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REVERSE REARRANGEMENT INEQUALITIES VIA MATRIX TECHNICS

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ABSTRACT. We give a reverse inequality to the most standard rearrangement inequality for sequences and we emphasize the usefulness of matrix methods to study classical inequalities.

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1. REVERSE REARRANGEMENT INEQUALITIES

We have the following reverse inequality to the most basic rearrangement inequality. Down arrows mean nonincreasing rearrangements.

Theorem 1.1. *Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be n-tuples of positive numbers with*

$$p \geq \frac{a_i}{b_i} \geq q, \quad i = 1, \dots, n,$$

for some $p, q > 0$. Then,

$$\sum_{i=1}^n a_i^{\downarrow} b_i^{\downarrow} \leq \frac{p+q}{2\sqrt{pq}} \sum_{i=1}^n a_i b_i.$$

The proof uses matrix arguments. Indeed, Theorem 1.1 is a byproduct of some matrix inequalities which are given in Section 2.

For the convenience of readers we recall some facts about the trace norm. Capital letters $A,\ B, \dots, Z,$ denote n-by-n matrices or operators on an n-dimensional Hilbert space $\mathcal{H}$. Let $X = U|X|$ be the polar decomposition of X, so U is unitary and $|X| = (X^*X)^{1/2}$. The trace norm of X is $\|X\|_1 = \operatorname{Tr}|X|$. One may easily check that the trace norm is a norm: For any X, Y, consider the polar decomposition $X + Y = U|X + Y|$. Then,

$$\|X+Y\|_1 = \operatorname{Tr}|X+Y| = \operatorname{Tr} U^*(X+Y) = \operatorname{Tr} U^*X + \operatorname{Tr} U^*Y. \tag{1.1}$$

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On the other hand, for all A,

$$|\operatorname{Tr} A| \le \operatorname{Tr} |A|, \tag{1.2}$$

as it is shown by computing $|\operatorname{Tr} A|$ in a basis of eigenvectors of $|A|$. From (1.1) and (1.2) we infer that $\|\cdot\|_1$ is a norm.

We need a simple fact: Given two diagonal positive matrices $X = \operatorname{diag}(x_i)$, $Y = \operatorname{diag}(y_i)$ and a permutation matrix V acting on the canonical basis $\{e_i\}$ by $Ve_i = e_{\sigma(i)}$, we have

$$\|XVY\|_1 = \sum x_i y_{\sigma(i)}. \tag{1.3}$$

Indeed, since $|XVY|^2 e_i = YV^*X^2VYe_i = (x^2_{\sigma(i)} y_i^2)e_i$, we obtain $|XVY|e_i = (x_{\sigma(i)}y_i)e_i$ so that (1.3) holds.

Proof of Theorem 1.1. Introduce the diagonal matrices $A = \operatorname{diag}(a_i)$ and $B = \operatorname{diag}(b_i)$. By the above discussion, we have

$$\sum_{i=1}^n a_i b_i = \|AB\|_1 \qquad \text{and} \qquad \sum_{i=1}^n a_i^{\downarrow} b_i^{\downarrow} = \|AVB\|_1$$

for some permutation matrix V. Hence we have to show that

$$\|AVB\|_1 \le \frac{p+q}{2\sqrt{pq}}\, \|AB\|_1.$$

To this end consider the spectral representation $V = \sum_i v_i h_i \otimes h_i$ where v_i are the eigenvalues and h_i the corresponding unit eigenvectors. We have

$$\begin{aligned}
\|AVB\|_1 &\le \sum_{i=1}^n \|A \cdot v_i h_i \otimes h_i \cdot B\|_1 \\
&= \sum_{i=1}^n \|Ah_i\|\,\|Bh_i\| \\
&\le \frac{p+q}{2\sqrt{pq}} \sum_{i=1}^n \langle Ah_i, Bh_i\rangle \\
&= \frac{p+q}{2\sqrt{pq}} \sum_{i=1}^n \langle h_i, ABh_i\rangle \\
&= \frac{p+q}{2\sqrt{pq}}\, \|AB\|_1,
\end{aligned}$$

where we have used the triangle inequality for the trace norm and Lemma 1.4 below. □

The following example shows that equality can occur.

Example 1.1. Consider couples $a_1 = 2$, $a_2 = 1$ and $b_1 = 1/2$, $b_2 = 1$; then with $p = 4$, $q = 1$,

$$\frac{p+q}{2\sqrt{pq}} = \frac{5}{4} = \frac{a_1b_2 + a_2b_1}{a_1b_1 + a_2b_2}.$$

From the above, one easily derives:

Corollary 1.2. *Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be n-tuples of positive numbers with*

$$b_i \le a_i \le pb_i, \quad i = 1, \dots, n,$$

for some $p > 0$*. Then,*

$$\sum_{i=1}^{n} a_i^{\downarrow} b_i^{\downarrow} \leq \frac{p+1}{2\sqrt{p}} \sum_{i=1}^{n} a_i b_i.$$

Moreover, for even n *and each* p*, there are* n*-tuples for which equality occurs.*

To obtain equality, consider an n-tuple $\{a_i\}$ for which the first half terms equal $\sqrt{p}$ and the second half ones equal 1, and an n-tuple $\{b_i\}$ for which the first half terms equal 1 and the second half ones equal $1/\sqrt{p}$.

We turn to the lemmas necessary to complete the proof of Theorem 1.1. Given a subspace $\mathcal{E} \subset \mathcal{H}$, denote by $Z_{\mathcal{E}}$ the compression of Z onto E, that is the restriction of EZ to $\mathcal{E}$ where E is the orthoprojection onto $\mathcal{E}$.

Lemma 1.3. *Let* $Z > 0$ *with extremal eigenvalues* a *and* b*. Then, for every norm one vector* h*,*

$$\|Zh\| \leq \frac{a+b}{2\sqrt{ab}} \langle h, Zh \rangle.$$

Proof. Let $\mathcal{E}$ be any subspace of $\mathcal{H}$ and let a' and b' be the extremal eigenvalues of $Z_{\mathcal{E}}$. Then $a \geq a' \geq b' \geq b$ and, setting $t = \sqrt{a/b}$, $t' = \sqrt{a'/b'}$, we have $t \geq t' \geq 1$. Since $t \longrightarrow t + 1/t$ increases on $[1, \infty)$ and

$$\frac{a+b}{2\sqrt{ab}} = \frac{1}{2}\left(t + \frac{1}{t}\right), \qquad \frac{a'+b'}{2\sqrt{a'b'}} = \frac{1}{2}\left(t' + \frac{1}{t'}\right),$$

we infer

$$\frac{a+b}{2\sqrt{ab}} \geq \frac{a'+b'}{2\sqrt{a'b'}}.$$

Therefore, it suffices to prove the lemma for $Z_{\mathcal{E}}$ with $\mathcal{E} = \operatorname{span}\{h, Zh\}$. Hence, we may assume $\dim \mathcal{H} = 2$, $Z = ae_1 \otimes e_1 + be_2 \otimes e_2$ and $h = xe_1 + (\sqrt{1-x^2})e_2$. Setting $x^2 = y$ we have

$$\frac{||Zh||}{\langle h, Zh \rangle} = \frac{\sqrt{a^2y + b^2(1-y)}}{ay + b(1-y)}.$$

The right hand side attains its maximum on $[0, 1]$ at $y = b/(a+b)$, and then

$$\frac{||Zh||}{\langle h, Zh \rangle} = \frac{a+b}{2\sqrt{ab}},$$

proving the lemma. □

Lemma 1.4. *Let* $A, B > 0$ *with* $AB = BA$ *and* $pI \geq AB^{-1} \geq qI$ *for some* $p, q > 0$*. Then, for every vector* h*,*

$$\|Ah\| \, \|Bh\| \leq \frac{p+q}{2\sqrt{pq}} \langle Ah, Bh \rangle.$$

Proof. Write $h = B^{-1}f$ and apply Lemma 1.3. □

Remark 1.5. Lemma 1.3 is nothing but a rephrasing of a Kantorovich inequality and Lemma 1.4 a rephrasing of Cassel's Inequality:
Cassel's inequality. For nonnegative n-tuples $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ with

$$p \geq \frac{a_i}{b_i} \geq q, \quad i = 1, \dots, n,$$

for some p, $q > 0$; it holds that

$$\left(\sum_{i=1}^{n} w_i a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} w_i b_i^2\right)^{\frac{1}{2}} \leq \frac{p+q}{2\sqrt{pq}} \sum_{i=1}^{n} w_i a_i b_i.$$

Of course it is a reverse inequality to the Cauchy-Schwarz inequality. To obtain it from Lemma 1.4, one simply takes $A = \mathrm{diag}(a_1, \ldots, a_n)$, $B = \mathrm{diag}(b_1, \ldots, b_n)$ and $h = (\sqrt{w_1}, \ldots, \sqrt{w_n})$. If one lets $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ then Cassel's inequality can be written as

$$\|a\|\,\|b\| \le \frac{p+q}{2\sqrt{pq}}\,\langle a, b\rangle \tag{1.4}$$

for a suitable inner product $\langle \cdot, \cdot\rangle$. It is then natural to search for conditions on a, b ensuring that the above inequality remains valid with Ua, Ub for all orthogonal matrices U. This motivates a remarkable extension of Cassel's inequality:

t*Dragomir's inequality*. For real vectors a, b such that $\langle a - qb, pb - a\rangle \ge 0$ for some scalars p, q with $pq > 0$, inequality (1.4) holds. For this inequality and its complex version see [4], [5], [6].

Taking squares in Cassel's inequality and using the convexity of t^2 we obtain:

$$\sum_{i=1}^n w_i a_i \sum_{i=1}^n w_i b_i \;\le\; \frac{(\sqrt{p}+\sqrt{q})^2}{4\sqrt{pq}}\; \sum_{i=1}^n w_i a_i b_i$$

for all nonnegative n-tuples $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ with $\sum_{i=1}^n w_i = 1$ and $p \ge a_i/b_i \ge q$ for some $p,\, q > 0$. Though weaker than Cassel's inequality, this is also a sharp inequality: Taking $b_i = 1/a_i$ we get the (sharp) Kantorovich inequality: If $p \ge a_i \ge q > 0$ and $\sum_{i=1}^n w_i = 1$, then

$$\sum_{i=1}^n w_i a_i \sum_{i=1}^n w_i a_i^{-1} \;\le\; \frac{(p+q)^2}{4pq}\,.$$

Let (Ω, P) be a probability space. The above discussions shows a sharp result:

Proposition 1.6. *Let $f(\omega)$ and $g(\omega)$ be measurable functions on Ω such that $p \ge f(\omega)/g(\omega) \ge q$ for some $p,\, q > 0$. Then,*

$$\int_\Omega f(\omega)\,\mathrm{d}P \int_\Omega g(\omega)\,\mathrm{d}P \le \frac{(\sqrt{p}+\sqrt{q})^2}{4\sqrt{pq}} \int_\Omega fg(\omega)\,\mathrm{d}P.$$

2. Related Matrix Inequalities and Comments

We dicovered the statements of Theorem 1.1 and its corollaries while investigating some matrix inequalities. Among those are inequalities for symmetric norms. Such a norm $\|\cdot\|$ is characterized by the property that $\|A\| = \|UAV\|$ for all A and all unitaries U, V. The most basic inequality for symmetric norms is

$$\|AB\| \le \|BA\|,$$

whenever the product AB is normal. In [1] (see also [2]) we established:

Theorem 2.1. *Let A, B such that $AB \ge 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then, for every symmetric norm, the following sharp inequality holds*

$$\|ZAB\| \le \frac{a+b}{2\sqrt{ab}}\,\|BZA\|.$$

By sharpness, we mean that we can find A and B such that equality occurs. Note that letting $A = B$ be a rank one projection $h \otimes h$ we recapture Lemma 1.3 which is the starting point of Theorem 1.1. From this theorem we derived several known Kantorovich type inequalities and also a sharp operator inequality:

Corollary 2.2. *Let $0 \le A \le I$ and let $Z > 0$ with extremal eigenvalues a and b. Then,*

$$AZA \le \frac{(a+b)^2}{4ab}\, Z.$$

Next, let us note that an immediate consequence of Theorem 1.1 is:

Corollary 2.3. *Let $Z \ge 0$ and let $A,\, B > 0$ with $AB = BA$ and $pI \ge AB^{-1} \ge qI$ for some $p,\, q > 0$. Then, for all symmetric norms,*

$$\|AZB\| \le \frac{p+q}{2\sqrt{pq}}\, \|ZAB\|.$$

Proof. From Theorem 2.1 we get

$$\|AZB\| = \|AB^{-1}(BZ \cdot B)\| \le \frac{p+q}{2\sqrt{pq}}\, \|ABZ\| = \frac{p+q}{2\sqrt{pq}}\, \|ZAB\|.$$

by the simple fact that $\|ST\| = \|TS\|$ for all Hermitians S, T, since $\|X\| = \|X^*\|$ for all X. $\square$

The previous theorem cannot be extended to normal operators Z, except in the case of the trace norm:

Theorem 2.4. *Let $A,\, B > 0$ with $AB = BA$ and $pI \ge AB^{-1} \ge qI$ for some $p,\, q > 0$ and let Z be normal. Then,*

$$\|AZB\|_1 \le \frac{p+q}{2\sqrt{pq}}\, \|ZAB\|_1.$$

The proof is quite similar to that of Theorem 1.1. Clearly Theorem 1.1 is a corollary of Theorem 2.4.

Some comments. One aim of the paper is to place stress on the power of matrix methods in dealing with classical inequalities. This is apparent in the quite natural statement and proof of Cassel's inequality via Lemma 1.4. We also note that from the matrix inequality of Theorem 2.4 we infer our reverse rearrangement inequality stated in Theorem 1.1. Having now at our disposal the good statement, it remains to find a direct proof without matrix arguments (in particular without using complex numbers via the spectral decomposition). A first immediate simplification consists in noting that we can assume that

$$a_1, \ldots, a_n = a_1^{\downarrow}, \ldots, a_n^{\downarrow} \qquad \text{and} \qquad b_1, \ldots, b_n = b_{\sigma(1)}^{\downarrow}, \ldots, b_{\sigma(n)}^{\downarrow}$$

for a permutation σ. By decomposing σ in cycles we may assume that σ is a cycle. Equivalently we may assume that

$$a_1, \ldots, a_n = a_{\sigma(1)}^{\downarrow}, \ldots, a_{\sigma(n)}^{\downarrow} \qquad \text{and} \qquad b_1, \ldots, b_n = b_{\sigma(2)}^{\downarrow}, \ldots, b_{\sigma(n)}^{\downarrow}, b_{\sigma(1)}^{\downarrow}$$

for a permutation σ. However, does it really simplify the problem ?

It is tempting to try to reduce the problem to the case $n = 2$. We have no idea of how to proceed. The case $n = 2$ can be easily solved by elementary methods as it is shown in the next proposition. The proof shows that the inequality of Theorem 1.1 is sharp (and equality can occur when n is even).

Proposition 2.5. *Let $a^* \ge a_* > 0$ and $b^* \ge b_* > 0$ with*

$$p \ge \frac{a^*}{b_*} \qquad \textit{and} \qquad \frac{a_*}{b^*} \ge q$$

for some $p,\, q > 0$. Then,

$$\frac{a^*b^* + a_*b_*}{a^*b_* + a_*b^*} \le \frac{p+q}{2\sqrt{pq}}.$$

Proof. First, fix a^*, a_* and renamed b^*, b_* by x, y respectively. We want to maximize

$$f(x,y) = \frac{a^*x + a_*y}{a^*y + a_*x}$$

on the domain

$$\Delta = \left\{ (x,y) \,:\, x \geq y,\ q \leq \frac{a_*}{x} \leq p,\ q \leq \frac{a^*}{y} \leq p \right\}$$

that is,

$$\Delta = \left\{ (x,y) \,:\, x \geq y,\ \frac{a_*}{p} \leq x \leq \frac{a_*}{q},\ \frac{a^*}{p} \leq y \leq \frac{a^*}{q} \right\}.$$

Thus Δ is a triangle (more precisely a half-square) with vertices

$$(a^*/p, a^*/p) \qquad (a_*/q, a_*/q) \qquad (a_*/q, a^*/p).$$

On Δ we have $\partial f/\partial x > 0$ and $\partial f/\partial y < 0$. This shows that f takes its maximun in Δ at $(a_*/q, a^*/p)$. The value is then

$$\frac{\frac{a^*a_*}{q} + \frac{a^*a_*}{p}}{\frac{a^*a^*}{p} + \frac{a_*a_*}{q}}.$$

Next, observe that in our inequality we can take $a^* = 1$. Hence, letting $a_* = t$, we have to check that

$$\max_{t\in[0,1]} \frac{(\frac{1}{p} + \frac{1}{q})t}{\frac{1}{p} + \frac{t^2}{q}} = \frac{p+q}{2\sqrt{pq}}.$$

Considering the derivative, we see that the maximum is attained at $t = \sqrt{q/p}$ and we obtain the expected value. □

We close with two open problems:

Problem 2.1. Find a direct proof of Theorem 1.1.

Problem 2.2. Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be n-tuples of positive numbers. Find a suitable bound for the difference

$$\sum_{i=1}^n a_i^{\downarrow} b_i^{\downarrow} - \sum_{i=1}^n a_i b_i.$$

In the research/survey paper [3] we consider matrix proofs and several extensions of some classical inequalities of Chebyshev, Grüss and Kantorovich type.

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