BOUNDS ON EXPECTATIONS OF RECORD RANGE AND RECORD INCREMENT FROM DISTRIBUTIONS WITH BOUNDED SUPPORT

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Abstract: In this paper, we consider the record statistics at the time when the nth record of any kind (either an upper or lower) is observed based on a sequence of independent random variables with identical continuous distributions of bounded support. We provide sharp upper bounds for expectations of record range and current upper record increment. We also present numerical evaluations of the so obtained bounds. The results may be of interest in estimating the expected lengths of the confidence intervals for quantiles as well as prediction intervals for record statistics.

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1. Introduction

Let $\{X_j, j \geq 1\}$ be a sequence of independent identically distributed (iid) continuous random variables (r.v.'s) on a bounded support $[a, b]$. Let $F(x)$, $F^{-1}(x)$, and $\mu = \int_0^1 F^{-1}(x)dx \in (a, b)$ denote the cumulative distribution function (cdf), quantile function and population mean respectively. Let $X_{j:n}$, $1 \leq j \leq n$, be the jth smallest value in the finite sequence $X_1, X_2, \ldots, X_n$. An observation X_j will be called an upper record value if its value exceeds that of all previous observations. That is; X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition deals with lower record values. The times at which the records occur are called record times.

The nth upper current record U_n^c is defined as the current value of upper records, in the X_n sequence when the nth value of either lower or upper record is observed. The nth lower current record L_n^c can be defined similarly. It can be noticed that $U_{n+1}^c = U_n^c$ iff $L_{n+1}^c < L_n^c$ and that $L_{n+1}^c = L_n^c$ if $U_{n+1}^c > U_n^c$. That is, the upper current record value is the largest observation seen to date at the time when the nth record (of either kind) is observed. According to the definition, $L_0^c = U_0^c = X_1$.

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of a sample of size $n \geq 1$. Define the sample range sequence by $I_n = X_{n:n} - X_{1:n}$, $n = 1, 2, \ldots$. Let R_n $(n = 1, 2, \ldots)$ be the nth record in the sequence of sample ranges, $\{I_n, n \geq 1\}$. In fact, R_n is the nth record range in the X_n sequence. It is also expressed by the current values of upper and lower records as

$$R_n = U_n^c - L_n^c, \quad n = 1, 2, \ldots. \tag{1.1}$$

By the definition, $R_0 = 0$ and $R_1 = I_2$ is the first record range. The current record values can be used (see, for example, [5]) in a general sequential method for model choice and outlier detection involving the record range. Let N denote the stopping time such that

$$N = Inf\{n > 0;\ R_n > c\}, \quad c \text{ is an arbitrary fixed value.}$$

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Hence, N gives the waiting time until the record range of an iid sample exceeds a given value c. In this context, the waiting time N is defined in terms of the current values of lower and upper records but not in terms of the number of observations. For populations of thicker tails, N would tend to be smaller.

Houchens [7] introduced the concept of current record statistics and derived the pdf of the nth upper and lower current record statistics. Ahmadi and Balakrishnan in [1] established confidence intervals for quantiles in terms of record range; in [2] they studied some reliability properties of certain current record statistics. Recently, Raqab [9] presented sharp upper bounds for the expected values of the gap between the nth upper current record and nth upper record value as well as upper sharp bounds for the current record increments from general distributions.

It is of interest to address the problem of sharp bounds for the expectations of current records and other related statistics from an iid sequence with continuous $F(x)$ supported on a finite $[a, b]$. In this paper, we use an approach of Rychlik [11] to provide sharp upper bounds for the expected record range and current upper record increments in the support interval lengths units $b - a$. The obtained bounds also depend on the parameter

$$\eta = \frac{b - \mu}{b - a} \in (0, 1),$$

which represents the relative distance of μ from the upper support point in the support length units.

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2. Auxiliary Results

We will present some auxiliary results that will be helpful in the subsequent results.

Lemma 2.1. *For* $n \geq 1$*, the marginal densities of* L_n^c *and* U_n^c *from the iid* $U(0,1)$ *sequence are respectively,*

$$f_{L_n^c}(x) = 2^n \left\{ 1 - x \sum_{j=0}^{n-1} \frac{[-\log x]^j}{j!} \right\}, \tag{2.1}$$

and

$$f_{U_n^c}(x) = 2^n \left\{ 1 - (1-x) \sum_{j=0}^{n-1} \frac{[-\log(1-x)]^j}{j!} \right\}. \tag{2.2}$$

Proof. Let V_k and W_k be the kth lower and upper current records, respectively from a sequence of iid $U(0,1)$ r.v.'s with joint pdf $f_k(v,w)$ and cdf $F_k(v,w)$. It is easily observed (see [7]) that

$$P\left(V_n \leq v^*, W_n > w^* | V_{n-1} = v, W_{n-1} = w\right) = \begin{cases} 1, & \text{if} \quad v^* \geq v, \ w* \leq w, \\ 0, & \text{if} \quad v^* < v, \ w* > w, \\ \frac{v^*}{(v+1-w)}, & \text{if} \quad v^* < v, \ w* \leq w, \\ \frac{1-w^*}{(v+1-w)}, & \text{if} \quad v^* \geq v, \ w* > w, \end{cases}$$

where $0 < v^* < w^* < 1$ and $0 < v < w < 1$, $n = 1, 2, \ldots$.

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Using integration, we obtain the unconditional probability as follows:

$$P(V_n \le v^*, W_n > w^*) \tag{2.3}$$
$$= \int_0^{v^*}\int_{w^*}^{1} f_{n-1}(x,y)dydx + \int_{w^*}^{1}\int_{v^*}^{y} \frac{v^*}{x+1-y} f_{n-1}(x,y)dxdy$$
$$+ \int_0^{v^*}\int_{x}^{w^*} \frac{1-w^*}{x+1-y} f_{n-1}(x,y)dydx.$$

From the identity

$$F_k(v^*, w^*) = P(V_k \le v^*) - P(V_k < v^*, W_k > w^*),$$

and the fact that the first integral in (2.3) is $P(V_{n-1} \le v^*, W_{n-1} > w^*)$, we have

$$F_n(v^*, w^*) = F_{n-1}(v^*, w^*) + P(V_n \le v^*) - P(V_{n-1} \le v^*) \tag{2.4}$$
$$- \int_{w^*}^{1}\int_{v^*}^{y} \frac{v^*}{x+1-y} f_{n-1}(x,y)\, dxdy$$
$$- \int_0^{v^*}\int_{x}^{w^*} \frac{1-w^*}{x+1-y} f_{n-1}(x,y)\, dydx.$$

Differentiating (2.4) with respect to v^* and w^*, we obtain recursively

$$f_n(v^*, w^*) = \int_{v^*}^{w^*} \frac{1}{x+1-w^*} f_{n-1}(x, w^*)dx + \tag{2.5}$$
$$\int_{v^*}^{w^*} \frac{1}{v^*+1-y} f_{n-1}(v^*, y)dy.$$

Using the recurrence relation in (2.5) and an inductive argument, we immediately

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have the joint pdf of V_n and W_n

$$f_n(l,u) = 2^n \frac{[-\log(1-u+l)]^{n-1}}{(n-1)!}, \quad 0 < l < u. \tag{2.6}$$

It follows from (2.6) that the marginal pdf's of L_n^c and U_n^c can be derived and obtained in the form of (2.1) and (2.2), respectively. The expressions in curly brackets in (2.1) and (2.2) represent the cdf's of $(n-1)$th lower and upper records, respectively in a sequence of iid $U(0,1)$ random variables (see [4] and [3]). ∎

Lemma 2.2 (Moriguti's Inequality). *Let $\overline{g}$ be the right derivative of the greatest convex function $\overline{G}(x) = \int_a^x \overline{g}(u)du$, not greater than the indefinite integral $G(x) = \int_a^x g(u)du$ of g. For every nondecreasing function τ on $[a,b]$ for which both integrals in (2.7) are finite, we have*

$$\int_a^b \tau(u)g(u)du \le \int_a^b \tau(u)\overline{g}(u)du. \tag{2.7}$$

The equality in (2.7) holds iff τ is constant on every open interval where $G > \overline{G}$.

Lemma 2.2 follows from [8, Theorem 1]. If $g \in L^2([a,b],dx)$ then $\overline{g}(x)$ is the projection of $g(x)$ onto the convex cone of nondecreasing functions in $L^2([a,b],dx)$ (cf. [10, pp. 12-16]).

The expected value of the nth record range can be written as

$$E(R_n) = \int_0^1 [F^{-1}(x) - \mu]\varphi_n(x)dx, \tag{2.8}$$

where

$$\varphi_n(u) = f_{U_n^c}(u) - f_{L_n^c}(u) \tag{2.9}$$

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represents the difference between the pdf's of the nth upper current record and nth lower current record from the $U(0,1)$ iid sequence. The following equality

$$\gamma(r,t) = \int_t^\infty \frac{x^{r-1}e^{-x}}{\Gamma(r)}dx = \sum_{j=0}^{r-1} \frac{t^j e^{-t}}{j!}, \tag{2.10}$$

represents the relationship between the incomplete gamma function and the sum of Poisson probabilities. The function defined by

$$\begin{aligned} \delta_{m,n}(x) &= f_{U_n^c}(x) - f_{U_m^c}(x) \\ &= \int_0^{-\log(1-x)} g_{m,n}(y)dy, \end{aligned} \tag{2.11}$$

where

$$g_{m,n}(y) = \left[\frac{2^n}{(n-1)!}\, y^{n-1} - \frac{2^m}{(m-1)!}\, y^{m-1}\right] e^{-y},$$

represents the difference between the pdf's of mth and nth upper current records ($1 \le m < n$) from the $U(0,1)$ iid sequence. Its respective expectation can be written as

$$E(I_{m,n}) = E(U_n^c - U_m^c) = \int_0^1 (F^{-1}(x) - \mu)\delta_{m,n}(x)dx. \tag{2.12}$$

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3. Main Results

We use several inequalities for the integral of the product of two functions such that one is given and the other one belongs to class of non-decreasing functions. We assume that all the integrals below are finite.

Theorem 3.1. *Let F be a continuous cdf with bounded support $[a, b]$. Then for $n \geq 1$,*

$$\begin{aligned} E(R_n) &\leq B_1(n) \\ &= (a-b)\left\{(1-2^n) + (1-\eta)^2 \sum_{j=0}^{n-1} (2^n - 2^j) \frac{[-\log(1-\eta)]^j}{j!} \right. \\ &\qquad \left. - \eta^2 \sum_{j=0}^{n-1} (2^j - 2^n) \frac{[-\log \eta]^j}{j!} \right\}. \end{aligned} \tag{3.1}$$

The equality in (3.1) is attained in the limit by the sequence of continuous distributions tending to the family of two-point distributions supported on a and b with probabilities η and $1-\eta$.

Proof. Combining (2.1), (2.2) and (2.10), we rewrite $\varphi_n(x)$ as

$$\varphi_n(x) = 2^n \left\{\gamma(n, -\log x) - \gamma(n, -\log(1-x)\right\}.$$

Therefore, the derivative of $\varphi_n(x)$ is

$$\varphi_n'(x) = 2^n \left(f_{U_n}(x) + f_{L_n}(x)\right) > 0.$$

where $f_{U_n}(x)$ and $f_{L_n}(x)$ are the pdf's of the nth upper and lower records from the $U(0, 1)$ iid sequence, respectively (see [3]). Since $\varphi_n(x)$ is a nondecreasing function

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on $[0,1]$ and $a-\mu < F^{-1}(x)-\mu < b-\mu$ with $a-\mu \le 0$ and $b-\mu \ge 0$, we have

$$\begin{aligned} E(R_n) &= \int_0^1 [F^{-1}(x)-\mu][\varphi_n(x)-\varphi_n(\eta)]dx \\ &= \int_0^\eta [F^{-1}(x)-\mu][\varphi_n(x)-\varphi_n(\eta)]dx \\ &\qquad + \int_\eta^1 [F^{-1}(x)-\mu][\varphi_n(x)-\varphi_n(\eta)]dx \\ &\le (a-\mu)\int_0^\eta [\varphi_n(x)-\varphi_n(\eta)]dx + (b-\mu)\int_\eta^1 [\varphi_n(x)-\varphi_n(\eta)]dx \\ &= (a-b)\Phi_n(\eta), \end{aligned} \tag{3.2}$$

where $\Phi_n(x)$ is the antiderivative of $\varphi_n(x)$. By definition, $\Phi_n(x)$ is the difference between the cdf's of the nth upper and lower current records $F_{U_n^c}$ and $F_{L_n^c}$, respectively.

From (2.1), the cdf $F_{L_n^c}(x)$ can be represented as

$$\begin{aligned} P\left(L_n^c \le u\right) &= \frac{2^n}{(n-1)!}\int_0^u \int_0^{-\log x} y^{n-1}e^{-y}\,dy\,dx \\ &= \frac{2^n}{(n-1)!}\left\{u\int_0^{-\log u} y^{n-1}e^{-y}\,dy + \int_{-\log u}^{\infty} y^{n-1}e^{-2y}\,dy\right\}. \end{aligned}$$

By (2.10), we have

$$F_{L_n^c}(u) = 2^n u + u^2 \sum_{j=0}^{n-1}(2^j-2^n)\frac{[-\log u)]^j}{j!}, \tag{3.3}$$

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Proceeding similarly, we write the cdf of U_n^c as

$$F_{U_n^c}(u) = 1 - 2^n(1-u) + (1-u)^2 \sum_{j=0}^{n-1} (2^n - 2^j) \frac{[-\log(1-u)]^j}{j!}. \tag{3.4}$$

Using (3.2), (3.3) and (3.4), we obtain (3.1). The inequality in (3.2) becomes equality if

$$F^{-1}(x) = \begin{cases} a, & \text{if} \quad 0 < x < \eta, \\ b, & \text{if} \quad \eta < x < 1, \end{cases}$$

which determines the family of two-point distributions. ∎

Now, we consider the bounds for the mean of current record increments $E(I_{m,n})$, $0 \le m < n$. The function $\delta_{m,n}(x)$ in (2.11) is not monotonic for $m \ge 1$ and $F^{-1} - \mu$ is nondecreasing. In order to get optimal evaluations for current record increments, we should analyze the variability of $\delta_{m,n}(x)$. Theorem 3.2 below allows us to establish sharp bounds on the expectations of current record increments for distributions with finite support.

Theorem 3.2. *For given* $1 \le m < n$, *there exists a unique* $\rho_{m,n} \in [\theta_{m,n}, 1]$ *defined as the solution to equation*

$$\begin{aligned} 2^n\gamma(n, -\log(1-u)) - 2^m\gamma(m, -\log(1-u)) + \gamma(m, -2\log(1-u)) \\ - \gamma(n, -2\log(1-u)) = 2^n - 2^m, \end{aligned} \tag{3.5}$$

such that for

$$\bar{\delta}_{m,n}(x) = \delta_{m,n}\left(\max\{x, \rho_{m,n}\}\right), \qquad 0 \le x \le 1,$$

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and every nondecreasing $\tau \in L^1([0,1], dx)$, *we have*

$$\int_0^1 \tau(x)\,\delta_{m,n}(x)du \leq \int_0^1 \tau(x)\,\overline{\delta}_{m,n}(x)dx \tag{3.6}$$

with the equality iff

$$\tau(u) = const, \quad 0 < x < \rho_{m,n}. \tag{3.7}$$

Proof. By simple analysis of the derivative of (2.11), $\delta_{m,n}$, $1 \leq m < n$, is decreasing-increasing. Precisely, $\delta_{m,n}(x)$ decreases on $(0, \theta_{m,n})$ and increases on $(\theta_{m,n}, 1)$, where $\theta_{m,n} = 1 - e^{-\frac{1}{2}\left[\frac{(n-1)!}{(m-1)!}\right]^{1/(n-m)}}$. By adding the facts $\delta_{m,n}(0) = 0$, $\delta_{m,n}(1) = 2^n - 2^m > 0$, we conclude that $\delta_{m,n}$ is negative-positive passing the horizontal axis at $\xi_{m,n}$ that satisfies

$$2^m\gamma(m, -\log(1-\xi_{m,n})) - 2^n\gamma(n, -\log(1-\xi_{m,n})) = 2^m - 2^n. \tag{3.8}$$

The antiderivative of $\delta_{m,n}(x)$ needed for the projection, $\Delta_{m,n}(x)$, is therefore concave decreasing, convex decreasing and convex increasing in $[0, \theta_{m,n}]$, $[\theta_{m,n}, \xi_{m,n}]$, and $[\xi_{m,n}, 1]$, respectively. Further, it is negative with $\Delta_{m,n}(0) = \Delta_{m,n}(1) = 0$. Thus its greatest convex minorant $\overline{\Delta}_{m,n}$ is given by

$$\overline{\Delta}_{m,n}(x) = \begin{cases} \delta_{m,n}(\rho_{m,n})x, & \text{if} \quad 0 \leq x \leq \rho_{m,n}, \\ \Delta_{m,n}(x), & \text{if} \quad \rho_{m,n} < x < 1. \end{cases} \tag{3.9}$$

where $\rho_{m,n}$ is determined by solving the equation

$$\Delta_{m,n}(x) = \delta_{m,n}(x)x. \tag{3.10}$$

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Using (2.10), Eq. (3.10) can be simplified and rewritten in the form

$$\int_0^u \int_0^{-\log(1-x)} g_{m,n}(y)dydx = \left\{2^n - 2^m + 2^m\gamma(m, -\log(1-u)) - 2^n\gamma(n, -\log(1-u))\right\} u,$$

which leads to (3.5). Note that Eq.(3.5) has to be solved numerically in order to find the numbers $\rho_{m,n}$'s. ∎

Theorem 3.3. *Let F be a continuous cdf with bounded support $[a,b]$. If $m = 0$, then*

$$E(I_{m,n}) \le B_2(m,n) = (b-a)\left\{(2^n-1)(1-\eta) - (1-\eta)^2 \sum_{j=0}^{n-1}(2^n - 2^j)\frac{[-\log(1-\eta)]^j}{j!}\right\}. \tag{3.11}$$

Let $1 \le m < n$ and $\rho_{m,n}$ be the unique solution of (3.5). If $a \le \mu \le a\rho_{m,n} + b(1-\rho_{m,n})$, we have

$$E(I_{m,n}) \le B_2(m,n) = (b-a)\left\{(2^n-2^m)(1-\eta) - (1-\eta)^2 \sum_{j=0}^{n-1}(2^n - 2^j)\frac{[-\log(1-\eta)]^j}{j!} + (1-\eta)^2 \sum_{j=0}^{m-1}(2^m - 2^j)\frac{[-\log(1-\eta)]^j}{j!}\right\}. \tag{3.12}$$

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If $a\rho_{m,n} + b(1-\rho_{m,n}) \le \mu \le b$, *then*

$$E(I_{m,n}) \le B_2(m,n)$$
$$= (b-a)\eta \left\{ 2^n(1-\rho_{m,n}) \sum_{j=0}^{n-1} \frac{[-\log(1-\rho_{m,n})]^j}{j!} \right.$$
$$(3.13) \qquad \left. - 2^m(1-\rho_{m,n}) \sum_{j=0}^{m-1} \frac{[-\log(1-\rho_{m,n})]^j}{j!} - (2^n - 2^m) \right\}.$$

The bounds (3.11) *and* (3.12) *are attained in limit by the probability distributions*

$$(3.14) \qquad P(X_1 = a) = \eta = 1 - P(X_1 = b).$$

The bound (3.13) *is attained in limit by the probability distribution*

$$(3.15) \qquad P\left(X_1 = \frac{\mu - b(1-\rho_{m,n})}{\rho_{m,n}}\right) = \rho_{m,n} = 1 - P(X_1 = b).$$

Proof. It follows from (2.12) and (2.7) that

$$E(I_{m,n}) = \int_0^1 [F^{-1}(x) - \mu][\delta_{m,n}(x) - \bar{\delta}_{m,n}(\eta)]dx$$
$$(3.16) \qquad \le \int_0^1 [F^{-1}(x) - \mu][\bar{\delta}_{m,n}(x) - \bar{\delta}_{m,n}(\eta)]dx$$
$$= \int_0^\eta [F^{-1}(x) - \mu][\bar{\delta}_{m,n}(x) - \bar{\delta}_{m,n}(\eta)]dx$$
$$+ \int_\eta^1 [F^{-1}(x) - \mu]\left[\bar{\delta}_{m,n}(x) - \bar{\delta}_{m,n}(\eta)\right]dx.$$

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Using the fact that $\overline{\delta}_{m,n}(x)$ is a nondecreasing function and $a < F^{-1}(x) < b$, we obtain

$$(3.17)\quad \begin{aligned} E(I_{m,n}) &\leq (a-\mu)\int_0^\eta [\overline{\delta}_{m,n}(x) - \overline{\delta}_{m,n}(\eta)]dx \\ &\quad + (b-\mu)\int_\eta^1 [\overline{\delta}_{m,n}(x) - \overline{\delta}_{m,n}(\eta)]dx \\ &= (a-b)\overline{\Delta}_{m,n}(\eta). \end{aligned}$$

For $m = 0, U_0^c = X_1$ and

$$\delta_{0,n}(x) = (2^n - 1) - 2^n\gamma(n, -\log(1-x)).$$

$\Delta_{0,n}(x)$ is non-increasing convex and non-decreasing convex on $(0,\nu)$ and $(\nu, 1)$, respectively where ν is the unique solution of

$$2^n\gamma(n, -\log(1-x)) = 2^n - 1.$$

Therefore,

$$(3.18)\quad E(I_{m,n}) \leq (b-a)\left(\eta - F_{U_n^c}(\eta)\right).$$

By (3.4) and (3.18), we immediately obtain (3.11).

Since $\overline{\delta}_{0,n}(x) = \delta_{0,n}(x)$, the inequality in (3.16) becomes equality for any distribution $F(x)$. The equality in (3.17) holds if

$$F^{-1}(x) - \mu = \begin{cases} a-\mu, & \text{if} \quad 0 \leq x < \eta, \\ b-\mu, & \text{if} \quad \eta \leq x < 1, \end{cases}$$

which determines the two-point distribution supported on a and b with probabilities η and $1-\eta$.

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For $1 \leq m < n$, the greatest convex minorant of the antiderivative $\Delta_{m,n}$ is defined in (3.9).

If $a \leq \mu \leq a\rho_{m,n} + b(1-\rho_{m,n})$, then $\overline{\Delta}_{m,n}(\eta) = \Delta_{m,n}(\eta)$. Consequently,

$$E(I_{m,n}) \leq (a-b)\Delta_{m,n}(\eta),$$

and by (3.4), we deduce (3.12). The inequality in (3.17) becomes equality if

$$F^{-1}(x) - \mu = \begin{cases} a-\mu, & \text{if} \quad 0 \leq x < \eta, \\ b-\mu, & \text{if} \quad \eta \leq x < 1, \end{cases}$$

which leads to the two-point distribution supported on a and b with probabilities η and $1-\eta$.

If $a\rho_{m,n} + b(1-\rho_{m,n}) \leq \mu \leq b$, then by (3.9), $\overline{\Delta}_{m,n}(\eta) = \delta_{m,n}(\rho_{m,n})\eta$. Hence,

$$E(I_{m,n}) \leq (a-b)\eta\delta_{m,n}(\rho_{m,n}).$$

From (3.7), the equality in (3.16) is attained if $F^{-1}(x) = c$ on $(0, \rho_{m,n})$ and the equality in (3.17) is attained if $F^{-1}(x) = b$ on $(\rho_{m,n}, 1)$. From the moment condition $E(X_1) = \mu$, we have $c = [\mu - b(1-\rho_{m,n})]/\rho_{m,n}$. This leads to the probability distribution (3.15). ∎

Remark 1. Maximization of the bounds in Theorems 3.1 and 3.3 with respect to $0 < \eta < 1$ leads to parameter free bounds. In the case of record range, a general bound independent of η is derived by maximizing the right hand side of (3.2),

$$q_1(\eta) = (a-b)\left(F_{U_n^c}(\eta) - F_{L_n^c}(\eta)\right).$$

It follows from the fact that $q_1(\eta)$ is a concave and symmetric about $1/2$ function with $q_1(0) = q_1(1) = 0$, the maximal bound is attained at $\eta = 1/2$. Substituting

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$\eta = 1/2$ in (3.1), we obtain

$$B_1(n) = (b-a)\left\{(2^n - 1) - \frac{1}{2}\sum_{j=0}^{n-1}(2^n - 2^j)\frac{(\log 2)^j}{j!}\right\}.$$

This bound is attained in limit by the two-point distribution

$$P(X = a) = P(X = b) = \frac{1}{2}.$$

For the current upper record increment, the value η maximizing the bound in Theorem 3.3 can be obtained by maximizing the right hand side of (3.17),

$$q_2(\eta) = (a-b)\overline{\Delta}_{m,n}(\eta).$$

It is easily checked that the bound is maximized by $0 < \eta < 1$ satisfying $\overline{\delta}_{m,n}(\eta) = 0$, or equivalently (3.8).

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4. Computational Results

We evaluate the values of the upper bounds for the expectations of the record range and current record increment based on three distributions $U(-2,3)$, standard exponential $\mathrm{Exp}(1)$ on $(0,\sqrt{3})$, and $N(1/2,1)$ on $(-1,3)$. The bounds obtained by Moriguti's inequalities are expressed in terms of the parameter $\eta = (b-\mu)/(b-a)$. The bound for the mean of the record range can be computed by evaluating (3.1). The ratio

$$D(n) = \frac{(b-a) - B_1(n)}{(b-a) - ER_n},$$

represents the relative distance of $B_1(n)$ from the support interval length with respect to the distance of ER_n from the support interval length. In Table 1, values of $D(n)$ are presented for $n = 1,2,\dots,8$. It is shown in Table 1 that the bounds $B_1(n)$, $n \geq 1$ tend to the length of support intervals as n gets large. These bounds tend to their respective limits faster than the exact expectations of the record range.

The numbers $\rho_{m,n}$ are determined numerically by solving (3.5). In fact, for $a\rho_{m,n} + b(1-\rho_{m,n}) \leq \mu < b$, the bounds for the current record increments can be determined by computing the values $\rho_{m,n}$'s and then evaluating the formula (3.13). If $a \leq \mu \leq \rho_{m,n} + b(1-\rho_{m,n})$, $\overline{\Delta}_{m,n}(\eta) = \Delta_{m,n}(\eta)$ and then the bounds can be obtained by (3.12). The evaluations of the bounds $B_2(m,n)$, $0 \leq m < n$ given in (3.11), (3.12) and (3.13) as well as the exact expectations of the record increments are used to compute the following ratio

$$H(m,n) = \frac{(b-a) - B_2(m,n)}{(b-a) - E(I_{m,n})}, \qquad 0 \leq m < n.$$

These ratios are presented in Table 2 for various choices of m and n. Clearly, for $m = 0$ and n getting large, the ratios tend to 1 and consequently, the bounds tend to the exact expectations. For fixed $m \geq 1$, the ratios decrease slowly as n increases.

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Table 1: Values of $D(n)$ for $n = 1, 2, ..., 8$.

	$D(n)$		
n	$U(-2,3)$	Exp(1)	$N(1/2,1)$
1	0.7500	0.7720	0.6846
2	0.4346	0.5003	0.3702
3	0.2021	0.2818	0.1677
4	0.0780	0.1395	0.0657
5	0.0256	0.0610	0.0227
6	0.0073	0.0238	0.0070
7	0.0018	0.0083	0.0020
8	0.0004	0.0026	0.0005

The standard exponential distribution is truncated on $(0, \sqrt{3})$ and the normal distribution $N(1/2, 1)$ is truncated on $(-1, 3)$.

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Table 2: Values of $H(m,n)$ for Various Choices of m and n.

		$H(m,n)$		
m	n	$U(-2,3)$	Exp(1)	$N(1/2,1)$
0	1	0.9000	0.9064	0.8641
	3	0.8176	0.7203	0.6564
	8	0.9625	0.8871	0.8052
	10	0.9830	0.9437	0.8777
1	2	0.9492	0.9172	0.8947
	5	0.8763	0.8347	0.7864
	8	0.8233	0.7946	0.7720
	10	0.7987	0.7703	0.7650
2	3	0.9614	0.9519	0.9363
	5	0.8805	0.8684	0.8469
	8	0.7785	0.7556	0.7680
	12	0.7003	0.6538	0.7132
3	4	0.9571	0.9559	0.9490
	6	0.8632	0.8564	0.8597
	10	0.7169	0.6772	0.7311
	12	0.6723	0.6166	0.6884
4	5	0.9523	0.9520	0.9534
	8	0.8062	0.7863	0.8204
	12	0.6723	0.6125	0.6867
	15	0.6176	0.5373	0.6241
5	6	0.9491	0.9469	0.9544
	8	0.8464	0.8293	0.8610
	12	0.6898	0.6299	0.7009
	15	0.6209	0.5374	0.6202
	20	0.5651	0.4611	0.5490

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