Journal of Inequalities in Pure and Applied Mathematics

ON THE APPROXIMATION OF LOCALLY BOUNDED FUNCTIONS BY OPERATORS OF BLEIMANN, BUTZER AND HAHN

JESÚS DE LA CAL AND VIJAY GUPTA

Departamento de Matemática Aplicada y Estadística e Investigación Operativa
Facultad de Ciencias
Universidad del País Vasco
Apartado 644, 48080 Bilbao, Spain
EMail: mepcaagj@lg.ehu.es

School of Applied Sciences
Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110045, India
EMail: vijay@nsit.ac.in

ISSN (electronic): 1443-5756
228-04

volume 6, issue 1, article 4, 2005.

Received 25 November, 2004; accepted 29 December, 2004.

Communicated by: A. Lupaş

Abstract

We estimate the rate of the pointwise approximation by operators of Bleimann, Butzer and Hahn of locally bounded functions, and of functions having a locally bounded derivative.

2000 Mathematics Subject Classification: 41A20, 41A25, 41A36.
Key words: Operators of Bleimann, Butzer and Hahn, Locally bounded function, Function of bounded variation, Total variation, Rate of convergence, Binomial distribution.

J. de la Cal was supported by the Spanish MCYT, Proyecto BFM2002-04163-C02-02, and by FEDER.

Contents

1. Introduction and Main Results

Bleimann, Butzer and Hahn [1] introduced the Bernstein type operator L_n over the interval $[0, \infty)$ given by

$$L_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) b_{n,k}(x), \qquad x \geq 0, \ n = 1, 2, \dots,$$

where f is a real function on $[0, \infty)$, and

$$b_{n,k}(x) := \binom{n}{k} p_x^k q_x^{n-k}, \qquad p_x := \frac{x}{1+x}, \quad q_x := 1 - p_x = \frac{1}{1+x}. \tag{1.1}$$

The approximation of uniformly continuous functions by these operators has been considered in [1] – [4]. For other properties of L_n (preservation of global smoothness, preservation of ϕ-variation, behavior of the iterates, etc.) we refer, for instance, to [4] – [10]. In some of the mentioned works, the results are achieved by using probabilistic methods. This comes from the fact that L_n is an operator of probabilistic type. We can actually write

$$L_n(f, x) = Ef(Z_{n,x}),$$

where E denotes mathematical expectation, and $Z_{n,x}$ is the random variable given by

$$Z_{n,x} := \frac{S_{n,x}}{n - S_{n,x} + 1}, \qquad S_{n,x} := \xi_{1,x} + \cdots + \xi_{n,x}, \tag{1.2}$$

where $\xi_{1,x},\ \xi_{2,x},\dots$ are independent random variables having the same Bernoulli distribution with parameter p_x, i.e.,

$$P(\xi_{k,x}=1)=p_x=1-P(\xi_{k,x}=0)$$

(so that $S_{n,x}$ has the binomial distribution with parameters $n,\ p_x$). This probabilistic representation also plays a significant role in the present paper (for a more refined representation useful for other purposes, see [5, 6]).

Here, we discuss the approximation of real functions f on the semi axis which are locally bounded, i.e., bounded on each finite subinterval of $[0,\infty)$. In such a case, we set, for $x>0$ and $h\geq 0$,

$$\omega_x^+(f;h):=\sup_{x\leq t\leq x+h}|f(t)-f(x)|,$$

$$\omega_x^-(f;h):=\sup_{(x-h)^+\leq t\leq x}|f(t)-f(x)|,$$

$$\omega_x(f;h):=\omega_x^+(f;h)+\omega_x^-(f;h),$$

where $(x-h)^+:=\max(x-h,0)$, and we observe that these functions are (nonnegative and) nondecreasing on $[0,\infty)$. In particular, every continuous function is locally bounded. Also, if f is locally of bounded variation, i.e., such that

$$\bigvee_a^b(f)<\infty,\qquad 0\leq a<b<\infty,$$

where $\bigvee_a^b(f)$ stands for the total variation of f on the interval $[a, b]$, then f is locally bounded, and we obviously have

$$\omega_x(f; h) \leq \bigvee_{x-h}^{x+h}(f), \qquad 0 \leq h \leq x.$$

This kind of problem has been already considered for other Bernstein-type operators (see, for instance, [11] – [14] and the references therein). Our main results are stated as follows.

Theorem 1.1. *Let g be a real locally bounded function on $[0, \infty)$ such that $g(t) = O(t^r) \quad (t \to \infty)$, for some $r = 1, 2, \ldots$. If g is continuous at $x > 0$, then, for n large enough, we have*

$$|L_n(g, x) - g(x)| \leq \frac{7(1+x)^2}{(n+2)x} \sum_{k=1}^{n} \omega_x\left(g; \frac{x}{\sqrt{k}}\right) + O_{r,x}\left(\frac{1}{n}\right). \tag{1.3}$$

In the following statements (and throughout the paper), we use the notations:

$$f^*(x) := f(x+) - f(x-)$$

$$\tilde{f}(x) := \frac{f(x+) + f(x-)}{2},$$

$$f_x := (f - f(x-))1_{[0,x)} + (f - f(x+))1_{(x,\infty)}$$

(1_A being the indicator function of the set A), provided that the lateral limits $f(x+)$ and $f(x-)$ exist (such a condition is fulfilled when f is locally of bounded variation). We also use the symbol $\lfloor a \rfloor$ to indicate the integral part of the real number a.

Theorem 1.2. *Let f be a real locally bounded function on $[0,\infty)$ such that $f(t) = O(t^r) \quad (t \to \infty)$, for some $r = 1, 2, \ldots$. If $x > 0$, and $f(x+)$ and $f(x-)$ exist, then we have for n large enough*

$$\left| L_n(f,x) - \tilde{f}(x) \right| \\ \leq \Delta_{n,x}(f_x) + \frac{1.6 + x + 2.6\,x^2}{\sqrt{nx}(1+x)} \cdot \frac{|f^*(x)|}{2} + \frac{\epsilon_{n,x}(1+x)}{\sqrt{2enx}} |f(x) - f(x-)|,$$

where $\Delta_{n,x}(f_x)$ is the right-hand side of (1.3) with g replaced by f_x, and

$$\epsilon_{n,x} := \begin{cases} 1 & \text{if } (n+1)p_x \in \{1, 2, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.3. *Let g be a real function on $[0,\infty)$ such that $g(t) = O(t^r) \quad (t \to \infty)$, for some $r = 1, 2, \ldots$, and having the form*

$$g(t) = c + \int_0^t f(u)\,du, \qquad t \geq 0,$$

where c is a constant and f is measurable and locally bounded on $[0,\infty)$. If $x > 0$, and $f(x+)$ and $f(x-)$ exist, then we have for n large enough

$$\left| L_n(g,x) - g(x) - \frac{\sqrt{x}(1+x)}{\sqrt{2\pi n}} f^*(x) \right| \\ \leq \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \omega_x\left(f_x; \frac{x}{k}\right) + |f^*(x)|\, o_x\left(n^{-1/2}\right) + O_{r,x}(n^{-1}).$$

The proofs of the preceding theorems are given in Sections 3 – 5. In Section 2, we collect the necessary auxiliary results. Some remarks on moments close the paper.

2. Auxiliary Results

In the following lemma, Φ denotes the standard normal distribution function, and $F^*_{n,x}$ stands for the distribution function of $S^*_{n,x} := (S_{n,x} - np_x) \big/ \sqrt{np_xq_x}$,, where $S_{n,x}$ is the same as in (1.2). Such a lemma is nothing but the application of the well-known Berry-Esseen theorem (cf. [15]) to the situation at hand.

Lemma 2.1. *We have, for $x > 0$ and $n \geq 1$,*

$$\sup_{-\infty<t<\infty} |F^*_{n,x}(t) - \Phi(t)| \leq \frac{0.8(p_x^3q_x + p_xq_x^3)}{\sqrt{n}(p_xq_x)^{3/2}} = \frac{0.8(1+x^2)}{\sqrt{nx}(1+x)}.$$

Lemma 2.2. *Let $x > 0$ and $n \geq 1$. Then, we have:*

(a)

$$L_n((\cdot - x)^2, x) = E(Z_{n,x} - x)^2 \leq \frac{3x(1+x)^2}{n+2}.$$

(b)

$$P(Z_{n,x} \leq x - h) + P(Z_{n,x} \geq x + h) \leq \frac{3x(1+x)^2}{(n+2)h^2}, \qquad h > 0.$$

(c)

$$|P(Z_{n,x} > x) - P(Z_{n,x} \leq x)| \leq \sqrt{\frac{x}{n}} + \frac{1.6(1+x^2)}{\sqrt{nx}(1+x)}.$$

(d)

$$L_n((\cdot - x), x) = E(Z_{n,x} - x) = -xp_x^n = o_x(n^{-1}), \qquad (n \to \infty).$$

(e)

$$L_n(|\cdot - x|, x) = E|Z_{n,x} - x| = \frac{\sqrt{2x}(1+x)}{\sqrt{\pi n}} + o_x(n^{-1/2}), \qquad (n \to \infty).$$

Proof. Part (a) was shown in [10]. Part (b) follows from (a) and the fact that, by Markov's inequality,

$$P(Z_{n,x} \leq x - h) + P(Z_{n,x} \geq x + h) = P(|Z_{n,x} - x| \geq h) \leq \frac{E(Z_{n,x} - x)^2}{h^2}.$$

To show (c), observe that

$$\begin{aligned}
&|P(Z_{n,x} > x) - P(Z_{n,x} \leq x)| \\
&\qquad = |1 - 2P(Z_{n,x} \leq x)| \\
&\qquad = |1 - 2P(S_{n,x} \leq (n+1)p_x)| \\
&\qquad = \left|1 - 2F^*_{n,x}\left(\sqrt{\frac{x}{n}}\right)\right| \\
&\qquad \leq 2\left|\Phi\left(\sqrt{\frac{x}{n}}\right) - F^*_{n,x}\left(\sqrt{\frac{x}{n}}\right)\right| + \left|1 - 2\Phi\left(\sqrt{\frac{x}{n}}\right)\right|.
\end{aligned}$$

Thus, the conclusion in part (c) follows from Lemma 2.1 and the fact that (cf. [16])

$$0 < 2\Phi(t) - 1 \leq \left(1 - e^{-t^2}\right)^{1/2} \leq t, \qquad (t > 0).$$

Part (d) is immediate. Finally, to show (e), let $m := \lfloor (n+1)p_x \rfloor$. We have

$$\begin{aligned}
&L_n(|\cdot - x|, x) - L_n((\cdot - x), x)\\
&\qquad = 2\sum_{k=0}^{m}\left(x - \frac{k}{n-k+1}\right) b_{n,k}(x)\\
&\qquad = 2x\sum_{k=0}^{m} b_{n,k}(x) - 2\sum_{k=1}^{m}\frac{n!}{(k-1)!(n-k+1)!}\, p_x^k q_x^{n-k}\\
&\qquad = 2x\sum_{k=0}^{m} b_{n,k}(x) - 2x\sum_{k=0}^{m-1} b_{n,k}(x)\\
&\qquad = 2x\, b_{n,m}(x)\\
&\qquad = \frac{\sqrt{2x}(1+x)}{\sqrt{\pi n}} + o_x(n^{-1/2}), \qquad (n \to \infty),
\end{aligned}$$

the last equality by [13, Lemma 1], and the conclusion follows from (d). □

Lemma 2.3. *Let $x > 0$ and $r = 1, 2, \ldots$. Then, we have for all integers n such that $(n+1)(p_{2x} - p_{3x/2}) \geq r$,*

$$\begin{aligned}
\sum_{k\in K}\frac{k^r}{(n-k+1)^r}\, b_{n,k}(x) &\leq 12\, r! \sum_{s=1}^{r}\left\{ {r \atop s} \right\}\frac{x^{s-1}(1+x)^{r-s+2}}{n+r-s+2}\cdot\frac{n!}{(n+r-s)!}\\
&= O_{r,x}(n^{-1}), \qquad (n \to \infty),
\end{aligned}$$

where the $\left\{ {r \atop s} \right\}$ are the Stirling numbers of the second kind, and K is the set of all integers k such that $n \geq k > (n-k+1)2x$ (i.e., $n \geq k > (n+1)p_{2x}$).

Proof. Using the well known identity

$$a^r = \sum_{s=1}^{r} \left\{ {r \atop s} \right\} a(a-1)\cdots(a-s+1),$$

we can write

$$\sum_{k\in K} \frac{k^r}{(n-k+1)^r} b_{n,k}(x) = \sum_{s=1}^{r} \left\{ {r \atop s} \right\} A_s, \tag{2.1}$$

where

$$\begin{aligned} A_s := & \sum_{k\in K} \frac{k(k-1)\cdots(k-s+1)}{(n-k+1)^r} b_{n,k}(x) \\ = & \sum_{k\in K} \frac{1}{(n-k+1)^r} \cdot \frac{n!}{(k-s)!(n-k)!} p_x^k q_x^{n-k}. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{(n-k+1)^r} = & \prod_{i=1}^{r} \left[\frac{1}{n-k+i} \frac{n-k+i}{n-k+1} \right] \\ = & \prod_{i=1}^{r} \left[\frac{1}{n-k+i} \left(1 + \frac{i-1}{n-k+1} \right) \right] \\ \leq & \prod_{i=1}^{r} \frac{i}{n-k+i} = \frac{r!(n-k)!}{(n-k+r)!}, \end{aligned}$$

we have

$$\begin{aligned}
A_s &\leq r! \sum_{k\in K} \frac{n!}{(k-s)!(n-k+r)!} p_x^k q_x^{n-k} \\
&= r! \sum_{l\in K_s} \frac{n!}{l!(n+r-s-l)!} p_x^{l+s} q_x^{n-l-s} \\
&= \frac{r!n!p_x^s q_x^{-r}}{(n+r-s)!} \sum_{l\in K_s} \binom{n+r-s}{l} \frac{x^l}{(1+x)^{n+r-s}} \\
&\leq \frac{r!n!p_x^s q_x^{-r}}{(n+r-s)!} \sum_{l\in K'} \binom{n+r-s}{l} \frac{x^l}{(1+x)^{n+r-s}},
\end{aligned}$$

where $K_s := \{k-s : k \in K\}$, and K' stands for the set of all integers l such that $n \geq l > (n-l+1)(3x/2)$ (observe that, by the assumption on n, we have $K_s \subset K'$). The probabilistic interpretation of the last sum together with Lemma 2.2(b) yield

$$\begin{aligned}
A_s &\leq \frac{r!n!x^s(1+x)^{r-s}}{(n+r-s)!} P\left(Z_{n+r-s,x} > \frac{3x}{2}\right) \\
&\leq \frac{12\, r!n!x^{s-1}(1+x)^{r-s+2}}{(n+r-s)!(n+r-s+2)}, \qquad (2.2)
\end{aligned}$$

and the conclusion follows from (2.1) and (2.2). □

Remark 1. *The same procedure as in the preceding proof leads to the following* *upper bound for the integral moments of* L_n *(or* $Z_{n,x}$*):*

$$\begin{aligned} L_n(t^r, x) &= E(Z_{n,x})^r \\ &= \sum_{k=0}^{n} \frac{k^r}{(n-k+1)^r} b_{n,k}(x) \\ &\le r! \sum_{s=1}^{r} \left\{ {r \atop s} \right\} \frac{n! x^s (1+x)^{r-s}}{(n+r-s)!}. \end{aligned}$$

3. Proof of Theorem 1.1

Without loss of generality, we assume that $g(x) = 0$. Denote by $K_{n,x}$ the distribution function of $Z_{n,x}$, i.e.,

$$K_{n,x}(t) := P(Z_{n,x} \le t) = \sum_{k \le (n-k+1)t} b_{n,k}(x) \qquad t \ge 0.$$

We can write $L_n(g, x)$ as the Lebesgue-Stieltjes integral

$$L_n(g,x) = Eg(Z_{n,x}) = \int_{[0,\infty)} g(t)\, dK_{n,x}(t) = \sum_{j=1}^{4} \int_{I_j} g(t)\, dK_{n,x}(t),$$

where

$$I_1 := \left[0, x - \frac{x}{\sqrt{n}}\right], \qquad I_2 := \left(x - \frac{x}{\sqrt{n}}, x + \frac{x}{\sqrt{n}}\right],$$

$$I_3 := \left(x + \frac{x}{\sqrt{n}}, 2x\right] \qquad \text{and} \qquad I_4 := (2x, \infty).$$

We obviously have

$$\begin{aligned} \int_{I_2} |g(t)|\, dK_{n,x}(t) &\le \omega_x\left(g; \frac{x}{\sqrt{n}}\right) \int_{I_2} dK_{n,x}(t) \\ &\le \omega_x\left(g; \frac{x}{\sqrt{n}}\right) \\ &\le \frac{1}{n} \sum_{k=1}^{n} \omega_x\left(g; \frac{x}{\sqrt{k}}\right). \end{aligned} \tag{3.1}$$

On The Approximation Of Locally Bounded Functions By Operators Of Bleimann, Butzer And Hahn

Jesús de la Cal and Vijay Gupta

On the other hand, from the asymptotic assumption on g, we have

$$|g(t)| \le M\,t^r, \qquad t \ge \alpha,$$

for some constants $M > 0$ and $\alpha \ge 2x$. Therefore,

$$\begin{aligned}\int_{I_4} |g(t)|\, dK_{n,x}(t) &= \left(\int_{(2x,\alpha]} + \int_{(\alpha,\infty)}\right) |g(t)|\, dK_{n,x}(t) \\ &\le \omega_x^+(g;\alpha - x) P(Z_{n,x} > 2x) + M \sum_{k>(n-k+1)\alpha} \frac{k^r}{(n-k+1)^r} b_{n,k}(x).\end{aligned}$$

By Lemma 2.2(b) and Lemma 2.3, this shows that

$$\int_{I_4} |g(t)|\, dK_{n,x}(t) = O_{r,x}(n^{-1}) \qquad (n \to \infty). \tag{3.2}$$

Finally, using Lemma 2.2(b) and integration by parts (follow the same procedure as in the proof of Theorem 1 in [13]), we obtain

$$\begin{aligned}\int_{I_1} |g(t)|\, dK_{n,x}(t) &\le \int_{I_1} \omega_x^-(g; x-t)\, dK_{n,x}(t) \\ &\le \frac{3x(1+x)^2}{(n+2)} \left[\frac{\omega_x^-(g;x)}{x^2} + 2\int_0^{x-x/\sqrt{n}} \frac{\omega_x^-(g;x-t)}{(x-t)^3}\, dt\right] \\ &\le \frac{6(1+x)^2}{(n+2)x} \sum_{k=1}^{n} \omega_x^- \left(g; \frac{x}{\sqrt{k}}\right), \end{aligned} \tag{3.3}$$

and, analogously,

$$\int_{I_3} |g(t)|\, dK_{n,x}(t) \leq \frac{6(1+x)^2}{(n+2)x} \sum_{k=1}^{n} \omega_x^+ \left(g; \frac{x}{\sqrt{k}} \right). \tag{3.4}$$

The conclusion follows from (3.1) – (3.4).

4. Proof of Theorem 1.2

We can write, for $t \geq 0$,

$$f(t) - \tilde{f}(x) = f_x(t) + \frac{f^*(x)}{2}\sigma_x(t) + (f(x) - \tilde{f}(x))\delta_x(t), \tag{4.1}$$

where $\sigma_x := -1_{[0,x)} + 1_{(x,\infty)}$, and $\delta_x := 1_{\{x\}}$ is Dirac's delta at x (this is the so called Bojanic-Vuilleumier-Cheng decomposition).

By Theorem 1.1, we have

$$|L_n(f_x, x)| \leq \Delta_{n,x}(f_x), \tag{4.2}$$

where $\Delta_{n,x}(f_x)$ is the right-hand side of (1.2) with g replaced by f_x. Moreover,

$$\begin{aligned} L_n(\sigma_x, x) &= P(Z_{n,x} > x) - P(Z_{n,x} < x) \\ &= (P(Z_{n,x} > x) - P(Z_{n,x} \leq x)) + P(Z_{n,x} = x), \end{aligned} \tag{4.3}$$

and

$$L_n(\delta_x, x) = P(Z_{n,x} = x). \tag{4.4}$$

Using Lemma 2.2(c) and the fact that (cf. [17, Theorem 1])

$$P(Z_{n,x} = x) = \begin{cases} \binom{n}{k} p_x^k q_x^{n-k} \leq \frac{(1+x)}{\sqrt{2enx}} & \text{if } (n+1)p_x = k \in \{1, 2, \dots, n\} \\ 0 & \text{otherwise,} \end{cases}$$

the conclusion readily follows from (4.1) – (4.4).

5. Proof of Theorem 1.3

Using the decomposition (4.1), it is easily checked that

$$L_n(g,x)-g(x)=\sum_{i=1}^{4}A_i(n,x), \tag{5.1}$$

where

$$A_1(n,x):=\tilde{f}(x)L_n((\cdot-x),x)+\frac{f^*(x)}{2}L_n(|\cdot-x|,x),$$

$$A_2(n,x):=\int_{[0,x]}\left(\int_t^x f_x(u)\,du\right)dK_{n,x}(t),$$

$$A_3(n,x):=\int_{(x,2x]}\left(\int_x^t f_x(u)\,du\right)dK_{n,x}(t),$$

$$A_4(n,x):=\int_{(2x,\infty)}\left(\int_x^t f_x(u)\,du\right)dK_{n,x}(t),$$

and $K_{n,x}(t)$ is the same as in the preceding proofs.

From Lemma 2.2(d,e), we have

$$A_1(n,x)=\frac{\sqrt{x}(1+x)}{\sqrt{2\pi n}}f^*(x)+f^*(x)\,o_x(n^{-1/2})+o_x(n^{-1}),\quad (n\to\infty). \tag{5.2}$$

Next, we estimate $A_2(n,x)$. By Fubini's theorem,

$$A_2(n,x)=\int_0^x K_{n,x}(u)\,f_x(u)\,du=\left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^{x}\right)K_{n,x}(u)\,f_x(u)\,du.$$

It is clear that

$$\left|\int_{x-x/\sqrt{n}}^{x} K_{n,x}(u)\, f_x(u)\, du\right| \le \int_{x-x/\sqrt{n}}^{x} |f_x(u)|\, du$$
$$\le \int_{x-x/\sqrt{n}}^{x} \omega_x^-(f_x; x-u)\, du$$
$$\le \frac{x}{\sqrt{n}}\, \omega_x^-\left(f_x; \frac{x}{\sqrt{n}}\right)$$
$$\le \frac{2x}{n} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \omega_x^-\left(f_x; \frac{x}{k}\right),$$

and, using Lemma 2.2(b),

$$\left|\int_{0}^{x-x/\sqrt{n}} K_{n,x}(u)\, f_x(u)\, du\right| \le \frac{3x(1+x)^2}{(n+2)} \int_0^{x-x/\sqrt{n}} \frac{|f_x(u)|}{(x-u)^2}\, du$$
$$\le \frac{3x(1+x)^2}{(n+2)} \int_0^{x-x/\sqrt{n}} \frac{\omega_x^-(f_x; x-u)}{(x-u)^2}\, du$$
$$\le \frac{3(1+x)^2}{(n+2)} \int_1^{\sqrt{n}} \omega_x^-\left(f_x; \frac{x}{t}\right) dt$$
$$\le \frac{3(1+x)^2}{n+2} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \omega_x^-\left(f_x; \frac{x}{k}\right).$$

We therefore conclude that

$$(5.3) \qquad |A_2(n,x)| \le \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \omega_x^- \left(f_x; \frac{x}{k}\right).$$

Similarly,

$$(5.4) \qquad |A_3(n,x)| \le \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \omega_x^+ \left(f_x; \frac{x}{k}\right).$$

Finally,

$$A_4(n,x) = \int_{(2x,\infty)} g(t)\, dK_{n,x}(t) - \int_{(2x,\infty)} [g(x) + f(x+)(t-x)]\, dK_{n,x}(t),$$

and, by the asymptotic assumption on g, Lemma 2.2(b) and Lemma 2.3, we obtain

$$(5.5) \qquad |A_4(n,x)| = O_{r,x}(n^{-1}), \qquad (n \to \infty).$$

The conclusion follows from (5.1) – (5.5).

6. Remarks on Moments

Fix $x > 0$, and let $g(\cdot) := |\cdot - x|^\beta$, with $\beta > 2$. Since

$$\omega_x(g,h) = 2\,h^\beta, \qquad 0 \le h \le x,$$

and

$$\sum_{k=1}^{n} k^{-\beta/2} = O(1), \qquad (n \to \infty),$$

we conclude from Theorem 1.1 that

$$L_n(|\cdot - x|^\beta, x) = O_{r,x}(n^{-1}), \qquad (n \to \infty).$$

In the case that $0 < \beta \le 2$, we have, by Jensen's inequality (or Hölder's inequality) and Lemma 2.2(a),

$$L_n(|\cdot - x|^\beta, x) = E|Z_{n,x} - x|^\beta \le \left(E(Z_{n,x} - x)^2\right)^{\beta/2} \le \left(\frac{3x(1+x)^2}{n+2}\right)^{\frac{\beta}{2}},$$

for all $n \ge 1$.

References

[1] G. BLEIMANN, P.L. BUTZER AND L. HAHN, A Bernstein-type operator approximating continuous functions on the semi axis, *Indag. Math.*, **42** (1980), 255–262.

[2] V. TOTIK, Uniform approximation by Bernstein-type operators, *Indag. Math.*, **46** (1984), 87–93.

[3] R.A. KHAN, A note on a Bernstein type operator of Bleimann, Butzer and Hahn, *J. Approx. Theory*, **53** (1988), 295–303.

[4] R.A. KHAN, Some properties of a Bernstein type operator of Bleimann, Butzer and Hahn, In *Progress in Approximation Theory*, (Edited by P. Nevai and A. Pinkus), pp. 497–504, Academic Press, New York (1991).

[5] J.A. ADELL AND J. DE LA CAL, Preservation of moduli of continuity for Bernstein-type operators, In *Approximation, Probability, and Related Fields*, (Edited by G. Anastassiou and S.T. Rachev), pp. 1–18, Plenum Press, New York (1994).

[6] J.A. ADELL AND J. DE LA CAL, Bernstein-type operators diminish the ϕ-variation, *Constr. Approx.*, **12** (1996), 489–507.

[7] J.A. ADELL, F.G. BADíA AND J. DE LA CAL, On the iterates of some Bernstein-type operators, *J. Math. Anal. Appl.*, **209** (1997), 529–541.

[8] B. DELLA VECCHIA, Some properties of a rational operator of Bernstein type, In *Progress in Approximation Theory*, (Edited by P. Nevai and A. Pinkus), pp. 177–185, Academic Press, New York (1991).

[9] U. ABEL AND M. IVAN, Some identities for the operator of Bleimann-Butzer-Hahn involving divided differences, *Calcolo*, **36** (1999), 143–160.

[10] U. ABEL AND M. IVAN, Best constant for a Bleimann-Butzer-Hahn moment estimation, *East J. Approx.*, **6** (2000), 1–7.

[11] S. GUO AND M. KHAN, On the rate of convergence of some operators on functions of bounded variation, *J. Approx. Theory*, **58** (1989), 90–101.

[12] V. GUPTA AND R.P. PANT, Rate of convergence for the modified Szász-Mirakyan operators on functions of bounded variation, *J. Math. Anal. Appl.*, **233** (1999), 476–483.

[13] X.M. ZENG AND F. CHENG, On the rates of approximation of Bernstein type operators, *J. Approx. Theory*, **109** (2001), 242–256.

[14] X.M. ZENG AND V. GUPTA, Rate of convergence of Baskakov-Bézier type operators for locally bounded functions, *Computers. Math. Applic.*, **44** (2002), 1445–1453.

[15] A.N. SHIRYAYEV, *Probability*, Springer, New York (1984).

[16] N.L. JOHNSON, S. KOTZ AND N. BALAKRISHNAN, *Continuous Univariate Distributions, Vol. 1, 2nd Edition*, Wiley, New York (1994).

[17] X.M. ZENG, Bounds for Bernstein basis functions and Meyer-König and Zeller basis functions, *J. Math. Anal. Appl.*, **219** (1998), 364–376.