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ON CERTAIN SUBCLASS OF p-VALENTLY BAZILEVIC FUNCTIONS

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Abstract
Contents
Home Page
Go Back
Close
Quit

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Abstract

We introduce a subclass $\mathcal{M}_p(\lambda, \mu, A, B)$ of p-valent analytic functions and derive certain properties of functions belonging to this class by using the techniques of Briot-Bouquet differential subordination. Further, the integral preserving properties of Bazilevic functions in a sector are also considered.

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Contents

1. Introduction

Let $\mathcal{A}_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. We denote $\mathcal{A}_1 = \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\alpha)$ of p-valently starlike of order α, if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \le \alpha < p; z \in E). \tag{1.2}$$

We write $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, the class of p-valently starlike functions in E.

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_p(\alpha)$ of p-valently convex of order α, if it satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \le \alpha < p; z \in E). \tag{1.3}$$

The class of p-valently convex functions in E is denoted by $\mathcal{K}_p$. It follows from (1.2) and (1.3) that

$$f \in \mathcal{K}_p(\alpha) \iff f \in \mathcal{S}_p^*(\alpha) \quad (0 \le \alpha < p).$$

Furthermore, a function $f \in \mathcal{A}_p$ is said to be p-valently Bazilevic of type μ and order α, if there exists a function $g \in \mathcal{S}_p^*$ such that

$$\Re\left\{\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}}\right\} > \alpha \quad (z \in E) \tag{1.4}$$

for some $\mu(\mu \geq 0)$ and $\alpha(0 \leq \alpha < p)$. We denote by $\mathcal{B}_p(\mu, \alpha)$, the subclass of $\mathcal{A}_p$ consisting of all such functions. In particular, a function in $\mathcal{B}_p(1, \alpha) = \mathcal{B}_p(\alpha)$ is said to be p-valently close-to-convex of order α in E.

For given arbitrary real numbers A and B $(-1 \leq B < A \leq 1)$, let

$$\mathcal{S}_p^*(A, B) = \left\{f \in \mathcal{A}_p : \frac{zf'(z)}{f(z)} \prec p\frac{1 + Az}{1 + Bz}, \quad z \in E\right\}, \tag{1.5}$$

where the symbol $\prec$ stands for subordination. In particular, we note that $\mathcal{S}_p^*\left(1 - \frac{2\alpha}{p}, -1\right) = \mathcal{S}_p^*(\alpha)$ is the class of p-valently starlike functions of order $\alpha(0 \leq \alpha < p)$. From (1.5), we observe that $f \in \mathcal{S}_p^*(A, B)$, if and only if

$$\left|\frac{zf'(z)}{f(z)} - \frac{p(1 - AB)}{1 - B^2}\right| < \frac{p(A - B)}{1 - B^2} \quad (-1 < B < A \leq 1; z \in E) \tag{1.6}$$

and

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{p(1 - A)}{2} \quad (B = -1; z \in E). \tag{1.7}$$

Let $\mathcal{M}_p(\lambda, \mu, A, B)$ denote the class of functions in $\mathcal{A}_p$ satisfying the condition

$$\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} + \lambda\left\{1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}\right\} \prec p\frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1;\, z \in E) \tag{1.8}$$

for some real $\mu (\mu \geq 0)$, $\lambda (\lambda > 0)$, and $g \in \mathcal{S}_p^*$. For convenience, we write

$$\begin{aligned}
&\mathcal{M}_p\left(\lambda, \mu, 1 - \frac{2\alpha}{p}, -1\right) \\
&= \mathcal{M}_p(\lambda, \mu, \alpha) \\
&= \left\{ f \in \mathcal{A}_p : \Re\left[\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} \right.\right. \\
&\qquad \left.\left. + \lambda\left\{1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}\right\}\right] > \alpha \right\}
\end{aligned}$$

for some $\alpha\, (0 \leq \alpha < p)$ and $z \in E$.

In the present paper, we derive various useful properties and characteristics of the class $\mathcal{M}_p(\lambda, \mu, A, B)$ by employing techniques involving Briot-Bouquet differential subordination. The integral preserving properties of Bazilevic functions in a sector are also considered. Relevant connections of the results presented here with those obtained in earlier works are pointed out.

2. Preliminaries

To establish our main results, we shall require the following lemmas.

Lemma 2.1 ([6]). *Let h be a convex function in E and let ω be analytic in E with $\Re\{\omega(z)\} \geq 0$. If q is analytic in E and $q(0) = h(0)$, then*

$$q(z) + \omega(z)\, zq^{'}(z) \prec h(z) \quad (z \in E)$$

implies

$$q(z) \prec h(z) \quad (z \in E).$$

Lemma 2.2. *If $-1 \leq B < A \leq 1, \beta > 0$ and the complex number γ satisfies $\Re(\gamma) \geq -\beta(1-A)/(1-B)$, then the differential equation*

$$q(z) + \frac{zq^{'}(z)}{\beta\, q(z) + \gamma} = \frac{1+Az}{1+Bz} \quad (z \in E)$$

has a univalent solution in E given by

(2.1)
$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}dt} - \dfrac{\gamma}{\beta}, & B \neq 0 \\ \dfrac{z^{\beta+\gamma}\exp(\beta\, Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta\, At)dt} - \dfrac{\gamma}{\beta}, & B = 0. \end{cases}$$

If $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in E and satisfies

(2.2)
$$\phi(z) + \frac{z\,\phi^{'}(z)}{\beta\phi(z)+\gamma} \prec \frac{1+Az}{1+Bz} \quad (z \in E),$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in E)$$

and $q(z)$ is the best dominant of (2.2).

The above lemma is due to Miller and Mocanu [7].

Lemma 2.3 ([12]). *Let ν be a positive measure on $[0,1]$. Let h be a complex-valued function defined on $E \times [0,1]$ such that $h(\cdot,t)$ is analytic in E for each $t \in [0,1]$, and $h(z,\cdot)$ is ν-integrable on $[0,1]$ for all $z \in E$. In addition, suppose that $\Re\{h(z,t)\} > 0$, $h(-r,t)$ is real and $\Re\{1/h(z,t)\} \geq 1/h(-r,t)$ for $|z| \leq r < 1$ and $t \in [0,1]$. If $h(z) = \int_0^1 h(z,t)\, d\nu(t)$, then $\Re\{1/h(z)\} \geq 1/h(-r)$.*

For real or complex numbers $a, b, c\, (c \neq 0, -1, -2, \dots)$, the hypergeometric function is defined by

$$(2.3) \qquad {}_2F_1(a,b;c;z) = 1 + \frac{a \cdot b}{c} \cdot \frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots .$$

We note that the series in (2.3) converges absolutely for $z \in E$ and hence represents an analytic function in E. Each of the identities (asserted by Lemma 2.3 below) is well-known [13].

Lemma 2.4. *For real numbers $a, b, c\, (c \neq 0, -1, -2, \dots)$, we have*

$$(2.4) \qquad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a,b;c;z) \qquad (c > b > 0)$$

$$ {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z) \tag{2.5} $$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right). \tag{2.6} $$

Lemma 2.5 ([10]). *Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic in* E *and* $p(z) \neq 0$ *in* E. *If there exists a point* $z_0 \in E$ *such that*

$$ |\arg p(z)| < \frac{\pi}{2}\eta \;(|z| < |z_0|) \quad \textit{and} \quad |\arg p(z_0)| = \frac{\pi}{2}\eta \;(0 < \eta \leq 1), \tag{2.7} $$

then we have

$$ \frac{z_0 p'(z_0)}{p(z_0)} = ik\eta, \tag{2.8} $$

where

$$ \begin{cases} k \geq \frac{1}{2}\left(x + \frac{1}{x}\right), & \textit{when } \arg p(z_0) = \frac{\pi}{2}\eta, \\ k \leq -\frac{1}{2}\left(x + \frac{1}{x}\right), & \textit{when } \arg p(z_0) = -\frac{\pi}{2}\eta, \end{cases} \tag{2.9} $$

and

$$ \left(p(z_0)\right)^{1/\eta} = \pm ix \;(x > 0). \tag{2.10} $$

3. Main Results

Theorem 3.1. *Let* $-1 \leq B < A \leq 1, \lambda > 0$ *and* $\mu \geq 0$. *If* $f \in \mathcal{M}_p(\lambda, \mu, A, B)$, *then*

$$\frac{zf'(z)}{p\, f(z)^{1-\mu} g(z)^{\mu}} \prec \frac{\lambda}{pQ(z)} = q(z) \quad (z \in E), \tag{3.1}$$

where

$$Q(z) = \begin{cases} \int_0^1 s^{\frac{p}{\lambda}-1} \left(\frac{1+Bsz}{1+Bz}\right)^{\frac{p(A-B)}{\lambda B}} ds, & B \neq 0, \\ \int_0^1 s^{\frac{p}{\lambda}-1} \exp\left(\frac{p}{\lambda}(s-1)Az\right) ds, & B = 0, \end{cases} \tag{3.2}$$

$$q(z) = \frac{1}{1+Bz} \quad \textit{when} \quad A = -\frac{\lambda B}{p}, \ B \neq 0,$$

and $q(z)$ *is the best dominant of* (3.1). *Furthermore, if* $A \leq -\lambda B/p$ *with* $-1 \leq B < 0$, *then*

$$\mathcal{M}_p(\lambda, \mu, A, B) \subset \mathcal{B}_p(\mu, \rho), \tag{3.3}$$

where

$$\rho = \rho(p, \lambda, A, B) = p\left\{ {}_2F_1\left(1, \frac{p(B-A)}{\lambda B}; \frac{p}{\lambda}+1; \frac{B}{B-1}\right)\right\}^{-1}.$$

The result is best possible.

Proof. Defining the function $\phi(z)$ by

$$\phi(z) = \frac{zf'(z)}{p\, f(z)^{1-\mu} g(z)^{\mu}} \quad (z \in E), \tag{3.4}$$

we note that $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in E. Taking the logarithmic differentiations in both sides of (3.4), we have

$$\frac{zf'(z)}{f(z)^{1-\mu} g(z)^{\mu}} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu \frac{zg'(z)}{g(z)} \right\} \\ = p\,\phi(z) + \lambda \frac{z\phi'(z)}{\phi(z)} \prec \frac{p(1+Az)}{1+Bz} \quad (z \in E). \tag{3.5}$$

Thus, $\phi(z)$ satisfies the differential subordination (2.2) and hence by using Lemma 2.2, we get

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in E),$$

where $q(z)$ is given by (2.1) for $\beta = p/\lambda$ and $\gamma = 0$, and is the best dominant of (3.5). This proves the assertion (3.1).

Next, we show that

$$\inf_{|z|<1} \left\{ \Re(q(z)) \right\} = q(-1). \tag{3.6}$$

If we set $a = p(B-A)/\lambda B$, $b = p/\lambda$, $c = (p/\lambda) + 1$, then $c > b > 0$. From

(3.2), by using (2.4), (2.5) and (2.6), we see that for $B \neq 0$

$$Q(z) = (1+Bz)^a \int_0^1 s^{b-1}(1+Bsz)^{-a}\,ds = \frac{\Gamma(b)}{\Gamma(c)}\,{}_2F_1\left(1, a; c; \frac{Bz}{Bz+1}\right). \tag{3.7}$$

To prove (3.6), we need to show that $\Re\{1/Q(z)\} \geq 1/Q(-1)$, $z \in E$. Since $A < -\lambda B/p$ implies $c > a > 0$, by using (2.4), (3.7) yields

$$Q(z) = \int_0^1 h(z,s)\,d\nu(s),$$

where

$$h(z,s) = \frac{1+Bz}{1+(1-s)Bz}\ (0 \leq s \leq 1) \quad \text{and}$$

$$d\nu(s) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} s^{a-1}(1-s)^{c-a-1}\,ds$$

which is a positive measure on $[0,1]$. For $-1 \leq B < 0$, it may be noted that $\Re\{h(z,s)\} > 0$, $h(-r,s)$ is real for $0 \leq r < 1, 0 \in [0,1]$ and

$$\Re\left\{\frac{1}{h(z,s)}\right\} = \Re\left\{\frac{1+(1-s)Bz}{1+Bz}\right\} \geq \frac{1-(1-s)Br}{1-Br} = \frac{1}{h(-r,s)}$$

for $|z| \leq r < 1$ and $s \in [0,1]$. Therefore, by using Lemma 2.3, we have

$$\Re\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-r)}, \quad |z| \leq r < 1$$

and by letting $r \to 1^-$, we obtain $\Re\{1/Q(z)\} \geq 1/Q(-1)$. Further, by taking $A \to (-\lambda B/p)^+$ for the case $A = (-\lambda B/p)$, and using (3.1), we get (3.3).

The result is best possible as the function $q(z)$ is the best dominant of (3.1). This completes the proof of Theorem 3.1. □

Setting $\mu = 1, A = 1 - (2\alpha/p)\left((p-\lambda)/2 \leq \alpha < p\right)$ and $B = -1$ in Theorem 3.1, we have

Corollary 3.2. *If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{\frac{zf'(z)}{g(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right)\right\} > \alpha \quad (\lambda > 0, z \in E)$$

for some $g \in \mathcal{S}_p^*$, then $f \in \mathcal{B}_p(\kappa(p,\lambda,\alpha))$, where

$$\kappa(p,\lambda,\alpha) = p\left\{{}_2F_1\left(1, \frac{2(p-\alpha)}{\lambda}; \frac{p}{\lambda}+1; \frac{1}{2}\right)\right\}^{-1}. \tag{3.8}$$

The result is best possible.

Taking $\mu = 0, A = 1 - (2\alpha/p)\left((p-\lambda)/2 \leq \alpha < p\right)$ and $B = -1$ in Theorem 3.1, we get

Corollary 3.3. *If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \quad (\lambda > 0, z \in E)$$

then $f \in \mathcal{S}_p^*(\kappa(p,\lambda,\alpha))$, where $\kappa(p,\lambda,\alpha)$ is given by (3.8). The result is best possible.

Putting $\lambda = 1$ in Corollary 3.3, we get

Corollary 3.4. *For $(p-1)/2 \le \alpha < p$, we have*

$$\mathcal{K}_p(\alpha) \subset \mathcal{S}_p^*(\varkappa(p,\alpha)),$$

where $\varkappa(p,\alpha) = p\left\{{}_2F_1(1, 2(p-\alpha); p+1; 1/2)\right\}^{-1}$. The result is best possible.

Remark 1.

1. *Noting that*

$$\left\{{}_2F_1\left(1, 2(1-\alpha); 2; \frac{1}{2}\right)\right\}^{-1} = \begin{cases} \frac{1-2\alpha}{2^{2(1-\alpha)}(1-2^{2\alpha-1})}, & \alpha \ne \frac{1}{2} \\ \frac{1}{2\ \ln\ 2}, & \alpha = \frac{1}{2}, \end{cases}$$

   Corollary 3.4 yields the corresponding result due to MacGregor [5] (see also [12]) for $p = 1$.

2. *It is proved [9] that if $p \ge 2$ and $f \in \mathcal{K}_p$, then f is p-valently starlike in E but is not necessarily p-valently starlike of order larger than zero in E. However, our Corollary 3.4 shows that if f is p-valently convex of order at least $(p-1)/2$, then f is p-valently starlike of order larger than zero in E.*

Theorem 3.5. *If $f \in \mathcal{B}_p(\mu,\alpha)$ for some $\mu\,(\mu > 0), \alpha\,(0 \le \alpha < p)$, then $f \in \mathcal{M}_p(\lambda,\mu,\alpha)$ for $|z| < R(p,\lambda,\alpha)$, where $\lambda > 0$ and*

$$R(p,\lambda,\alpha) = \begin{cases} \frac{(p+\lambda-\alpha)-\sqrt{(p+\lambda-\alpha)^2-p(p-2\alpha)}}{p-2\alpha}, & \alpha \ne \frac{p}{2}; \\ \frac{p}{p+2\lambda}, & \alpha = \frac{p}{2}. \end{cases} \tag{3.9}$$

The bound $R(p,\lambda,\alpha)$ is best possible.

Proof. From (1.4), we get

$$\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} = \alpha + (p-\alpha)u(z) \quad (z \in E), \tag{3.10}$$

where $u(z) = 1 + u_1 z + u_2 z^2 + \cdots$ is analytic and has a positive real part in E. Differentiating (3.10) logarithmically, we deduce that

$$\begin{aligned} \Re\left\{\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right.\right. \\ \left.\left. -(1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}\right)\right\} - \alpha \\ = (p-\alpha)\Re\left\{u(z) + \frac{\lambda\, z\, u'(z)}{\alpha + (p-\alpha)u(z)}\right\} \\ \geq (p-\alpha)\Re\left\{u(z) - \frac{\lambda\,|z\, u'(z)|}{|\alpha + (p-\alpha)u(z)|}\right\}. \end{aligned} \tag{3.11}$$

Using the well-known estimates [5]

$$|z\,u'(z)| \leq \frac{2r}{1-r^2}\Re\{u(z)\} \quad \text{and} \quad \Re\{u(z)\} \geq \frac{1-r}{1+r} \quad (|z| = r < 1)$$

in (3.11), we get

$$\begin{aligned} \Re\left\{\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} + \lambda\left(1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}\right)\right\} - \alpha \\ \geq (p-\alpha)\,\Re\{u(z)\}\left\{1 - \frac{2\lambda\, r}{\alpha(1-r^2) + (p-\alpha)(1-r)^2}\right\}, \end{aligned}$$

which is certainly positive if $r < R(p,\lambda,\alpha)$, where $R(p,\lambda,\alpha)$ is given by (3.9).

To show that the bound $R(p,\lambda,\alpha)$ is best possible, we consider the function $f \in \mathcal{A}_p$ defined by

$$\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} = \alpha + (p-\alpha)\frac{1-z}{1+z} \quad (0 \le \alpha < p, z \in E)$$

for some $g \in \mathcal{S}_p^*$. Noting that

$$\begin{aligned} &\Re\left\{\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} + \lambda\left(1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}\right)\right\} - \alpha \\ &= (p-\alpha)\left[\frac{1-z}{1+z} + \frac{2\lambda z}{\alpha(1-z^2) + (p-\alpha)(1+z)^2}\right] \\ &= 0 \end{aligned}$$

for $z = -R(p,\lambda,\alpha)$, we conclude that the bound is best possible. This proves Theorem 3.5. □

For $\mu = 0$ and $\lambda = 1$, Theorem 3.5 yields:

Corollary 3.6. *If $f \in \mathcal{S}_p^*(\alpha)$ $(0 \le \alpha < p)$, then $f \in K_p(\alpha)$ in $|z| < \xi(p,\alpha)$, where*

$$\xi(p,\alpha) = \begin{cases} \frac{(p+1-\alpha) - \sqrt{\alpha^2 + 2(p-\alpha) + 1}}{p-2\alpha}, & \alpha \ne \frac{p}{2}; \\ \frac{p}{p+2}, & \alpha = \frac{p}{2}. \end{cases}$$

The bound $\xi(p,\alpha)$ is best possible.

Theorem 3.7. *If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{\frac{f(z)}{z^p}\right\} > 0 \quad \textit{and} \quad \left|\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} - p\right| < p \quad (0 \le \mu, z \in E)$$

for $g \in \mathcal{S}_p^*$, then f is p-valently convex(univalent) in $|z| < \widetilde{R}(p,\mu)$, where

$$\widetilde{R}(p,\mu) = \frac{3 + 2\mu(p-1) - \sqrt{(3+2\mu(p-1))^2 - 4p(2\mu p - p - 1)}}{2(2\mu p - p - 1)}.$$

The bound $\widetilde{R}(p,\mu)$ is best possible.

Proof. Letting

$$h(z) = \frac{zf'(z)}{p\,f(z)^{1-\mu}g(z)^{\mu}} - 1 \quad (z \in E),$$

we note that $h(z)$ is analytic in E, $h(0) = 0$ and $|h(z)| < 1$ for $z \in E$. Thus, by applying Schwarz's Lemma we get

$$h(z) = z\,\psi(z),$$

where $\psi(z)$ is analytic in E and $|\psi(z)| \le 1$ for $z \in E$. Therefore,

$$zf'(z) = pf(z)^{1-\mu}g(z)^{\mu}(1 + z\psi(z)). \tag{3.12}$$

Making use of logarithmic differentiation in (3.12), we obtain

$$1 + \frac{zf''(z)}{f'(z)} = (1-\mu)\frac{zf'(z)}{f(z)} + \mu\frac{zg'(z)}{g(z)} + \frac{z(\psi(z) + z\psi'(z))}{1 + z\psi(z)}. \tag{3.13}$$

Setting $\phi(z) = f(z)/z^p = 1 + c_1 z + c_2 z^2 + \cdots, \Re\{\phi(z)\} > 0$ for $z \in E$, we get

$$\frac{zf'(z)}{f(z)} = p + \frac{z\phi'(z)}{\phi(z)}$$

so that by (3.13),

$$1 + \frac{zf''(z)}{f'(z)} = (1-\mu)p + (1-\mu)\frac{z\phi'(z)}{\phi(z)} + \mu\frac{zg'(z)}{g(z)} + \frac{z(\psi(z) + z\psi'(z))}{1 + z\psi(z)}. \tag{3.14}$$

Now, by using the well-known estimates [1]

$$\Re\left\{\frac{z\phi'(z)}{\phi(z)}\right\} \geq -\frac{2r}{1-r^2}, \; \Re\left\{\frac{zg'(z)}{g(z)}\right\} \geq -\frac{p(1-r)}{1+r} \quad \text{and}$$

$$\Re\left\{\frac{\psi(z) + z\psi'(z)}{1 + z\psi(z)}\right\} \geq -\frac{1}{1-r}$$

for $|z| = r < 1$ in (3.14), we deduce that

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \frac{(2\mu p - p - 1)r^2 - \{3 + 2\mu(p-1)\}r + p}{1-r^2}$$

which is certainly positive if $r < \widetilde{R}(p, \mu)$. □

It is easily seen that the bound $\widetilde{R}(p, \mu)$ is sharp for the functions $f, g \in \mathcal{A}_p$ defined in E by

$$\frac{zf'(z)}{p\,f(z)^{1-\mu}g(z)^{\mu}} = \frac{1}{1+z}, \; g(z) = \frac{z^p}{(1+z)^2} \quad (0 \leq \mu, z \in E).$$

Choosing $\mu = 0$ in Theorem 3.7, we have

Corollary 3.8. *If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{\frac{f(z)}{z^p}\right\} > 0 \quad \textit{and} \quad \left|\frac{zf'(z)}{f(z)} - p\right| < p \quad (z \in E)$$

then f is p-valently convex in $|z| < \left\{\sqrt{9+4p(p+1)} - 3\right\}\Big/ 2(p+1)$. The result is best possible.

For a function $f \in \mathcal{A}_p$, we define the integral operator $F_{\mu,\delta}$ as follows:

$$\text{(3.15)} \qquad F_{\mu,\delta}(f) = F_{\mu,\delta}(f)(z) = \left(\frac{\delta + p\mu}{z^\delta}\int_0^z t^{\delta-1} f(t)^\mu dt\right)^{\frac{1}{\mu}} \quad (z \in E),$$

where μ and δ are real numbers with $\mu > 0$, $\delta > -p\mu$.

The following lemma will be required for the proof of Theorem 3.12 below.

Lemma 3.9. *Let $g \in \mathcal{S}_p^*(A,B)$, μ and δ are real numbers with $\mu > 0$, $\delta > \max\left\{-p\mu, -\frac{p\mu(1-A)}{(1-B)}\right\}$. Then $F_{\mu,\delta}(g) \in \mathcal{S}_p^*(A,B)$.*

The proof of the above lemma follows by using Lemma 2.2 followed by a simple calculation.

Theorem 3.10. *Let μ and δ be real numbers with $\mu > 0$, $\delta > \max\left\{-p\mu, -\frac{p\mu(1-A)}{(1-B)}\right\}$ $(-1 \le B < A \le 1)$ and let $f \in f \in \mathcal{A}_p$. If*

$$\left|\arg\left(\frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} - \alpha\right)\right| < \frac{\pi}{2}\beta \qquad (0 \le \alpha < p; 0 < \beta \le 1)$$

for some $g \in \mathcal{S}_p^*(A,B)$*, then*

$$\left|\arg\left(\frac{z(F_{\mu,\delta})'(f)}{F_{\mu,\delta}(f)^{1-\mu}F_{\mu,\delta}(g)^{\mu}}-\alpha\right)\right|<\frac{\pi}{2}\eta,$$

where $F_{\mu,\delta}(f)$ *is the operator given by (3.15) and* η $(0<\eta\le 1)$ *is the solution of the equation*

$$\beta=\begin{cases}\eta+\frac{2}{\pi}\tan^{-1}\left(\frac{(1+B)\eta\sin\left(\frac{\pi}{2}(1-t(A,B,\delta,\mu,p))\right)}{(1+B)\delta+\mu p(1+A)+(1+B)\eta\cos\left(\frac{\pi}{2}(1-t(A,B,\delta,\mu,p))\right)}\right), & B\neq -1;\\ \eta, & B=-1,\end{cases} \tag{3.16}$$

and

$$t(A,B,\delta,\mu,p)=\frac{2}{\pi}\sin^{-1}\left(\frac{\mu p(A-B)}{\delta(1-B^2)+\mu p(1-AB)}\right). \tag{3.17}$$

Proof. Let us put

$$q(z)=\frac{1}{p-\alpha}\left(\frac{z(F_{\mu,\delta})'(f)}{F_{\mu,\delta}(f)^{1-\mu}F_{\mu,\delta}(g)^{\mu}}-\beta\right)=\frac{\Phi(z)}{\Psi(z)},$$

where

$$\Phi(z)=\frac{1}{p-\alpha}\left\{z^{\delta}f(z)^{\mu}-\delta\int_0^z t^{\delta-1}f(t)^{\mu}dt-\mu\,\alpha\int_0^z t^{\delta-1}g(t)^{\mu}dt\right\}$$

and

$$\Psi(z) = \mu \int_0^z t^{\delta-1} g(t)^\mu dt.$$

Then $q(z)$ is analytic in E and $q(0) = 1$. By a simple calculation, we get

$$\begin{aligned}\frac{\Phi'(z)}{\Psi'(z)} &= q(z)\left(1 + \frac{S(z)}{zS'(z)} \frac{zq'(z)}{q(z)}\right) \\ &= \frac{1}{p-\beta}\left(\frac{zf'(z)}{f(z)^{1-\mu} g(z)^\mu} - \alpha\right).\end{aligned}$$

Since $F_{\mu,\delta}(g) \in \mathcal{S}_p^*(A, B)$, by (1.6) and (1.7), we have

$$\frac{zS'(z)}{S(z)} = \delta + \mu \frac{z(F_{\mu,\delta})'(g)}{F_{\mu,\delta}(g)} = \rho e^{i\pi\theta/2}, \tag{3.18}$$

where

$$\begin{cases} \delta + \frac{\mu p(1-A)}{1-B} < \rho < \delta + \frac{\mu p(1+A)}{1+B} \\ -t(A, B, \delta, \mu, p) < \theta < t(A, B, \delta, \mu, p) \text{ for } B \neq -1 \end{cases}$$

when $t(A, B, \delta, \mu, p)$ is given by (3.17), and

$$\begin{cases} \delta + \frac{\mu p(1-A)}{2} < \rho < \infty \\ -1 < \theta < 1 \text{ for } B = -1. \end{cases}$$

Further, taking $\omega(z) = S(z)/zS'(z)$ in Lemma 2.1, we note that $q(z) \neq 0$ in E. If there exists a point $z_0 \in E$ such that the condition (2.7) is satisfied, then (by Lemma 2.5) we obtain (2.8) under the restrictions (2.9) and (2.10).

At first, suppose that $q(z_0)^{\frac{1}{\eta}} = ix \; (x > 0)$. For the case $B \neq -1$, by (3.18), we obtain

$$\begin{aligned}
&\arg\left(\frac{z_0 f'(z_0)}{f(z_0)^{1-\mu} g(z_0)^{\mu}} - \alpha\right) \\
&\quad = \arg\left(q(z_0)\right) + \arg\left(1 + \frac{1}{\delta + \mu \frac{z_0 (F_{\mu,\delta}(g))'(z_0)}{F_{\mu,\delta}(g)(z_0)}} \cdot \frac{z_0 q'(z_0)}{q(z_0)}\right) \\
&\quad = \frac{\pi}{2}\eta + \arg\left(1 + \left(\rho e^{i\pi\theta/2}\right)^{-1} i\eta k\right) \\
&\quad = \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\eta k \sin(\pi(1-\theta)/2)}{\rho + \cos(\pi(1-\theta)/2)}\right) \\
&\quad \geq \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\eta \sin\left(\pi(1 - t(A,B,\delta,\mu,p))/2\right)}{\delta + \frac{\mu p(1+A)}{1+B} + \eta \cos\left(\pi(1 - t(A,B,\delta,\mu,p))/2\right)}\right) \\
&\quad = \frac{\pi}{2}\beta,
\end{aligned}$$

where β and $t(A,B,\delta,\mu,p)$ are given by (3.16) and (3.17), respectively. Similarly, for the case $B = -1$, we have

$$\arg\left(\frac{z f'(z)}{f(z)^{1-\mu} g(z)^{\mu}} - \alpha\right) \geq \frac{\pi}{2}\eta.$$

This is a contradiction to the assumption of our theorem.

Next, suppose that $q(z_0)^{\frac{1}{\eta}} = -ix\ (x > 0)$. For the case $B \neq -1$, applying the same method as above,we have

$$\begin{aligned}\arg\left(\frac{z_0 f'(z_0)}{f(z_0)^{1-\mu}g(z_0)^{\mu}} - \alpha\right)\\ &\leq -\frac{\pi}{2}\eta - \tan^{-1}\left(\frac{\eta\sin\left(\pi(1-t(A,B,\delta,\mu,p))/2\right)}{\delta + \frac{\mu p(1+A)}{1+B} + \eta\cos\left(\pi(1-t(A,B,\delta,\mu,p))/2\right)}\right)\\ &= -\frac{\pi}{2}\beta,\end{aligned}$$

where β and $t(A,B,\delta,\mu,p)$ are given by (3.16) and (3.17), respectively and for the case $B = -1$, we have

$$\arg\left(\frac{zf'(z)}{f(z)^{1-\mu}g(z)^{\mu}} - \alpha\right) \leq -\frac{\pi}{2}\eta,$$

which contradicts the assumption. Thus, we complete the proof of the theorem. □

Letting $\mu = 1, B \to A$ and $g(z) = z^p$ in Theorem 3.10, we have

Corollary 3.11. *Let $\delta > -p$ and $f \in \mathcal{A}_p$. If*

$$\left|\arg\left(\frac{f'(z)}{z^{p-1}} - \alpha\right)\right| < \frac{\pi}{2}\beta \qquad (0 \leq \alpha < p; 0 < \beta \leq 1),$$

then

$$\left|\arg\left(\frac{F'_{1,\delta}(f)}{z^{p-1}}-\alpha\right)\right|<\frac{\pi}{2}\eta,$$

where $F_{1,\delta}(f)$ *is the integral operator given by (3.15) for* $\mu=1$ *and* η $(0<\eta\leq 1)$ *is the solution of the equation*

$$\beta=\eta+\frac{2}{\pi}\tan^{-1}\left(\frac{\eta}{\delta+p}\right).$$

Theorem 3.12. *Let* $\lambda>0$*. If* $f\in\mathcal{A}$ *satisfies the condition*

$$\gamma\left\{\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}\right\} + \lambda\left\{1+\frac{zf''(z)}{f'(z)}-(1-\mu)\frac{zf'(z)}{f(z)}-\mu\frac{zg'(z)}{g(z)}\right\}\neq it\ (z\in E) \tag{3.19}$$

for some $\mu(\mu\geq 0)$, $\gamma(\gamma>0)$ *and* $g\in\mathcal{S}_p^*$*, where* t *is a real number with* $|t|\geq\sqrt{\lambda(\lambda+2p\gamma)}$*, then*

$$\Re\left\{\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}\right\}>0\quad(z\in E).$$

Proof. Let

$$\phi(z)=\frac{zf'(z)}{p\,f^{1-\mu}(z)g^{\mu}(z)}\quad(z\in E),$$

where $\phi(0) = 1$. From (3.19), we easily have $\phi(z) \neq 0$ in E. In fact, if ϕ has a zero of order m at $z = z_1 \in E$, then ϕ can be written as

$$\phi(z) = (z - z_1)^m \, q(z) \quad (m \in \mathbb{N}),$$

where $q(z)$ is analytic in E and $q(z_1) \neq 0$. Hence, we have

$$\begin{aligned} \gamma \left\{ \frac{zf'(z)}{f^{1-\mu}(z) g^{\mu}(z)} \right\} &+ \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - (1-\mu) \frac{zf'(z)}{f(z)} - \mu \frac{zg'(z)}{g(z)} \right\} \\ &= p\,\gamma \phi(z) + \lambda \frac{z\phi'(z)}{\phi(z)} \\ &= p\,\gamma (z - z_1)^m \, q(z) + \lambda \frac{mz}{z - z_1} + \lambda \frac{zq'(z)}{q(z)}. \end{aligned} \tag{3.20}$$

But the imaginary part of (3.20) can take any infinite values when $z \to z_1$ in a suitable direction. This contradicts (3.19). Thus, if there exists a point $z_0 \in E$ such that

$$\Re\{p(z)\} > 0 \text{ for } |z| < |z_0|, \quad \Re\{p(z_0)\} > 0 \text{ and } p(z_0) = i\ell \ (\ell \neq 0),$$

then we have $p(z_0) \neq 0$. From Lemma 2.5 and (3.20), we get

$$p\,\gamma \phi(z_0) + \lambda \frac{z_0 \phi'(z_0)}{\phi(z_0)} = i(p\,\gamma \ell + \lambda\, k),$$

$$p\,\gamma \ell + \lambda\, k \geq \frac{1}{2} \left(\frac{\lambda}{\ell} + (\lambda + 2p\,\gamma)\ell \right) \geq \sqrt{\lambda(\lambda + 2p\,\gamma)} \text{ when } \ell > 0,$$

and

$$p\gamma\ell + \lambda k \leq -\frac{1}{2}\left(\frac{\lambda}{|\ell|} + (\lambda + 2p\gamma)|\ell|\right) \leq -\sqrt{\lambda(\lambda + 2p\gamma)} \quad \text{when } \ell < 0,$$

which contradicts (3.19). Therefore, we have $\Re\{\phi(z)\} > 0$ in E. This completes the proof of the theorem. □

Taking $g(z) = z^p$ and $\mu = 1$ in Theorem 3.12, we have

Corollary 3.13. *Let $\lambda > 0$. If $f \in \mathcal{A}_p$ satisfies the condition*

$$\gamma \frac{f'(z)}{z^{p-1}} + \lambda \left\{1 + \frac{zf''(z)}{f'(z)} - p\right\} \neq it \qquad (z \in E)$$

for some γ $(\gamma > 0)$, where t is a real number with $|t| \geq \sqrt{\lambda(\lambda + 2p\gamma)}$, then

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0 \qquad (z \in E).$$

Corollary 3.14. *Let $\lambda > 0$. If $f \in \mathcal{A}_p$ satisfies the condition*

$$\left|\gamma\left\{\frac{f'(z)}{z^{p-1}} - p\right\} + \lambda\left\{1 + \frac{zf''(z)}{f'(z)} - p\right\}\right| < \lambda + \gamma p \quad (z \in E)$$

for some γ $(\gamma > 0)$, then

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0 \quad (z \in E).$$

Remark 2. *From a result of Nunokawa [9] and Saitoh and Nunokawa [11], it follows that, if* $f \in \mathcal{A}_p$ *satisfies the hypothesis of Corollary 3.13 or Corollary 3.14, then* f *is* p*-valent in* E *and* p*-valently convex in the disc* $|z| < (\sqrt{p+1} - 1)/p$.

Letting $\gamma = 1, \mu = 0$ in Theorem 3.12, we get the following result due to Dinggong [4] which in turn yields the work of Cho and Kim [3] for $p = 1$.

Corollary 3.15. *Let* $\lambda > 0$. *If* $f \in \mathcal{A}_p$ *satisfies the condition*

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \neq it \qquad (z \in E),$$

where t *is a real number with* $|t| \geq \sqrt{\lambda(\lambda + 2p)}$, *then* $f \in S_p^*$.

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