, and CD_S are respectively the base, side, and top of the Saccheri quadrilateral; the angles at vertices A_S, B, A_L, and D_L are $\pi/2$.

It follows from the discussion in Subsubsection 2.1.1 that the group G acts transitively on the set of all Saccheri quadrilaterals with any given values of the base and the side, as well as on the set of all Lambert quadrilaterals with any given values of the base and the short side. That is, all Saccheri quadrilaterals with any given values of the base and the side are G*-congruent* to each other, and so, they have the same geodesic distances between any two of their corresponding vertices. The same holds for all Lambert quadrilaterals with any given values of the base and the short side.

2.2. **Main Results.**

2.2.1. *Lambert quadrilaterals.* In view of the conclusions of Subsection 2.1, any Lambert quadrilateral is G-congruent, for some

$$k \in (0, 1) \quad \text{and} \quad \theta \in (0, \pi/2),$$

to the particular Lambert quadrilateral $ABCD$ with vertices

$$A = k\,i, \quad B = i, \quad C = e^{i\theta}, \quad D = k\,e^{i\psi}, \quad \text{where} \quad \psi := \arccos\big(\operatorname{ch}(\ln k)\cos\theta\big)$$

(see Figure 2.2), so that, by (2.1),

$$AB = \ln\frac{1}{k}, \quad BC = \operatorname{arcch} c, \quad CD = \operatorname{arcch}\frac{1+k^2}{q}, \quad , \tag{2.3}$$
$$AD = \operatorname{arcch}\frac{2\,c\,k}{q}, \quad AC = \operatorname{arcch}\frac{c\,(1+k^2)}{2\,k}, \quad BD = \operatorname{arcch}\frac{c\,(1+k^2)}{q},$$

$$\text{where} \quad q := \sqrt{(1+k^2)^2 - c^2\,(1-k^2)^2} \quad \text{and} \quad c := 1/\sin\theta. \tag{2.4}$$

(One can verify, using (2.2) and (2.3), that indeed $\angle A = \angle B = \angle C = \pi/2$.) Then one may refer to AB as the base, of length $\ln(1/k)$, and to BC as the short side, of length $\operatorname{arcch} c$. Note that, for the point D to exist in H^2, one must have $\operatorname{ch}(\ln k)\cos\theta < 1$, which is equivalent to

$$1 < c < c_k, \quad \text{where} \quad c_k := \frac{1+k^2}{1-k^2}.$$

Let us fix (the length of) the base AB (so that $k \in (0,1)$ is fixed) and let c increase from 1 to c_k, so that the short side $BC = \operatorname{arcch} c$ increases from 0 to $\operatorname{arcch} c_k$. The goal here is to determine the monotonicity patterns of $\binom{6}{2} = 15$ completely representative pairwise ratios $r = CD/AD,\ CD/BD, \dots,\ BC/AB$ of the $\binom{4}{2} = 6$ (geodesic) distances between the four vertices A, B, C, D. For each pair of such distances, it is enough to consider only one of the two mutually reciprocal ratios; indeed, for example, the monotonicity pattern of the ratio CD/AD determines that of AD/CD. All the ratios r will be expressed as functions of c. (We do not distinguish in terminology or notation between a segment of a geodesic and its length.)

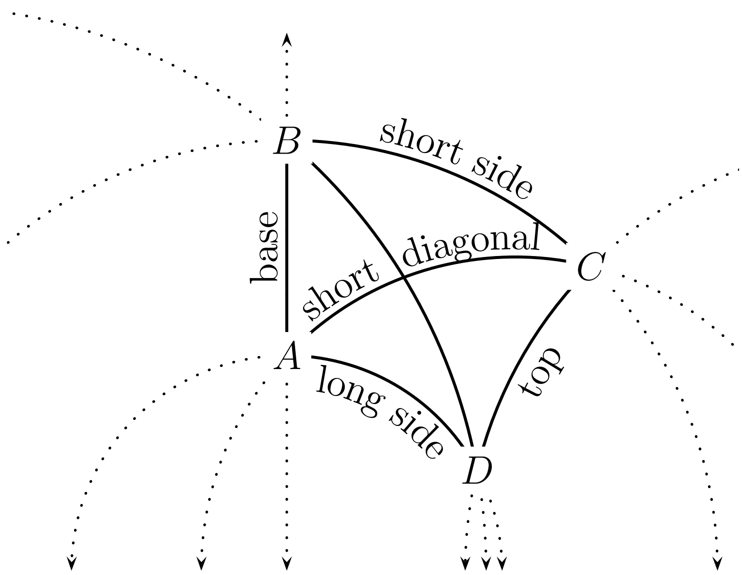

Figure 2.2: A Lambert quadrilateral: $\angle A = \angle B = \angle C = \pi/2$

Theorem 2.1. *The monotonicity patterns of the 15 representative ratios $r(c)$ are given by Table 2.1, where $k_* := \sqrt{2} - 1$.*

r	Pattern for each k in			$r(1+)$	$r(c_k-)$	Comments
	$(0,1)$	$(0,k_*]$	$(k_*,1)$			
CD/AC		$\nearrow$	$\searrow\nearrow$	1	∞	
CD/AD		$\searrow$	$\searrow\nearrow$	∞	1	
CD/BC	$\searrow\nearrow$			∞	∞	$\big(\exists c\in(1,c_k)\ r(c)=1\big) \iff k\geqslant 1/\sqrt{3}$
CD/BD	$\searrow\nearrow$			1	1	
CD/AB	$\nearrow$			1	∞	
AC/AD	$\searrow$			∞	0	
AC/BC	$\searrow$			∞	>1	
AC/BD	$\searrow$			1	0	
AC/AB	$\nearrow$			1	>1	
BD/AD	$\searrow$			∞	1	
BD/BC	$\searrow\nearrow$			∞	∞	$\forall k\in(0,1)\ \forall c\in(1,c_k)\ r(c)>1$
BD/AB	$\nearrow$			1	∞	
AD/BC	$\nearrow$			>1	∞	
AD/AB	$\nearrow$			0	∞	
BC/AB	$\nearrow$			0	$r(c_k-)$	$r(c_k-)>1 \iff k>k_*$

Table 2.1: Monotonicity patterns for the ratios in the Lambert quadrilateral

One simple corollary here is that, of the two sides (BC and AD) of the Lambert quadrilateral, BC is indeed always the shorter one (this is obvious from Figure 2.2 as well). Also, of the two diagonals (AC and BD) of the quadrilateral, AC is always the shorter one.

What is perhaps surprising is that the monotonicity patterns of two ratios, CD/AC (*top*-to-*short-diagonal*) and CD/AD (*top*-to-*long-side*), turn out to depend on (the fixed length of) the base $AB = \ln(1/k)$ of the quadrilateral. When the base AB is smaller than $\ln(1/k_*) = \ln(1+\sqrt{2})$, these two ratios are not monotonic.

Three other ratios — CD/BC (*top*-to-*short-side*), CD/BD (*top*-to-*long-diagonal*), and BD/BC (*long-diagonal*-to-*short-side*) — are not monotonic for any given base; however, this should not be surprising, since for each of these three ratios r one has $r(1+) = r(c_k-)$.

In particular, it follows that of all the 5 ratios of the *top* to the other lengths, only the trivial one, the ratio CD/AB of the *top* to the fixed base, is monotonic for every given base.

Another small-base peculiarity shows up for two ratios, CD/BC (*top*-to-*short-side*) and BC/AB (*short-side*-to-*base*); namely, these ratios take on values to both sides of 1 iff the base is small enough – smaller than $\ln\sqrt{3}$ in the case of CD/BC and smaller than $\ln(1/k_*) = \ln(1+\sqrt{2})$ in the case of BC/AB.

Proof of Theorem 2.1. From (2.3), it is clear that the 5 ratios of BC, CD, AD, AC, and BD to the fixed AB are increasing (in c), and the inequality $BC/AB > 1$ can be rewritten as $\operatorname{ch} BC > \operatorname{ch} AB$, which is equivalent to $k > k_*$. The monotonicity pattern for $AC/AD =$

$(AC/BD)(BD/AD)$ obviously follows from those for AC/BD and BD/AD. It remains to consider the other 9 of the 15 ratios.

In terms of the expression q, defined by (2.4), and the expressions

$$q_1 := \sqrt{(c^2-1)(1+k^2)^2 + (1-k^2)^2}, \quad q_2 := \sqrt{c^2-1}, \tag{2.5}$$

$$q_3 := \sqrt{2(c^2-1)(1+k^4) + (1-k^2)^2}, \tag{2.6}$$

one computes the ratios, ρ, of the derivatives of the distances with respect to c:

$$\frac{(CD)'}{(AC)'} = \frac{(1-k^2)\,q_1}{q^2}, \quad \frac{(CD)'}{(AD)'} = \frac{(1-k^2)\,q_2}{2\,k}, \quad \frac{(CD)'}{(BD)'} = \frac{(1-k^2)\,q_3}{(1+k^2)^2},$$

$$\frac{(AC)'}{(BC)'} = \frac{(1+k^2)\,q_2}{q_1}, \quad \frac{(AC)'}{(BD)'} = \frac{q^2\,q_3}{(1+k^2)^2\,q_1}, \quad \frac{(BD)'}{(AD)'} = \frac{(1+k^2)^2\,q_2}{2\,k\,q_3},$$

$$\frac{(AD)'}{(BC)'} = \frac{2\,k\,(1+k^2)}{q^2}, \quad \frac{(CD)'}{(BC)'} = \frac{(CD)'}{(AC)'}\,\frac{(AC)'}{(BC)'}, \quad \frac{(BD)'}{(BC)'} = \frac{(BD)'}{(AD)'}\,\frac{(AD)'}{(BC)'}.$$

Of these 9 ratios, it is now clear that 8 ratios (except $(AC)'/(BD)'$) are increasing (in c). Hence, by the first line of Table 1.1, each of the corresponding 8 ratios, r, of distances, $CD/AC, \dots, AD/BC$ (except for AC/BD), has one of these three patterns: $\nearrow$, $\searrow$, or $\searrow\nearrow$. (It can be shown that $(AC)'/(BD)'$ is $\searrow$ or $\nearrow\searrow$, depending on whether the base, AB, is large enough; however, this fact will not be used in this paper.)

Now let us consider each of the 8 "unexceptional" ratios separately, after which the "exceptional" ratio, AC/BD, will be considered.

(1) $r(c) = CD/AC$: Here it is obvious that $r(1+) = 1$ and $r(c_k-) = \infty$. This excludes the pattern $r \searrow$. To discriminate between the possibilities $r \searrow$ and $r \searrow\nearrow$, it suffices to determine whether there exists some $c \in (1, c_k)$ such that $r(c) = 1$ or, equivalently, $\operatorname{ch} CD = \operatorname{ch} AC$. Now it is easy to complete the proof of Theorem 2.1 for the ratio $r(c) = CD/AC$.

(2) $r(c) = CD/AD$: Here it is obvious that $r(1+) = \infty$. By l'Hospital's rule for limits, $r(c_k-) = \rho(c_k-) = 1$. This excludes the pattern $r \nearrow$. Moreover, it is easy to see, as in the previous case, that there exists some $c \in (1, c_k)$ such that $r(c) = 1$ iff $k > k_*$.

(3) $r(c) = CD/BC$: Here $r(1+) = r(c_k-) = \infty$. Hence, $r \searrow\nearrow$. Moreover, it is easy to see that there exists some $c \in (1, c_k)$ such that $r(c) = 1$ iff $k \geqslant 1/\sqrt{3}$.

(4) $r(c) = CD/BD$: Here $r(1+) = 1$. By l'Hospital's rule for limits, $r(c_k-) = \rho(c_k-) = 1$. Hence, $r \searrow\nearrow$.

(5) $r(c) = AC/BC$: Here, with $\mu := 2k\,(1+k^2)$ and $\nu := \sqrt{1 + 14k^4 + k^8}$, one has the following at $c = c_k-$:

$$r' \cdot \frac{2k\,\nu\,BC^2}{(1-k^2)AC} = \mu\,\frac{BC}{AC} - \nu < \mu - \nu,$$

since, in view of (2.3), $BC < AC$. But $\mu^2 - \nu^2 = -(1-k^2)^4 < 0$. Hence, $r'(c_k-) < 0$, so that $r \searrow$ in a left neighborhood of c_k. Thus, $r \searrow$.

(6) $r(c) = BD/AD$: Here $r(1+) = \infty$. By l'Hospital's rule for limits, $r(c_k-) = \rho(c_k-) = 1$. In view of (2.3), here $r > 1$ on $(1, c_k)$. Hence, r is decreasing on $(1, c_k)$ from ∞ to 1.

(7) $r(c) = BD/BC$: Here $r(1+) = r(c_k-) = \infty$. Hence, $r \searrow\nearrow$ on $(1, c_k)$. Also, in view of (2.3), one has here $r > 1$ on $(1, c_k)$.

(8) $r(c) = AD/BC$: Here, by the special-case rule for monotonicity, $r \nearrow$. By l'Hospital's rule, $r(1+) = \rho(1+) = (1+k^2)/(2k) > 1$. Also, it is obvious that $r(c_k-) = \infty$.

It remains to consider the 9th ratio,

¶ $r(c) = AC/BD$: Here, as was stated, $\rho(c) := (AC)'/(BD)'$ is non-monotonic in c for k in a left neighborhhood of 1. This makes it more difficult to act as in the cases considered above, since the root c of the equation $\rho'(c) = 0$ depends on k. However, what helps here is that the monotonicity pattern of r turns out to be simple, as will be proved in a moment: $r \searrow$. One can use the following lemma, whose proof is based on the special-case rule for monotonicity stated in Section 1.

Lemma 2.2. *For $x \geqslant 1$, let*

$$\lambda(x) := \frac{\sqrt{x^2-1}\,\operatorname{arcch} x}{x^3}, \quad \alpha(x) := \frac{x^2-1}{x^3}, \quad \beta(x) := \frac{\sqrt{x^2-1}}{x^3}.$$

Then for all u and v in $(1, \infty)$

$$\frac{\lambda(v)}{\lambda(u)} \leqslant \max\left(\frac{\alpha(v)}{\alpha(u)}, \frac{\beta(v)}{\beta(u)}\right).$$

Proof of Lemma 2.2. Obviously, $\lambda/\beta = \operatorname{arcch} \nearrow$. Hence, $\frac{\lambda(v)}{\lambda(u)} \leqslant \frac{\beta(v)}{\beta(u)}$ if $1 < v \leqslant u$. It remains to consider the case when $1 < u < v$. Note that

$$\frac{(\operatorname{arcch} x)'}{\left(\sqrt{x^2-1}\right)'} = \frac{1}{x}$$

is decreasing in $x > 1$. Hence, by the special-case rule for monotonicity,

$$\frac{\lambda(x)}{\alpha(x)} = \frac{\operatorname{arcch} x}{\sqrt{x^2-1}}$$

is decreasing in $x > 1$. Hence, $\frac{\lambda(v)}{\lambda(u)} < \frac{\alpha(v)}{\alpha(u)}$ if $1 < u < v$. □

Let us now return to the consideration of the ratio $r(c) = AC/BD$. It suffices to show that $r'(c) < 0$ for all $k \in (0, 1)$ and $c \in (1, c_k)$. One has the identity

$$r'(c)\,\frac{2\,BD^2\,k\,\sqrt{u^2-1}\,\sqrt{v^2-1}}{(1+k^2)\,\lambda(u)\,v^3} = \frac{\lambda(v)}{\lambda(u)} - K,$$

where

$$u := \frac{c\,(1+k^2)}{2\,k}, \quad v := \frac{c\,(1+k^2)}{\sqrt{(1+k^2)^2 - c^2\,(1-k^2)^2}}, \quad K := \left(\frac{1+k^2}{2\,k}\right)^2.$$

Therefore and in view of Lemma 2.2, it suffices to show that the expressions

$$P := \left(\left(\frac{\alpha(v)}{\alpha(u)}\right)^2 - K^2\right)\,\alpha(u)^2\,\frac{4\,c^6\,k^2\,(1+k^2)^6}{(1-k^2)^2} \quad \text{and}$$

$$Q := \left(\left(\frac{\beta(v)}{\beta(u)}\right)^2 - K^2\right)\,\beta(u)^2\,\frac{c^6\,(1+k^2)^6}{(1-k^2)^2}$$

are negative for all $k \in (0, 1)$ and $c \in (1, c_k)$. But this can be done in a completely algorithmic manner, since P and Q are polynomials in k and c, and c_k is a rational function of k [21, 12, 10]. With Mathematica, one can use the command `Reduce[P>=0 && 1<c<ck && 0<k<1]` (where `ck` stands for c_k), which outputs `False`, meaning that indeed $P < 0$ for all $k \in (0, 1)$ and $c \in (1, c_k)$; similarly, for Q in place of P.

Theorem 2.1 is proved. □

2.2.2. *Saccheri quadrilaterals.* Let $ABCD$ be a Saccheri quadrilateral. Here one may assume that

$$A = k\,i, \quad B = i, \quad C = e^{i\theta}, \quad D = k\,e^{i\theta},$$

where again $0 < k < 1$ and $0 < \theta < \pi/2$, so that the angles at vertices A and B are right, and $BC = AD$, so that $BD = AC$. Let us refer here to $AB = \ln(1/k)$ as the base and to $BC = AD = \operatorname{arcch} c$ as the side, where again $c := 1/\sin\theta$. Here c varies from 1 to ∞.

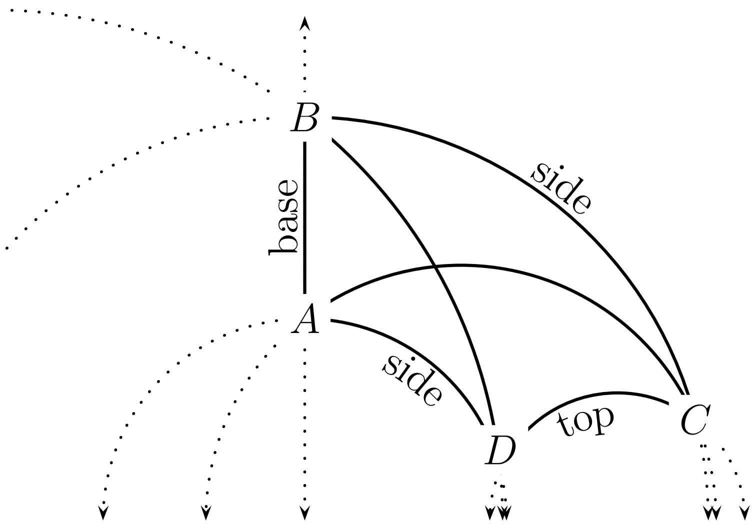

Figure 2.3: A Saccheri quadrilateral: $\angle A = \angle B = \pi/2$ and $\angle C = \angle D$, whence $AD = BC$ and $AC = BD$

Again, let us fix the base $AB = \ln(1/k)$ (so that $k \in (0,1)$ is fixed); also, let c increase from 1 to ∞, so that the side $BC = AD = \operatorname{arcch} c$ increases from 0 to ∞. Here, taking into account the equalities $BC = AD$ and $BD = AC$, we have to determine the monotonicity patterns of $\binom{4}{2} = 6$ completely representative pairwise ratios.

Theorem 2.3. *The monotonicity patterns of the 6 ratios $r(c)$ are given by Table 2.2.*

Thus, the diagonal $AC = BD$ always exceeds both the base AB and the side $AD = BC$. Also, the top CD always exceeds the base.

Recently it was observed by Pambuccian [13] that the ratio $CD/BD = CD/AC$ of the top of a Saccheri quadrilateral to its diagonal may be less than or greater than or equal to 1. The second line of Table 2.2 provides more information in that respect. In particular, one can see now that the *top*-to-*diagonal* ratio can be less than 1 only if the base AB is smaller than $\ln(2+\sqrt{3})$. On the other hand, this ratio is always less than 2.

Similarly to the case of the Lambert quadrilateral, the monotonicity patterns of two ratios, CD/AD (*top*-to-*side*) and CD/BD (*top*-to-*diagonal*), turn out to depend on the base $AB = \ln(1/k)$ of the quadrilateral. When the base is smaller than the threshold value $\ln(1/k_{**})$, these two ratios are not monotonic. However, in contrast with Lambert quadrilaterals, here the threshold values for these two ratios are different from each other. Yet, for Saccheri quadrilaterals as well, it is the small base values that may result in non-monotonic patterns.

r	Pattern for each k in			$r(1+)$	$r(\infty-)$	k_{**}
	$(0,1)$	$(0,k_{**}]$	$(k_{**},1)$			
CD/AD		$\searrow$	$\searrow\nearrow$	∞	2	$k_*^2 = 3-2\sqrt{2}$
CD/BD		$\nearrow$	$\searrow\nearrow$	1	2	$2-\sqrt{3}$
CD/AB	$\nearrow$			1	∞	
AD/BD	$\nearrow$			0	1	
AD/AB	$\nearrow$			0	∞	
BD/AB	$\nearrow$			1	∞	

Table 2.2: Monotonicity patterns for the ratios in the Saccheri quadrilateral

Proof of Theorem 2.3. In view of (2.1), here one has

$$AB = \ln\frac{1}{k}, \quad AD = BC = \operatorname{arcch} c, \quad CD = \operatorname{arcch}\frac{c^2\,(1-k)^2 + 2\,k}{2\,k}, \tag{2.7}$$
$$AC = BD = \operatorname{arcch}\frac{c\,(1+k^2)}{2\,k}.$$

From these expressions, the statements of Theorem 2.3 concerning the three ratios of the top (CD), side ($AD = AC$), and diagonal ($AC = BD$) to the fixed base (AB) are obvious. It remains to consider the other three ratios.

¶ $r(c) = CD/AD$: This case follows immediately from the case of the *top*-to-*long-side* ratio for the Lambert quadrilateral, which latter is a "half" of a Saccheri one; see Figure 2.1. Indeed, if the side of a Saccheri quadrilateral equals the long side of a Lambert quadrilateral and the base of the Saccheri quadrilateral is twice the base of the Lambert quadrilateral, then the top of the Saccheri quadrilateral is twice the top of the Lambert quadrilateral.

¶ $r(c) = CD/BD$: Here (recall (2.5)) $\rho(c) = 2\,(1-k)\,q_1\,/\,((1+k^2)\,q_4)$, where $q_4 := \sqrt{(c^2-1)(1-k)^2+(1+k)^2}$. Hence, $\rho \nearrow$, and so, $r \nearrow$ or $r \searrow$ or $r \nearrow\searrow$. Obviously, $r(1+) = 1$. By l'Hospital's rule, $r(\infty-) = \rho(\infty-) = 2$. Moreover, it is easy to see that ($\exists\, c > 1\ r(c) = 1$) iff $2-\sqrt{3} < k < 1$. This proves the second line of Table 2.2.

¶ $r(c) = AD/BD$: Here $\rho(c) = q_1\,/\,((1+k^2)\,q_2)$, so that $\rho \searrow$. Obviously, $r(1+) = 0$. By l'Hospital's rule, $r(\infty-) = \rho(\infty-) = 1$. Also, (2.7) implies $r < 1$. It follows that $r \nearrow$.

Theorem 2.3 is proved. □

2.3. **Conclusion.** It seems quite likely that one could similarly examine the monotonicity patterns of these ratios for the Lambert and Saccheri quadrilaterals under conditions other than that of a fixed base. Likewise, one could examine the monotonicity patterns of other ratios of distances, in this or other Riemann geometries.

REFERENCES

[1] G.D. ANDERSON, S.-L. QIU, M.K. VAMANAMURTHY AND M. VUORINEN, Generalized elliptic integrals and modular equations, *Pacific J. Math.*, **192** (2000), 1–37.

[2] G.D. ANDERSON, M.K. VAMANAMURTHY AND M. VUORINEN, Inequalities for quasiconformal mappings in space, *Pacific J. Math.*, **160** (1993), 1–18.

[3] G.D. ANDERSON, M.K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Wiley, New York 1997.

[4] G.D. ANDERSON, M.K. VAMANAMURTHY AND M. VUORINEN, Topics in special functions, *Papers on Analysis, Rep. Univ. Jyväskylä Dep. Math. Stat.*, Vol. 83, Univ. Jyväskylä, Jyväskylä, 2001, pp. 5–26.

[5] G.D. ANDERSON, M.K. VAMANAMURTHY AND M. VUORINEN, *Monotonicity of some functions in calculus*, Preprint, 2005.

[6] A.F. BEARDON, *The Geometry of Discrete Groups.* Graduate Texts in Mathematics, vol. 91. Springer-Verlag, New York, 1983.

[7] A.F. BEARDON, *A primer on Riemann surfaces*, London Mathematical Society Lecture Note Series, Vol. 78, Cambridge University Press, Cambridge, 1984.

[8] I. CHAVEL, *Riemannian Geometry – A Modern Introduction*, Cambridge Univ. Press, Cambridge, 1993.

[9] J. CHEEGER, M. GROMOV AND M. TAYLOR, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, *J. Diff. Geom.*, **17** (1982), 15–54.

[10] G.E. COLLINS, Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition, *Lecture Notes In Computer Science*, Vol. 33, 1975, pp. 134–183.

[11] M. GROMOV, Isoperimetric inequalities in Riemannian manifolds, *Asymptotic theory of finite dimensional spaces, Lecture Notes Math.*, Vol. 1200, Appendix I, Springer, Berlin, 1986, pp. 114–129.

[12] S. ŁOJASIEWICZ, *Ensembles semi-analytiques*, Inst. Hautes Etudes Sci., Bures-sur-Yvette, 1964.

[13] V. PAMBUCCIAN, Saccheri quadrilateral, *Amer. Math. Monthly*, **112** (2005), 88–89.

[14] I. PINELIS, Extremal probabilistic problems and Hotelling's T^2 test under symmetry condition, *Preprint*, 1991. A shorter version of the preprint appeared in *Ann. Statist.*, **22** (1994), 357–368.

[15] I. PINELIS, L'Hospital type results for monotonicity, with applications, *J. Inequal. Pure Appl. Math.*, **3**(1) (2002), Art. 5. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=158`]

[16] I. PINELIS, L'Hospital type rules for oscillation, with applications, *J. Inequal. Pure Appl. Math.*, **2**(3) (2001), Art. 33. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=149`]

[17] I. PINELIS, Monotonicity properties of the relative error of a Padé approximation for Mills' ratio, *J. Inequal. Pure Appl. Math.*, **3**(2) (2002), Art. 20. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=172`]

[18] I. PINELIS, L'Hospital type rules for monotonicity: Applications to probability inequalities for sums of bounded random variables, *J. Inequal. Pure Appl. Math.*, **3**(1) (2002), Art. 7. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=160`]

[19] I. PINELIS, L'Hospital rules for monotonicity and the Wilker-Anglesio inequality, *Amer. Math. Monthly*, **111** (2004), 905–909.

[20] I. PINELIS, L'Hospital-type rules for monotonicity and limits: Discrete case, *Preprint*, 2005.

[21] A. TARSKI, *A Decision Method for Elementary Algebra and Geometry*, University of California Press, Berkeley, 1951.