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ON CERTAIN SUBCLASS OF p -VALENTLY BAZILEVIC FUNCTIONS

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Abstract

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Abstract

We introduce a subclass $M_p(\lambda, \mu, A, B)$ of p -valent analytic functions and derive certain properties of functions belonging to this class by using the techniques of Briot-Bouquet differential subordination. Further, the integral preserving properties of Bazilevic functions in a sector are also considered.

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1. Introduction

Let \mathcal{A}_p be the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. We denote $\mathcal{A}_1 = \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\alpha)$ of p -valently starlike of order α , if it satisfies

$$(1.2) \quad \Re \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in E).$$

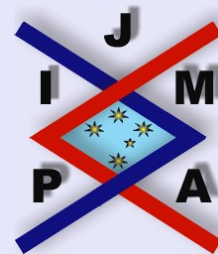
We write $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, the class of p -valently starlike functions in E .

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_p(\alpha)$ of p -valently convex of order α , if it satisfies

$$(1.3) \quad \Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in E).$$

The class of p -valently convex functions in E is denoted by \mathcal{K}_p . It follows from (1.2) and (1.3) that

$$f \in \mathcal{K}_p(\alpha) \iff f \in \mathcal{S}_p^*(\alpha) \quad (0 \leq \alpha < p).$$



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Furthermore, a function $f \in \mathcal{A}_p$ is said to be p -valently Bazilevic of type μ and order α , if there exists a function $g \in \mathcal{S}_p^*$ such that

$$(1.4) \quad \Re \left\{ \frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} \right\} > \alpha \quad (z \in E)$$

for some $\mu (\mu \geq 0)$ and $\alpha (0 \leq \alpha < p)$. We denote by $\mathcal{B}_p(\mu, \alpha)$, the subclass of \mathcal{A}_p consisting of all such functions. In particular, a function in $\mathcal{B}_p(1, \alpha) = \mathcal{B}_p(\alpha)$ is said to be p -valently close-to-convex of order α in E .

For given arbitrary real numbers A and $B (-1 \leq B < A \leq 1)$, let

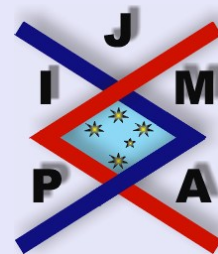
$$(1.5) \quad \mathcal{S}_p^*(A, B) = \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{f(z)} \prec p \frac{1 + Az}{1 + Bz}, z \in E \right\},$$

where the symbol \prec stands for subordination. In particular, we note that $\mathcal{S}_p^* \left(1 - \frac{2\alpha}{p}, -1 \right) = \mathcal{S}_p^*(\alpha)$ is the class of p -valently starlike functions of order $\alpha (0 \leq \alpha < p)$. From (1.5), we observe that $f \in \mathcal{S}_p^*(A, B)$, if and only if

$$(1.6) \quad \left| \frac{zf'(z)}{f(z)} - \frac{p(1 - AB)}{1 - B^2} \right| < \frac{p(A - B)}{1 - B^2} \quad (-1 < B < A \leq 1; z \in E)$$

and

$$(1.7) \quad \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{p(1 - A)}{2} \quad (B = -1; z \in E).$$



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Let $\mathcal{M}_p(\lambda, \mu, A, B)$ denote the class of functions in \mathcal{A}_p satisfying the condition

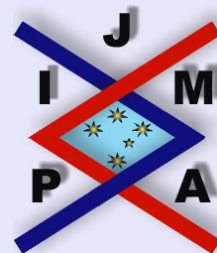
$$(1.8) \quad \frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)} \right\} \prec p \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1; z \in E)$$

for some real $\mu(\mu \geq 0)$, $\lambda(\lambda > 0)$, and $g \in \mathcal{S}_p^*$. For convenience, we write

$$\begin{aligned} \mathcal{M}_p \left(\lambda, \mu, 1 - \frac{2\alpha}{p}, -1 \right) &= \mathcal{M}_p(\lambda, \mu, \alpha) \\ &= \left\{ f \in \mathcal{A}_p : \Re \left[\frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)} \right\} \right] > \alpha \right\} \end{aligned}$$

for some $\alpha(0 \leq \alpha < p)$ and $z \in E$.

In the present paper, we derive various useful properties and characteristics of the class $\mathcal{M}_p(\lambda, \mu, A, B)$ by employing techniques involving Briot-Bouquet differential subordination. The integral preserving properties of Bazilevic functions in a sector are also considered. Relevant connections of the results presented here with those obtained in earlier works are pointed out.



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2. Preliminaries

To establish our main results, we shall require the following lemmas.

Lemma 2.1 ([6]). *Let h be a convex function in E and let ω be analytic in E with $\Re\{\omega(z)\} \geq 0$. If q is analytic in E and $q(0) = h(0)$, then*

$$q(z) + \omega(z)zq'(z) \prec h(z) \quad (z \in E)$$

implies

$$q(z) \prec h(z) \quad (z \in E).$$

Lemma 2.2. *If $-1 \leq B < A \leq 1, \beta > 0$ and the complex number γ satisfies $\Re(\gamma) \geq -\beta(1 - A)/(1 - B)$, then the differential equation*

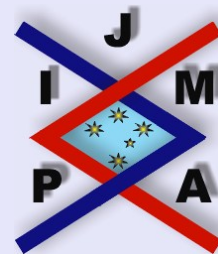
$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1 + Az}{1 + Bz} \quad (z \in E)$$

has a univalent solution in E given by

$$(2.1) \quad q(z) = \begin{cases} \frac{z^{\beta+\gamma}(1 + Bz)^{\beta(A-B)/B}}{\beta \int_0^z t^{\beta+\gamma-1}(1 + Bt)^{\beta(A-B)/B} dt} - \frac{\gamma}{\beta}, & B \neq 0 \\ \frac{z^{\beta+\gamma} \exp(\beta Az)}{\beta \int_0^z t^{\beta+\gamma-1} \exp(\beta At) dt} - \frac{\gamma}{\beta}, & B = 0. \end{cases}$$

If $\phi(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in E and satisfies

$$(2.2) \quad \phi(z) + \frac{z\phi'(z)}{\beta\phi(z) + \gamma} \prec \frac{1 + Az}{1 + Bz} \quad (z \in E),$$



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then

$$\phi(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in E)$$

and $q(z)$ is the best dominant of (2.2).

The above lemma is due to Miller and Mocanu [7].

Lemma 2.3 ([12]). Let ν be a positive measure on $[0, 1]$. Let h be a complex-valued function defined on $E \times [0, 1]$ such that $h(\cdot, t)$ is analytic in E for each $t \in [0, 1]$, and $h(z, \cdot)$ is ν -integrable on $[0, 1]$ for all $z \in E$. In addition, suppose that $\Re\{h(z, t)\} > 0$, $h(-r, t)$ is real and $\Re\{1/h(z, t)\} \geq 1/h(-r, t)$ for $|z| \leq r < 1$ and $t \in [0, 1]$. If $h(z) = \int_0^1 h(z, t) d\nu(t)$, then $\Re\{1/h(z)\} \geq 1/h(-r)$.

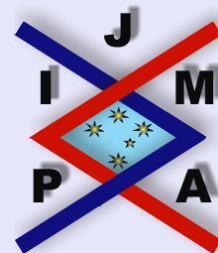
For real or complex numbers a, b, c ($c \neq 0, -1, -2, \dots$), the hypergeometric function is defined by

$$(2.3) \quad {}_2F_1(a, b; c; z) = 1 + \frac{a \cdot b}{c} \cdot \frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \dots$$

We note that the series in (2.3) converges absolutely for $z \in E$ and hence represents an analytic function in E . Each of the identities (asserted by Lemma 2.3 below) is well-known [13].

Lemma 2.4. For real numbers a, b, c ($c \neq 0, -1, -2, \dots$), we have

$$(2.4) \quad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z) \quad (c > b > 0)$$



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$$(2.5) \quad {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$$

$$(2.6) \quad {}_2F_1(a, b; c; z) = (1 - z)^{-a} {}_2F_1\left(a, c - b; c; \frac{z}{z - 1}\right).$$

Lemma 2.5 ([10]). Let $p(z) = 1 + c_1z + c_2z^2 + \dots$ be analytic in E and $p(z) \neq 0$ in E . If there exists a point $z_0 \in E$ such that

$$(2.7) \quad |\arg p(z)| < \frac{\pi}{2}\eta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg p(z_0)| = \frac{\pi}{2}\eta \quad (0 < \eta \leq 1),$$

then we have

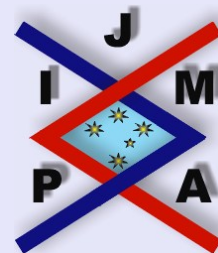
$$(2.8) \quad \frac{z_0 p'(z_0)}{p(z_0)} = ik\eta,$$

where

$$(2.9) \quad \begin{cases} k \geq \frac{1}{2} \left(x + \frac{1}{x}\right), & \text{when } \arg p(z_0) = \frac{\pi}{2}\eta, \\ k \leq -\frac{1}{2} \left(x + \frac{1}{x}\right), & \text{when } \arg p(z_0) = -\frac{\pi}{2}\eta, \end{cases}$$

and

$$(2.10) \quad (p(z_0))^{1/\eta} = \pm ix \quad (x > 0).$$



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3. Main Results

Theorem 3.1. Let $-1 \leq B < A \leq 1$, $\lambda > 0$ and $\mu \geq 0$. If $f \in \mathcal{M}_p(\lambda, \mu, A, B)$, then

$$(3.1) \quad \frac{zf'(z)}{pf(z)^{1-\mu}g(z)^\mu} \prec \frac{\lambda}{pQ(z)} = q(z) \quad (z \in E),$$

where

$$(3.2) \quad Q(z) = \begin{cases} \int_0^1 s^{\frac{p}{\lambda}-1} \left(\frac{1+Bsz}{1+Bz}\right)^{\frac{p(A-B)}{\lambda B}} ds, & B \neq 0, \\ \int_0^1 s^{\frac{p}{\lambda}-1} \exp\left(\frac{p}{\lambda}(s-1)Az\right) ds, & B = 0, \end{cases}$$

$$q(z) = \frac{1}{1+Bz} \text{ when } A = -\frac{\lambda B}{p}, B \neq 0,$$

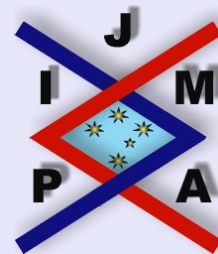
and $q(z)$ is the best dominant of (3.1). Furthermore, if $A \leq -\lambda B/p$ with $-1 \leq B < 0$, then

$$(3.3) \quad \mathcal{M}_p(\lambda, \mu, A, B) \subset \mathcal{B}_p(\mu, \rho),$$

where

$$\rho = \rho(p, \lambda, A, B) = p \left\{ {}_2F_1 \left(1, \frac{p(B-A)}{\lambda B}; \frac{p}{\lambda} + 1; \frac{B}{B-1} \right) \right\}^{-1}.$$

The result is best possible.



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Proof. Defining the function $\phi(z)$ by

$$(3.4) \quad \phi(z) = \frac{zf'(z)}{pf(z)^{1-\mu}g(z)^\mu} \quad (z \in E),$$

we note that $\phi(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in E . Taking the logarithmic differentiations in both sides of (3.4), we have

$$(3.5) \quad \frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)} \right\} \\ = p\phi(z) + \lambda \frac{z\phi'(z)}{\phi(z)} \prec \frac{p(1+Az)}{1+Bz} \quad (z \in E).$$

Thus, $\phi(z)$ satisfies the differential subordination (2.2) and hence by using Lemma 2.2, we get

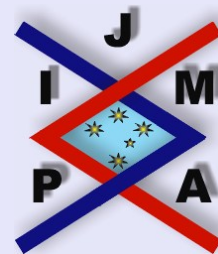
$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in E),$$

where $q(z)$ is given by (2.1) for $\beta = p/\lambda$ and $\gamma = 0$, and is the best dominant of (3.5). This proves the assertion (3.1).

Next, we show that

$$(3.6) \quad \inf_{|z|<1} \{\Re(q(z))\} = q(-1).$$

If we set $a = p(B-A)/\lambda B$, $b = p/\lambda$, $c = (p/\lambda) + 1$, then $c > b > 0$. From



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(3.2), by using (2.4), (2.5) and (2.6), we see that for $B \neq 0$

$$(3.7) \quad \begin{aligned} Q(z) &= (1 + Bz)^a \int_0^1 s^{b-1} (1 + Bs z)^{-a} ds \\ &= \frac{\Gamma(b)}{\Gamma(c)} {}_2F_1 \left(1, a; c; \frac{Bz}{Bz + 1} \right). \end{aligned}$$

To prove (3.6), we need to show that $\Re\{1/Q(z)\} \geq 1/Q(-1)$, $z \in E$. Since $A < -\lambda B/p$ implies $c > a > 0$, by using (2.4), (3.7) yields

$$Q(z) = \int_0^1 h(z, s) d\nu(s),$$

where

$$\begin{aligned} h(z, s) &= \frac{1 + Bz}{1 + (1 - s)Bz} \quad (0 \leq s \leq 1) \quad \text{and} \\ d\nu(s) &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(c - a)} s^{a-1} (1 - s)^{c-a-1} ds \end{aligned}$$

which is a positive measure on $[0, 1]$. For $-1 \leq B < 0$, it may be noted that $\Re\{h(z, s)\} > 0$, $h(-r, s)$ is real for $0 \leq r < 1$, $0 \in [0, 1]$ and

$$\Re \left\{ \frac{1}{h(z, s)} \right\} = \Re \left\{ \frac{1 + (1 - s)Bz}{1 + Bz} \right\} \geq \frac{1 - (1 - s)Br}{1 - Br} = \frac{1}{h(-r, s)}$$

for $|z| \leq r < 1$ and $s \in [0, 1]$. Therefore, by using Lemma 2.3, we have

$$\Re \left\{ \frac{1}{Q(z)} \right\} \geq \frac{1}{Q(-r)}, \quad |z| \leq r < 1$$



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and by letting $r \rightarrow 1^-$, we obtain $\Re\{1/Q(z)\} \geq 1/Q(-1)$. Further, by taking $A \rightarrow (-\lambda B/p)^+$ for the case $A = (-\lambda B/p)$, and using (3.1), we get (3.3).

The result is best possible as the function $q(z)$ is the best dominant of (3.1). This completes the proof of Theorem 3.1. \square

Setting $\mu = 1$, $A = 1 - (2\alpha/p) ((p - \lambda)/2 \leq \alpha < p)$ and $B = -1$ in Theorem 3.1, we have

Corollary 3.2. *If $f \in \mathcal{A}_p$ satisfies*

$$\Re \left\{ \frac{zf'(z)}{g(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right) \right\} > \alpha \quad (\lambda > 0, z \in E)$$

for some $g \in \mathcal{S}_p^*$, then $f \in \mathcal{B}_p(\kappa(p, \lambda, \alpha))$, where

$$(3.8) \quad \kappa(p, \lambda, \alpha) = p \left\{ {}_2F_1 \left(1, \frac{2(p - \alpha)}{\lambda}; \frac{p}{\lambda} + 1; \frac{1}{2} \right) \right\}^{-1}.$$

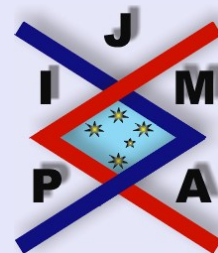
The result is best possible.

Taking $\mu = 0$, $A = 1 - (2\alpha/p) ((p - \lambda)/2 \leq \alpha < p)$ and $B = -1$ in Theorem 3.1, we get

Corollary 3.3. *If $f \in \mathcal{A}_p$ satisfies*

$$\Re \left\{ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha \quad (\lambda > 0, z \in E)$$

then $f \in \mathcal{S}_p^*(\kappa(p, \lambda, \alpha))$, where $\kappa(p, \lambda, \alpha)$ is given by (3.8). The result is best possible.



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Putting $\lambda = 1$ in Corollary 3.3, we get

Corollary 3.4. For $(p - 1)/2 \leq \alpha < p$, we have

$$\mathcal{K}_p(\alpha) \subset \mathcal{S}_p^*(\varkappa(p, \alpha)),$$

where $\varkappa(p, \alpha) = p \{ {}_2F_1(1, 2(p - \alpha); p + 1; 1/2) \}^{-1}$. The result is best possible.

Remark 1.

1. Noting that

$$\left\{ {}_2F_1 \left(1, 2(1 - \alpha); 2; \frac{1}{2} \right) \right\}^{-1} = \begin{cases} \frac{1-2\alpha}{2^{2(1-\alpha)}(1-2^{2\alpha-1})}, & \alpha \neq \frac{1}{2} \\ \frac{1}{2 \ln 2}, & \alpha = \frac{1}{2}, \end{cases}$$

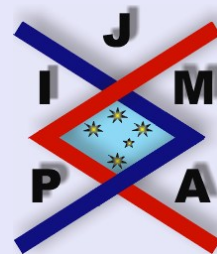
Corollary 3.4 yields the corresponding result due to MacGregor [5] (see also [12]) for $p = 1$.

2. It is proved [9] that if $p \geq 2$ and $f \in \mathcal{K}_p$, then f is p -valently starlike in E but is not necessarily p -valently starlike of order larger than zero in E . However, our Corollary 3.4 shows that if f is p -valently convex of order at least $(p - 1)/2$, then f is p -valently starlike of order larger than zero in E .

Theorem 3.5. If $f \in \mathcal{B}_p(\mu, \alpha)$ for some $\mu (\mu > 0)$, $\alpha (0 \leq \alpha < p)$, then $f \in \mathcal{M}_p(\lambda, \mu, \alpha)$ for $|z| < R(p, \lambda, \alpha)$, where $\lambda > 0$ and

$$(3.9) \quad R(p, \lambda, \alpha) = \begin{cases} \frac{(p+\lambda-\alpha) - \sqrt{(p+\lambda-\alpha)^2 - p(p-2\alpha)}}{p-2\alpha}, & \alpha \neq \frac{p}{2}; \\ \frac{p}{p+2\lambda}, & \alpha = \frac{p}{2}. \end{cases}$$

The bound $R(p, \lambda, \alpha)$ is best possible.



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Proof. From (1.4), we get

$$(3.10) \quad \frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} = \alpha + (p - \alpha)u(z) \quad (z \in E),$$

where $u(z) = 1 + u_1z + u_2z^2 + \dots$ is analytic and has a positive real part in E . Differentiating (3.10) logarithmically, we deduce that

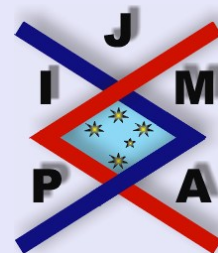
$$(3.11) \quad \begin{aligned} & \Re \left\{ \frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} - (1 - \mu) \frac{zf'(z)}{f(z)} - \mu \frac{zg'(z)}{g(z)} \right) \right\} - \alpha \\ &= (p - \alpha) \Re \left\{ u(z) + \frac{\lambda z u'(z)}{\alpha + (p - \alpha)u(z)} \right\} \\ &\geq (p - \alpha) \Re \left\{ u(z) - \frac{\lambda |z u'(z)|}{|\alpha + (p - \alpha)u(z)|} \right\}. \end{aligned}$$

Using the well-known estimates [5]

$$|z u'(z)| \leq \frac{2r}{1 - r^2} \Re\{u(z)\} \quad \text{and} \quad \Re\{u(z)\} \geq \frac{1 - r}{1 + r} \quad (|z| = r < 1)$$

in (3.11), we get

$$\begin{aligned} & \Re \left\{ \frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} - (1 - \mu) \frac{zf'(z)}{f(z)} - \mu \frac{zg'(z)}{g(z)} \right) \right\} - \alpha \\ &\geq (p - \alpha) \Re\{u(z)\} \left\{ 1 - \frac{2\lambda r}{\alpha(1 - r^2) + (p - \alpha)(1 - r)^2} \right\}, \end{aligned}$$



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which is certainly positive if $r < R(p, \lambda, \alpha)$, where $R(p, \lambda, \alpha)$ is given by (3.9).

To show that the bound $R(p, \lambda, \alpha)$ is best possible, we consider the function $f \in \mathcal{A}_p$ defined by

$$\frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} = \alpha + (p - \alpha)\frac{1 - z}{1 + z} \quad (0 \leq \alpha < p, z \in E)$$

for some $g \in \mathcal{S}_p^*$. Noting that

$$\begin{aligned} & \Re \left\{ \frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} - (1 - \mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)} \right) \right\} - \alpha \\ &= (p - \alpha) \left[\frac{1 - z}{1 + z} + \frac{2\lambda z}{\alpha(1 - z^2) + (p - \alpha)(1 + z)^2} \right] \\ &= 0 \end{aligned}$$

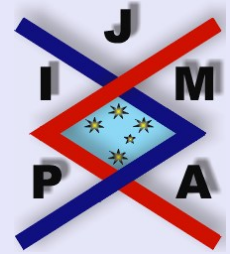
for $z = -R(p, \lambda, \alpha)$, we conclude that the bound is best possible. This proves Theorem 3.5. \square

For $\mu = 0$ and $\lambda = 1$, Theorem 3.5 yields:

Corollary 3.6. *If $f \in \mathcal{S}_p^*(\alpha)$ ($0 \leq \alpha < p$), then $f \in K_p(\alpha)$ in $|z| < \xi(p, \alpha)$, where*

$$\xi(p, \alpha) = \begin{cases} \frac{(p+1-\alpha) - \sqrt{\alpha^2 + 2(p-\alpha) + 1}}{p-2\alpha}, & \alpha \neq \frac{p}{2}; \\ \frac{p}{p+2}, & \alpha = \frac{p}{2}. \end{cases}$$

The bound $\xi(p, \alpha)$ is best possible.



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Theorem 3.7. If $f \in \mathcal{A}_p$ satisfies

$$\Re \left\{ \frac{f(z)}{z^p} \right\} > 0 \quad \text{and} \quad \left| \frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} - p \right| < p \quad (0 \leq \mu, z \in E)$$

for $g \in \mathcal{S}_p^*$, then f is p -valently convex(univalent) in $|z| < \tilde{R}(p, \mu)$, where

$$\tilde{R}(p, \mu) = \frac{3 + 2\mu(p-1) - \sqrt{(3 + 2\mu(p-1))^2 - 4p(2\mu p - p - 1)}}{2(2\mu p - p - 1)}.$$

The bound $\tilde{R}(p, \mu)$ is best possible.

Proof. Letting

$$h(z) = \frac{zf'(z)}{pf(z)^{1-\mu}g(z)^\mu} - 1 \quad (z \in E),$$

we note that $h(z)$ is analytic in E , $h(0) = 0$ and $|h(z)| < 1$ for $z \in E$. Thus, by applying Schwarz's Lemma we get

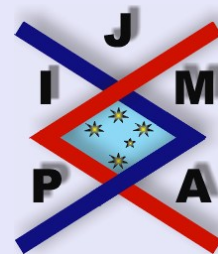
$$h(z) = z\psi(z),$$

where $\psi(z)$ is analytic in E and $|\psi(z)| \leq 1$ for $z \in E$. Therefore,

$$(3.12) \quad zf'(z) = pf(z)^{1-\mu}g(z)^\mu(1 + z\psi(z)).$$

Making use of logarithmic differentiation in (3.12), we obtain

$$(3.13) \quad 1 + \frac{zf''(z)}{f'(z)} = (1 - \mu) \frac{zf'(z)}{f(z)} + \mu \frac{zg'(z)}{g(z)} + \frac{z(\psi(z) + z\psi'(z))}{1 + z\psi(z)}.$$



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Setting $\phi(z) = f(z)/z^p = 1 + c_1z + c_2z^2 + \dots$, $\Re\{\phi(z)\} > 0$ for $z \in E$, we get

$$\frac{zf'(z)}{f(z)} = p + \frac{z\phi'(z)}{\phi(z)}$$

so that by (3.13),

$$(3.14) \quad 1 + \frac{zf''(z)}{f'(z)} = (1-\mu)p + (1-\mu)\frac{z\phi'(z)}{\phi(z)} + \mu\frac{zg'(z)}{g(z)} + \frac{z(\psi(z) + z\psi'(z))}{1+z\psi(z)}.$$

Now, by using the well-known estimates [1]

$$\Re\left\{\frac{z\phi'(z)}{\phi(z)}\right\} \geq -\frac{2r}{1-r^2}, \quad \Re\left\{\frac{zg'(z)}{g(z)}\right\} \geq -\frac{p(1-r)}{1+r} \quad \text{and}$$

$$\Re\left\{\frac{\psi(z) + z\psi'(z)}{1+z\psi(z)}\right\} \geq -\frac{1}{1-r}$$

for $|z| = r < 1$ in (3.14), we deduce that

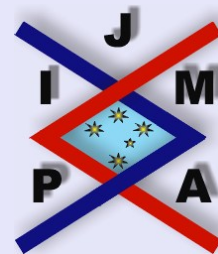
$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \frac{(2\mu p - p - 1)r^2 - \{3 + 2\mu(p - 1)\}r + p}{1 - r^2}$$

which is certainly positive if $r < \tilde{R}(p, \mu)$. □

It is easily seen that the bound $\tilde{R}(p, \mu)$ is sharp for the functions $f, g \in \mathcal{A}_p$ defined in E by

$$\frac{zf'(z)}{pf(z)^{1-\mu}g(z)^\mu} = \frac{1}{1+z}, \quad g(z) = \frac{z^p}{(1+z)^2} \quad (0 \leq \mu, z \in E).$$

Choosing $\mu = 0$ in Theorem 3.7, we have



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Corollary 3.8. If $f \in \mathcal{A}_p$ satisfies

$$\Re \left\{ \frac{f(z)}{z^p} \right\} > 0 \quad \text{and} \quad \left| \frac{zf'(z)}{f(z)} - p \right| < p \quad (z \in E)$$

then f is p -valently convex in $|z| < \left\{ \sqrt{9 + 4p(p+1)} - 3 \right\} / 2(p+1)$. The result is best possible.

For a function $f \in \mathcal{A}_p$, we define the integral operator $F_{\mu,\delta}$ as follows:

$$(3.15) \quad F_{\mu,\delta}(f) = F_{\mu,\delta}(f)(z) = \left(\frac{\delta + p\mu}{z^\delta} \int_0^z t^{\delta-1} f(t)^\mu dt \right)^{\frac{1}{\mu}} \quad (z \in E),$$

where μ and δ are real numbers with $\mu > 0$, $\delta > -p\mu$.

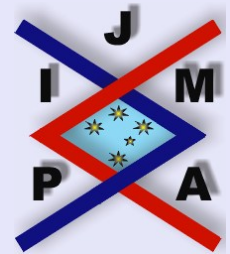
The following lemma will be required for the proof of Theorem 3.12 below.

Lemma 3.9. Let $g \in \mathcal{S}_p^*(A, B)$, μ and δ are real numbers with $\mu > 0$, $\delta > \max \left\{ -p\mu, -\frac{p\mu(1-A)}{(1-B)} \right\}$. Then $F_{\mu,\delta}(g) \in \mathcal{S}_p^*(A, B)$.

The proof of the above lemma follows by using Lemma 2.2 followed by a simple calculation.

Theorem 3.10. Let μ and δ be real numbers with $\mu > 0$, $\delta > \max \left\{ -p\mu, -\frac{p\mu(1-A)}{(1-B)} \right\}$ ($-1 \leq B < A \leq 1$) and let $f \in \mathcal{A}_p$. If

$$\left| \arg \left(\frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (0 \leq \alpha < p; 0 < \beta \leq 1)$$



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for some $g \in \mathcal{S}_p^*(A, B)$, then

$$\left| \arg \left(\frac{z(F_{\mu,\delta})'(f)}{F_{\mu,\delta}(f)^{1-\mu}F_{\mu,\delta}(g)^\mu} - \alpha \right) \right| < \frac{\pi}{2}\eta,$$

where $F_{\mu,\delta}(f)$ is the operator given by (3.15) and η ($0 < \eta \leq 1$) is the solution of the equation

$$(3.16) \quad \beta = \begin{cases} \eta + \frac{2}{\pi} \tan^{-1} \left(\frac{(1+B)\eta \sin \left(\frac{\pi}{2}(1-t(A,B,\delta,\mu,p)) \right)}{(1+B)\delta + \mu p(1+A) + (1+B)\eta \cos \left(\frac{\pi}{2}(1-t(A,B,\delta,\mu,p)) \right)} \right), & B \neq -1; \\ \eta, & B = -1, \end{cases}$$

and

$$(3.17) \quad t(A, B, \delta, \mu, p) = \frac{2}{\pi} \sin^{-1} \left(\frac{\mu p(A - B)}{\delta(1 - B^2) + \mu p(1 - AB)} \right).$$

Proof. Let us put

$$q(z) = \frac{1}{p - \alpha} \left(\frac{z(F_{\mu,\delta})'(f)}{F_{\mu,\delta}(f)^{1-\mu}F_{\mu,\delta}(g)^\mu} - \beta \right) = \frac{\Phi(z)}{\Psi(z)},$$

where

$$\Phi(z) = \frac{1}{p - \alpha} \left\{ z^\delta f(z)^\mu - \delta \int_0^z t^{\delta-1} f(t)^\mu dt - \mu \alpha \int_0^z t^{\delta-1} g(t)^\mu dt \right\}$$



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and

$$\Psi(z) = \mu \int_0^z t^{\delta-1} g(t)^\mu dt.$$

Then $q(z)$ is analytic in E and $q(0) = 1$. By a simple calculation, we get

$$\begin{aligned} \frac{\Phi'(z)}{\Psi'(z)} &= q(z) \left(1 + \frac{S(z)}{zS'(z)} \frac{zq'(z)}{q(z)} \right) \\ &= \frac{1}{p-\beta} \left(\frac{zf'(z)}{f(z)^{1-\mu}g(z)^\mu} - \alpha \right). \end{aligned}$$

Since $F_{\mu,\delta}(g) \in \mathcal{S}_p^*(A, B)$, by (1.6) and (1.7), we have

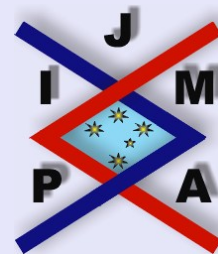
$$(3.18) \quad \frac{zS'(z)}{S(z)} = \delta + \mu \frac{z(F_{\mu,\delta})'(g)}{F_{\mu,\delta}(g)} = \rho e^{i\pi\theta/2},$$

where

$$\begin{cases} \delta + \frac{\mu p(1-A)}{1-B} < \rho < \delta + \frac{\mu p(1+A)}{1+B} \\ -t(A, B, \delta, \mu, p) < \theta < t(A, B, \delta, \mu, p) \text{ for } B \neq -1 \end{cases}$$

when $t(A, B, \delta, \mu, p)$ is given by (3.17), and

$$\begin{cases} \delta + \frac{\mu p(1-A)}{2} < \rho < \infty \\ -1 < \theta < 1 \text{ for } B = -1. \end{cases}$$



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Further, taking $\omega(z) = S(z)/zS'(z)$ in Lemma 2.1, we note that $q(z) \neq 0$ in E . If there exists a point $z_0 \in E$ such that the condition (2.7) is satisfied, then (by Lemma 2.5) we obtain (2.8) under the restrictions (2.9) and (2.10).

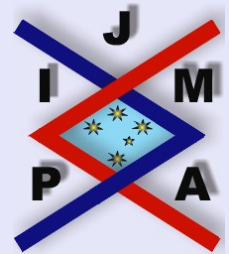
At first, suppose that $q(z_0)^{\frac{1}{\eta}} = ix$ ($x > 0$). For the case $B \neq -1$, by (3.18), we obtain

$$\begin{aligned} & \arg \left(\frac{z_0 f'(z_0)}{f(z_0)^{1-\mu} g(z_0)^\mu} - \alpha \right) \\ &= \arg(q(z_0)) + \arg \left(1 + \frac{1}{\delta + \mu \frac{z_0 (F_{\mu, \delta}(g))'(z_0)}{F_{\mu, \delta}(g)(z_0)}} \cdot \frac{z_0 q'(z_0)}{q(z_0)} \right) \\ &= \frac{\pi}{2} \eta + \arg \left(1 + (\rho e^{i\pi\theta/2})^{-1} i \eta k \right) \\ &= \frac{\pi}{2} \eta + \tan^{-1} \left(\frac{\eta k \sin(\pi(1-\theta)/2)}{\rho + \cos(\pi(1-\theta)/2)} \right) \\ &\geq \frac{\pi}{2} \eta + \tan^{-1} \left(\frac{\eta \sin(\pi(1-t(A, B, \delta, \mu, p))/2)}{\delta + \frac{\mu p(1+A)}{1+B} + \eta \cos(\pi(1-t(A, B, \delta, \mu, p))/2)} \right) \\ &= \frac{\pi}{2} \beta, \end{aligned}$$

where β and $t(A, B, \delta, \mu, p)$ are given by (3.16) and (3.17), respectively. Similarly, for the case $B = -1$, we have

$$\arg \left(\frac{z f'(z)}{f(z)^{1-\mu} g(z)^\mu} - \alpha \right) \geq \frac{\pi}{2} \eta.$$

This is a contradiction to the assumption of our theorem.



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Next, suppose that $q(z_0)^{\frac{1}{\eta}} = -ix$ ($x > 0$). For the case $B \neq -1$, applying the same method as above, we have

$$\begin{aligned} & \arg \left(\frac{z_0 f'(z_0)}{f(z_0)^{1-\mu} g(z_0)^\mu} - \alpha \right) \\ & \leq -\frac{\pi}{2} \eta - \tan^{-1} \left(\frac{\eta \sin(\pi(1 - t(A, B, \delta, \mu, p))/2)}{\delta + \frac{\mu p(1+A)}{1+B} + \eta \cos(\pi(1 - t(A, B, \delta, \mu, p))/2)} \right) \\ & = -\frac{\pi}{2} \beta, \end{aligned}$$

where β and $t(A, B, \delta, \mu, p)$ are given by (3.16) and (3.17), respectively and for the case $B = -1$, we have

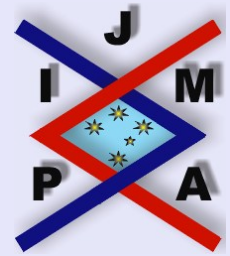
$$\arg \left(\frac{z f'(z)}{f(z)^{1-\mu} g(z)^\mu} - \alpha \right) \leq -\frac{\pi}{2} \eta,$$

which contradicts the assumption. Thus, we complete the proof of the theorem. \square

Letting $\mu = 1$, $B \rightarrow A$ and $g(z) = z^p$ in Theorem 3.10, we have

Corollary 3.11. *Let $\delta > -p$ and $f \in \mathcal{A}_p$. If*

$$\left| \arg \left(\frac{f'(z)}{z^{p-1}} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (0 \leq \alpha < p; 0 < \beta \leq 1),$$



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then

$$\left| \arg \left(\frac{F'_{1,\delta}(f)}{z^{p-1}} - \alpha \right) \right| < \frac{\pi}{2} \eta,$$

where $F_{1,\delta}(f)$ is the integral operator given by (3.15) for $\mu = 1$ and η ($0 < \eta \leq 1$) is the solution of the equation

$$\beta = \eta + \frac{2}{\pi} \tan^{-1} \left(\frac{\eta}{\delta + p} \right).$$

Theorem 3.12. Let $\lambda > 0$. If $f \in \mathcal{A}$ satisfies the condition

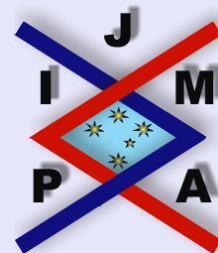
$$(3.19) \quad \gamma \left\{ \frac{zf'(z)}{f^{1-\mu}(z)g^\mu(z)} \right\} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - (1-\mu) \frac{zf'(z)}{f(z)} - \mu \frac{zg'(z)}{g(z)} \right\} \neq it \quad (z \in E)$$

for some μ ($\mu \geq 0$), γ ($\gamma > 0$) and $g \in \mathcal{S}_p^*$, where t is a real number with $|t| \geq \sqrt{\lambda(\lambda + 2p\gamma)}$, then

$$\Re \left\{ \frac{zf'(z)}{f^{1-\mu}(z)g^\mu(z)} \right\} > 0 \quad (z \in E).$$

Proof. Let

$$\phi(z) = \frac{zf'(z)}{p f^{1-\mu}(z)g^\mu(z)} \quad (z \in E),$$



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where $\phi(0) = 1$. From (3.19), we easily have $\phi(z) \neq 0$ in E . In fact, if ϕ has a zero of order m at $z = z_1 \in E$, then ϕ can be written as

$$\phi(z) = (z - z_1)^m q(z) \quad (m \in \mathbb{N}),$$

where $q(z)$ is analytic in E and $q(z_1) \neq 0$. Hence, we have

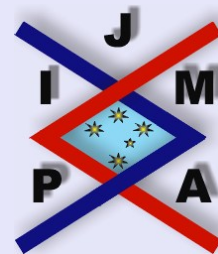
$$\begin{aligned} & \gamma \left\{ \frac{zf'(z)}{f^{1-\mu}(z)g^\mu(z)} \right\} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)} \right\} \\ & = p\gamma\phi(z) + \lambda\frac{z\phi'(z)}{\phi(z)} \\ (3.20) \quad & = p\gamma(z - z_1)^m q(z) + \lambda\frac{mz}{z - z_1} + \lambda\frac{zq'(z)}{q(z)}. \end{aligned}$$

But the imaginary part of (3.20) can take any infinite values when $z \rightarrow z_1$ in a suitable direction. This contradicts (3.19). Thus, if there exists a point $z_0 \in E$ such that

$$\Re\{p(z)\} > 0 \text{ for } |z| < |z_0|, \quad \Re\{p(z_0)\} > 0 \text{ and } p(z_0) = i\ell \ (\ell \neq 0),$$

then we have $p(z_0) \neq 0$. From Lemma 2.5 and (3.20), we get

$$\begin{aligned} p\gamma\phi(z_0) + \lambda\frac{z_0\phi'(z_0)}{\phi(z_0)} & = i(p\gamma\ell + \lambda k), \\ p\gamma\ell + \lambda k & \geq \frac{1}{2} \left(\frac{\lambda}{\ell} + (\lambda + 2p\gamma)\ell \right) \geq \sqrt{\lambda(\lambda + 2p\gamma)} \text{ when } \ell > 0, \end{aligned}$$



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and

$$p\gamma\ell + \lambda k \leq -\frac{1}{2} \left(\frac{\lambda}{|\ell|} + (\lambda + 2p\gamma)|\ell| \right) \leq -\sqrt{\lambda(\lambda + 2p\gamma)} \quad \text{when } \ell < 0,$$

which contradicts (3.19). Therefore, we have $\Re\{\phi(z)\} > 0$ in E . This completes the proof of the theorem. \square

Taking $g(z) = z^p$ and $\mu = 1$ in Theorem 3.12, we have

Corollary 3.13. *Let $\lambda > 0$. If $f \in \mathcal{A}_p$ satisfies the condition*

$$\gamma \frac{f'(z)}{z^{p-1}} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - p \right\} \neq it \quad (z \in E)$$

for some γ ($\gamma > 0$), where t is a real number with $|t| \geq \sqrt{\lambda(\lambda + 2p\gamma)}$, then

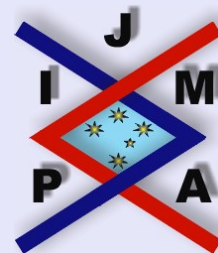
$$\Re \left\{ \frac{f'(z)}{z^{p-1}} \right\} > 0 \quad (z \in E).$$

Corollary 3.14. *Let $\lambda > 0$. If $f \in \mathcal{A}_p$ satisfies the condition*

$$\left| \gamma \left\{ \frac{f'(z)}{z^{p-1}} - p \right\} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} - p \right\} \right| < \lambda + \gamma p \quad (z \in E)$$

for some γ ($\gamma > 0$), then

$$\Re \left\{ \frac{f'(z)}{z^{p-1}} \right\} > 0 \quad (z \in E).$$



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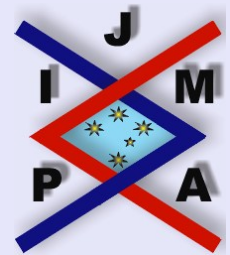
Remark 2. From a result of Nunokawa [9] and Saitoh and Nunokawa [11], it follows that, if $f \in \mathcal{A}_p$ satisfies the hypothesis of Corollary 3.13 or Corollary 3.14, then f is p -valent in E and p -valently convex in the disc $|z| < (\sqrt{p+1} - 1)/p$.

Letting $\gamma = 1, \mu = 0$ in Theorem 3.12, we get the following result due to Dingdong [4] which in turn yields the work of Cho and Kim [3] for $p = 1$.

Corollary 3.15. Let $\lambda > 0$. If $f \in \mathcal{A}_p$ satisfies the condition

$$(1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \neq it \quad (z \in E),$$

where t is a real number with $|t| \geq \sqrt{\lambda(\lambda + 2p)}$, then $f \in S_p^*$.



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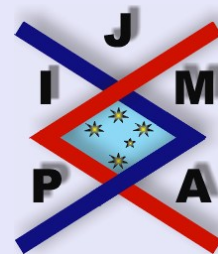
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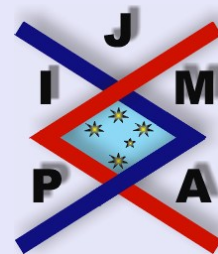
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