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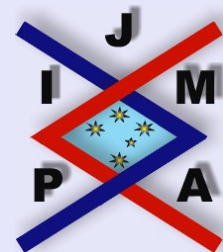
ON THE APPROXIMATION OF LOCALLY BOUNDED FUNCTIONS BY OPERATORS OF BLEIMANN, BUTZER AND HAHN

JESÚS DE LA CAL AND VIJAY GUPTA

Departamento de Matemática Aplicada y Estadística e Investigación Operativa
Facultad de Ciencias
Universidad del País Vasco
Apartado 644, 48080 Bilbao, Spain
EMail: mepcaagj@lg.ehu.es

School of Applied Sciences
Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110045, India
EMail: vijay@nsit.ac.in

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Abstract

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Abstract

We estimate the rate of the pointwise approximation by operators of Bleimann, Butzer and Hahn of locally bounded functions, and of functions having a locally bounded derivative.

2000 Mathematics Subject Classification: 41A20, 41A25, 41A36.

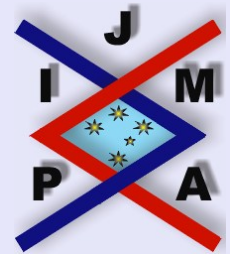
Key words: Operators of Bleimann, Butzer and Hahn, Locally bounded function, Function of bounded variation, Total variation, Rate of convergence, Binomial distribution.

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1. Introduction and Main Results

Bleimann, Butzer and Hahn [1] introduced the Bernstein type operator L_n over the interval $[0, \infty)$ given by

$$L_n(f, x) := \sum_{k=0}^n f\left(\frac{k}{n-k+1}\right) b_{n,k}(x), \quad x \geq 0, \quad n = 1, 2, \dots,$$

where f is a real function on $[0, \infty)$, and

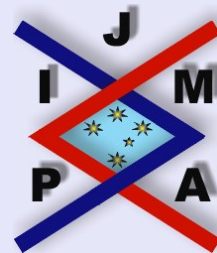
$$(1.1) \quad b_{n,k}(x) := \binom{n}{k} p_x^k q_x^{n-k}, \quad p_x := \frac{x}{1+x}, \quad q_x := 1 - p_x = \frac{1}{1+x}.$$

The approximation of uniformly continuous functions by these operators has been considered in [1] – [4]. For other properties of L_n (preservation of global smoothness, preservation of ϕ -variation, behavior of the iterates, etc.) we refer, for instance, to [4] – [10]. In some of the mentioned works, the results are achieved by using probabilistic methods. This comes from the fact that L_n is an operator of probabilistic type. We can actually write

$$L_n(f, x) = Ef(Z_{n,x}),$$

where E denotes mathematical expectation, and $Z_{n,x}$ is the random variable given by

$$(1.2) \quad Z_{n,x} := \frac{S_{n,x}}{n - S_{n,x} + 1}, \quad S_{n,x} := \xi_{1,x} + \dots + \xi_{n,x},$$



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where $\xi_{1,x}, \xi_{2,x}, \dots$ are independent random variables having the same Bernoulli distribution with parameter p_x , i.e.,

$$P(\xi_{k,x} = 1) = p_x = 1 - P(\xi_{k,x} = 0)$$

(so that $S_{n,x}$ has the binomial distribution with parameters n, p_x). This probabilistic representation also plays a significant role in the present paper (for a more refined representation useful for other purposes, see [5, 6]).

Here, we discuss the approximation of real functions f on the semi axis which are locally bounded, i.e., bounded on each finite subinterval of $[0, \infty)$. In such a case, we set, for $x > 0$ and $h \geq 0$,

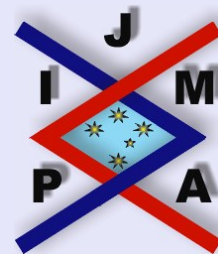
$$\omega_x^+(f; h) := \sup_{x \leq t \leq x+h} |f(t) - f(x)|,$$

$$\omega_x^-(f; h) := \sup_{(x-h)^+ \leq t \leq x} |f(t) - f(x)|,$$

$$\omega_x(f; h) := \omega_x^+(f; h) + \omega_x^-(f; h),$$

where $(x-h)^+ := \max(x-h, 0)$, and we observe that these functions are (non-negative and) nondecreasing on $[0, \infty)$. In particular, every continuous function is locally bounded. Also, if f is locally of bounded variation, i.e., such that

$$\bigvee_a^b(f) < \infty, \quad 0 \leq a < b < \infty,$$



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where $\bigvee_a^b(f)$ stands for the total variation of f on the interval $[a, b]$, then f is locally bounded, and we obviously have

$$\omega_x(f; h) \leq \bigvee_{x-h}^{x+h}(f), \quad 0 \leq h \leq x.$$

This kind of problem has been already considered for other Bernstein-type operators (see, for instance, [11] – [14] and the references therein). Our main results are stated as follows.

Theorem 1.1. *Let g be a real locally bounded function on $[0, \infty)$ such that $g(t) = O(t^r)$ ($t \rightarrow \infty$), for some $r = 1, 2, \dots$. If g is continuous at $x > 0$, then, for n large enough, we have*

$$(1.3) \quad |L_n(g, x) - g(x)| \leq \frac{7(1+x)^2}{(n+2)x} \sum_{k=1}^n \omega_x \left(g; \frac{x}{\sqrt{k}} \right) + O_{r,x} \left(\frac{1}{n} \right).$$

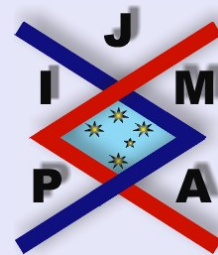
In the following statements (and throughout the paper), we use the notations:

$$f^*(x) := f(x+) - f(x-)$$

$$\tilde{f}(x) := \frac{f(x+) + f(x-)}{2},$$

$$f_x := (f - f(x-))1_{[0,x)} + (f - f(x+))1_{(x,\infty)}$$

(1_A being the indicator function of the set A), provided that the lateral limits $f(x+)$ and $f(x-)$ exist (such a condition is fulfilled when f is locally of bounded variation). We also use the symbol $[a]$ to indicate the integral part of the real number a .



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Theorem 1.2. Let f be a real locally bounded function on $[0, \infty)$ such that $f(t) = O(t^r)$ ($t \rightarrow \infty$), for some $r = 1, 2, \dots$. If $x > 0$, and $f(x+)$ and $f(x-)$ exist, then we have for n large enough

$$\begin{aligned} & \left| L_n(f, x) - \tilde{f}(x) \right| \\ & \leq \Delta_{n,x}(f_x) + \frac{1.6 + x + 2.6x^2}{\sqrt{nx}(1+x)} \cdot \frac{|f^*(x)|}{2} + \frac{\epsilon_{n,x}(1+x)}{\sqrt{2enx}} |f(x) - f(x-)|, \end{aligned}$$

where $\Delta_{n,x}(f_x)$ is the right-hand side of (1.3) with g replaced by f_x , and

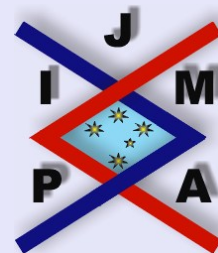
$$\epsilon_{n,x} := \begin{cases} 1 & \text{if } (n+1)p_x \in \{1, 2, \dots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.3. Let g be a real function on $[0, \infty)$ such that $g(t) = O(t^r)$ ($t \rightarrow \infty$), for some $r = 1, 2, \dots$, and having the form

$$g(t) = c + \int_0^t f(u) du, \quad t \geq 0,$$

where c is a constant and f is measurable and locally bounded on $[0, \infty)$. If $x > 0$, and $f(x+)$ and $f(x-)$ exist, then we have for n large enough

$$\begin{aligned} & \left| L_n(g, x) - g(x) - \frac{\sqrt{x}(1+x)}{\sqrt{2\pi n}} f^*(x) \right| \\ & \leq \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \omega_x \left(f_x; \frac{x}{k} \right) + |f^*(x)| o_x(n^{-1/2}) + O_{r,x}(n^{-1}). \end{aligned}$$



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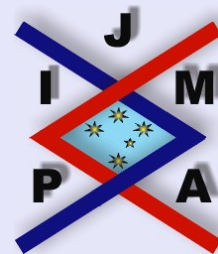
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The proofs of the preceding theorems are given in Sections 3 – 5. In Section 2, we collect the necessary auxiliary results. Some remarks on moments close the paper.



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2. Auxiliary Results

In the following lemma, Φ denotes the standard normal distribution function, and $F_{n,x}^*$ stands for the distribution function of $S_{n,x}^* := (S_{n,x} - np_x) / \sqrt{np_x q_x}$, where $S_{n,x}$ is the same as in (1.2). Such a lemma is nothing but the application of the well-known Berry-Esseen theorem (cf. [15]) to the situation at hand.

Lemma 2.1. *We have, for $x > 0$ and $n \geq 1$,*

$$\sup_{-\infty < t < \infty} |F_{n,x}^*(t) - \Phi(t)| \leq \frac{0.8(p_x^3 q_x + p_x q_x^3)}{\sqrt{n}(p_x q_x)^{3/2}} = \frac{0.8(1+x^2)}{\sqrt{nx}(1+x)}.$$

Lemma 2.2. *Let $x > 0$ and $n \geq 1$. Then, we have:*

(a)

$$L_n((\cdot - x)^2, x) = E(Z_{n,x} - x)^2 \leq \frac{3x(1+x)^2}{n+2}.$$

(b)

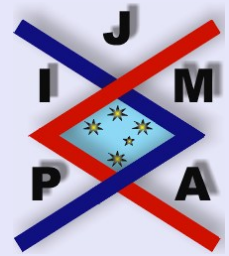
$$P(Z_{n,x} \leq x - h) + P(Z_{n,x} \geq x + h) \leq \frac{3x(1+x)^2}{(n+2)h^2}, \quad h > 0.$$

(c)

$$|P(Z_{n,x} > x) - P(Z_{n,x} \leq x)| \leq \sqrt{\frac{x}{n}} + \frac{1.6(1+x^2)}{\sqrt{nx}(1+x)}.$$

(d)

$$L_n((\cdot - x), x) = E(Z_{n,x} - x) = -xp_x^n = o_x(n^{-1}), \quad (n \rightarrow \infty).$$



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(e)

$$L_n(|\cdot - x|, x) = E|Z_{n,x} - x| = \frac{\sqrt{2x}(1+x)}{\sqrt{\pi n}} + o_x(n^{-1/2}), \quad (n \rightarrow \infty).$$

Proof. Part (a) was shown in [10]. Part (b) follows from (a) and the fact that, by Markov's inequality,

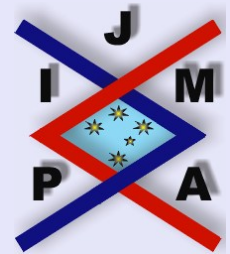
$$P(Z_{n,x} \leq x - h) + P(Z_{n,x} \geq x + h) = P(|Z_{n,x} - x| \geq h) \leq \frac{E(Z_{n,x} - x)^2}{h^2}.$$

To show (c), observe that

$$\begin{aligned} & |P(Z_{n,x} > x) - P(Z_{n,x} \leq x)| \\ &= |1 - 2P(Z_{n,x} \leq x)| \\ &= |1 - 2P(S_{n,x} \leq (n+1)p_x)| \\ &= \left| 1 - 2F_{n,x}^* \left(\sqrt{\frac{x}{n}} \right) \right| \\ &\leq 2 \left| \Phi \left(\sqrt{\frac{x}{n}} \right) - F_{n,x}^* \left(\sqrt{\frac{x}{n}} \right) \right| + \left| 1 - 2\Phi \left(\sqrt{\frac{x}{n}} \right) \right|. \end{aligned}$$

Thus, the conclusion in part (c) follows from Lemma 2.1 and the fact that (cf. [16])

$$0 < 2\Phi(t) - 1 \leq \left(1 - e^{-t^2}\right)^{1/2} \leq t, \quad (t > 0).$$



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Part (d) is immediate. Finally, to show (e), let $m := \lfloor (n + 1)p_x \rfloor$. We have

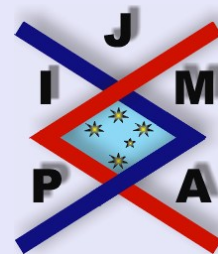
$$\begin{aligned}
 & L_n(|\cdot - x|, x) - L_n((\cdot - x), x) \\
 &= 2 \sum_{k=0}^m \left(x - \frac{k}{n - k + 1} \right) b_{n,k}(x) \\
 &= 2x \sum_{k=0}^m b_{n,k}(x) - 2 \sum_{k=1}^m \frac{n!}{(k - 1)!(n - k + 1)!} p_x^k q_x^{n-k} \\
 &= 2x \sum_{k=0}^m b_{n,k}(x) - 2x \sum_{k=0}^{m-1} b_{n,k}(x) \\
 &= 2x b_{n,m}(x) \\
 &= \frac{\sqrt{2x}(1 + x)}{\sqrt{\pi n}} + o_x(n^{-1/2}), \quad (n \rightarrow \infty),
 \end{aligned}$$

the last equality by [13, Lemma 1], and the conclusion follows from (d). \square

Lemma 2.3. *Let $x > 0$ and $r = 1, 2, \dots$. Then, we have for all integers n such that $(n + 1)(p_{2x} - p_{3x/2}) \geq r$,*

$$\begin{aligned}
 \sum_{k \in K} \frac{k^r}{(n - k + 1)^r} b_{n,k}(x) &\leq 12 r! \sum_{s=1}^r \left\{ \begin{matrix} r \\ s \end{matrix} \right\} \frac{x^{s-1}(1 + x)^{r-s+2}}{n + r - s + 2} \cdot \frac{n!}{(n + r - s)!} \\
 &= O_{r,x}(n^{-1}), \quad (n \rightarrow \infty),
 \end{aligned}$$

where the $\left\{ \begin{matrix} r \\ s \end{matrix} \right\}$ are the Stirling numbers of the second kind, and K is the set of all integers k such that $n \geq k > (n - k + 1)2x$ (i.e., $n \geq k > (n + 1)p_{2x}$).



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Proof. Using the well known identity

$$a^r = \sum_{s=1}^r \left\{ \begin{matrix} r \\ s \end{matrix} \right\} a(a-1)\cdots(a-s+1),$$

we can write

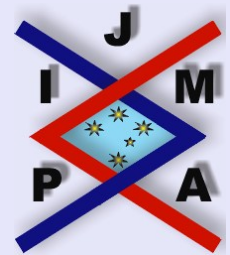
$$(2.1) \quad \sum_{k \in K} \frac{k^r}{(n-k+1)^r} b_{n,k}(x) = \sum_{s=1}^r \left\{ \begin{matrix} r \\ s \end{matrix} \right\} A_s,$$

where

$$\begin{aligned} A_s &:= \sum_{k \in K} \frac{k(k-1)\cdots(k-s+1)}{(n-k+1)^r} b_{n,k}(x) \\ &= \sum_{k \in K} \frac{1}{(n-k+1)^r} \cdot \frac{n!}{(k-s)!(n-k)!} P_x^k Q_x^{n-k}. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{(n-k+1)^r} &= \prod_{i=1}^r \left[\frac{1}{n-k+i} \frac{n-k+i}{n-k+1} \right] \\ &= \prod_{i=1}^r \left[\frac{1}{n-k+i} \left(1 + \frac{i-1}{n-k+1} \right) \right] \\ &\leq \prod_{i=1}^r \frac{i}{n-k+i} = \frac{r!(n-k)!}{(n-k+r)!}, \end{aligned}$$



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we have

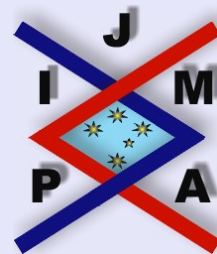
$$\begin{aligned}
 A_s &\leq r! \sum_{k \in K} \frac{n!}{(k-s)!(n-k+r)!} p_x^k q_x^{n-k} \\
 &= r! \sum_{l \in K_s} \frac{n!}{l!(n+r-s-l)!} p_x^{l+s} q_x^{n-l-s} \\
 &= \frac{r! n! p_x^s q_x^{-r}}{(n+r-s)!} \sum_{l \in K_s} \binom{n+r-s}{l} \frac{x^l}{(1+x)^{n+r-s}} \\
 &\leq \frac{r! n! p_x^s q_x^{-r}}{(n+r-s)!} \sum_{l \in K'} \binom{n+r-s}{l} \frac{x^l}{(1+x)^{n+r-s}},
 \end{aligned}$$

where $K_s := \{k-s : k \in K\}$, and K' stands for the set of all integers l such that $n \geq l > (n-l+1)(3x/2)$ (observe that, by the assumption on n , we have $K_s \subset K'$). The probabilistic interpretation of the last sum together with Lemma 2.2(b) yield

$$\begin{aligned}
 A_s &\leq \frac{r! n! x^s (1+x)^{r-s}}{(n+r-s)!} P \left(Z_{n+r-s, x} > \frac{3x}{2} \right) \\
 (2.2) \quad &\leq \frac{12 r! n! x^{s-1} (1+x)^{r-s+2}}{(n+r-s)!(n+r-s+2)},
 \end{aligned}$$

and the conclusion follows from (2.1) and (2.2). \square

Remark 1. *The same procedure as in the preceding proof leads to the following*



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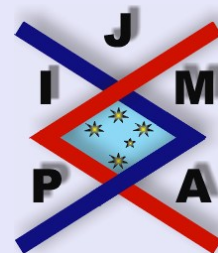
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upper bound for the integral moments of L_n (or $Z_{n,x}$):

$$\begin{aligned} L_n(t^r, x) &= E(Z_{n,x})^r \\ &= \sum_{k=0}^n \frac{k^r}{(n-k+1)^r} b_{n,k}(x) \\ &\leq r! \sum_{s=1}^r \left\{ \begin{matrix} r \\ s \end{matrix} \right\} \frac{n! x^s (1+x)^{r-s}}{(n+r-s)!}. \end{aligned}$$



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3. Proof of Theorem 1.1

Without loss of generality, we assume that $g(x) = 0$. Denote by $K_{n,x}$ the distribution function of $Z_{n,x}$, i.e.,

$$K_{n,x}(t) := P(Z_{n,x} \leq t) = \sum_{k \leq (n-k+1)t} b_{n,k}(x) \quad t \geq 0.$$

We can write $L_n(g, x)$ as the Lebesgue-Stieltjes integral

$$L_n(g, x) = Eg(Z_{n,x}) = \int_{[0, \infty)} g(t) dK_{n,x}(t) = \sum_{j=1}^4 \int_{I_j} g(t) dK_{n,x}(t),$$

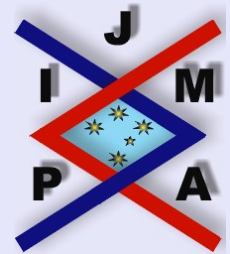
where

$$I_1 := \left[0, x - \frac{x}{\sqrt{n}}\right], \quad I_2 := \left(x - \frac{x}{\sqrt{n}}, x + \frac{x}{\sqrt{n}}\right),$$

$$I_3 := \left(x + \frac{x}{\sqrt{n}}, 2x\right] \quad \text{and} \quad I_4 := (2x, \infty).$$

We obviously have

$$\begin{aligned} \int_{I_2} |g(t)| dK_{n,x}(t) &\leq \omega_x \left(g; \frac{x}{\sqrt{n}}\right) \int_{I_2} dK_{n,x}(t) \\ &\leq \omega_x \left(g; \frac{x}{\sqrt{n}}\right) \\ (3.1) \quad &\leq \frac{1}{n} \sum_{k=1}^n \omega_x \left(g; \frac{x}{\sqrt{k}}\right). \end{aligned}$$



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On the other hand, from the asymptotic assumption on g , we have

$$|g(t)| \leq M t^r, \quad t \geq \alpha,$$

for some constants $M > 0$ and $\alpha \geq 2x$. Therefore,

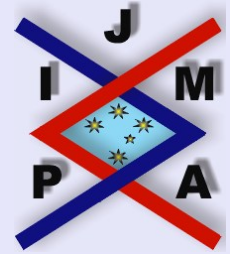
$$\begin{aligned} & \int_{I_4} |g(t)| dK_{n,x}(t) \\ &= \left(\int_{(2x, \alpha]} + \int_{(\alpha, \infty)} \right) |g(t)| dK_{n,x}(t) \\ &\leq \omega_x^+(g; \alpha - x) P(Z_{n,x} > 2x) + M \sum_{k > (n-k+1)\alpha} \frac{k^r}{(n-k+1)^r} b_{n,k}(x). \end{aligned}$$

By Lemma 2.2(b) and Lemma 2.3, this shows that

$$(3.2) \quad \int_{I_4} |g(t)| dK_{n,x}(t) = O_{r,x}(n^{-1}) \quad (n \rightarrow \infty).$$

Finally, using Lemma 2.2(b) and integration by parts (follow the same procedure as in the proof of Theorem 1 in [13]), we obtain

$$\begin{aligned} & \int_{I_1} |g(t)| dK_{n,x}(t) \leq \int_{I_1} \omega_x^-(g; x-t) dK_{n,x}(t) \\ &\leq \frac{3x(1+x)^2}{(n+2)} \left[\frac{\omega_x^-(g; x)}{x^2} + 2 \int_0^{x-x/\sqrt{n}} \frac{\omega_x^-(g; x-t)}{(x-t)^3} dt \right] \\ (3.3) \quad &\leq \frac{6(1+x)^2}{(n+2)x} \sum_{k=1}^n \omega_x^-\left(g; \frac{x}{\sqrt{k}}\right), \end{aligned}$$



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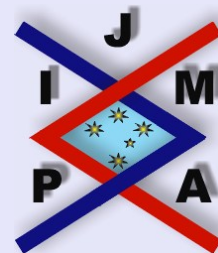
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and, analogously,

$$(3.4) \quad \int_{I_3} |g(t)| dK_{n,x}(t) \leq \frac{6(1+x)^2}{(n+2)x} \sum_{k=1}^n \omega_x^+ \left(g; \frac{x}{\sqrt{k}} \right).$$

The conclusion follows from (3.1) – (3.4).



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4. Proof of Theorem 1.2

We can write, for $t \geq 0$,

$$(4.1) \quad f(t) - \tilde{f}(x) = f_x(t) + \frac{f^*(x)}{2} \sigma_x(t) + (f(x) - \tilde{f}(x)) \delta_x(t),$$

where $\sigma_x := -1_{[0,x)} + 1_{(x,\infty)}$, and $\delta_x := 1_{\{x\}}$ is Dirac's delta at x (this is the so called Bojanic-Vuilleumier-Cheng decomposition).

By Theorem 1.1, we have

$$(4.2) \quad |L_n(f_x, x)| \leq \Delta_{n,x}(f_x),$$

where $\Delta_{n,x}(f_x)$ is the right-hand side of (1.2) with g replaced by f_x . Moreover,

$$(4.3) \quad \begin{aligned} L_n(\sigma_x, x) &= P(Z_{n,x} > x) - P(Z_{n,x} < x) \\ &= (P(Z_{n,x} > x) - P(Z_{n,x} \leq x)) + P(Z_{n,x} = x), \end{aligned}$$

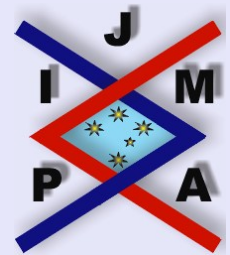
and

$$(4.4) \quad L_n(\delta_x, x) = P(Z_{n,x} = x).$$

Using Lemma 2.2(c) and the fact that (cf. [17, Theorem 1])

$$P(Z_{n,x} = x) = \begin{cases} \binom{n}{k} p_x^k q_x^{n-k} \leq \frac{(1+x)}{\sqrt{2enx}} & \text{if } (n+1)p_x = k \in \{1, 2, \dots, n\} \\ 0 & \text{otherwise,} \end{cases}$$

the conclusion readily follows from (4.1) – (4.4).



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5. Proof of Theorem 1.3

Using the decomposition (4.1), it is easily checked that

$$(5.1) \quad L_n(g, x) - g(x) = \sum_{i=1}^4 A_i(n, x),$$

where

$$A_1(n, x) := \tilde{f}(x)L_n((\cdot - x), x) + \frac{f^*(x)}{2}L_n(|\cdot - x|, x),$$

$$A_2(n, x) := \int_{[0,x]} \left(\int_t^x f_x(u) du \right) dK_{n,x}(t),$$

$$A_3(n, x) := \int_{(x,2x]} \left(\int_x^t f_x(u) du \right) dK_{n,x}(t),$$

$$A_4(n, x) := \int_{(2x,\infty)} \left(\int_x^t f_x(u) du \right) dK_{n,x}(t),$$

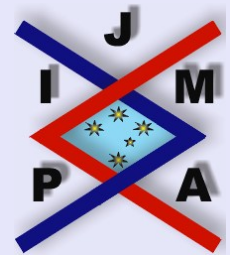
and $K_{n,x}(t)$ is the same as in the preceding proofs.

From Lemma 2.2(d,e), we have

$$(5.2) \quad A_1(n, x) = \frac{\sqrt{x}(1+x)}{\sqrt{2\pi n}} f^*(x) + f^*(x) o_x(n^{-1/2}) + o_x(n^{-1}), \quad (n \rightarrow \infty).$$

Next, we estimate $A_2(n, x)$. By Fubini's theorem,

$$A_2(n, x) = \int_0^x K_{n,x}(u) f_x(u) du = \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x \right) K_{n,x}(u) f_x(u) du.$$



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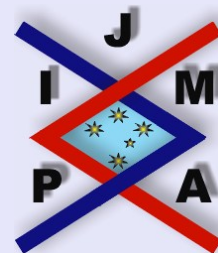
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It is clear that

$$\begin{aligned}
 \left| \int_{x-x/\sqrt{n}}^x K_{n,x}(u) f_x(u) du \right| &\leq \int_{x-x/\sqrt{n}}^x |f_x(u)| du \\
 &\leq \int_{x-x/\sqrt{n}}^x \omega_x^-(f_x; x-u) du \\
 &\leq \frac{x}{\sqrt{n}} \omega_x^-\left(f_x; \frac{x}{\sqrt{n}}\right) \\
 &\leq \frac{2x}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \omega_x^-\left(f_x; \frac{x}{k}\right),
 \end{aligned}$$

and, using Lemma 2.2(b),

$$\begin{aligned}
 \left| \int_0^{x-x/\sqrt{n}} K_{n,x}(u) f_x(u) du \right| &\leq \frac{3x(1+x)^2}{(n+2)} \int_0^{x-x/\sqrt{n}} \frac{|f_x(u)|}{(x-u)^2} du \\
 &\leq \frac{3x(1+x)^2}{(n+2)} \int_0^{x-x/\sqrt{n}} \frac{\omega_x^-(f_x; x-u)}{(x-u)^2} du \\
 &\leq \frac{3(1+x)^2}{(n+2)} \int_1^{\sqrt{n}} \omega_x^-\left(f_x; \frac{x}{t}\right) dt \\
 &\leq \frac{3(1+x)^2}{n+2} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \omega_x^-\left(f_x; \frac{x}{k}\right).
 \end{aligned}$$



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We therefore conclude that

$$(5.3) \quad |A_2(n, x)| \leq \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \omega_x^- \left(f_x; \frac{x}{k} \right).$$

Similarly,

$$(5.4) \quad |A_3(n, x)| \leq \frac{5(1+x)^2}{n+2} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \omega_x^+ \left(f_x; \frac{x}{k} \right).$$

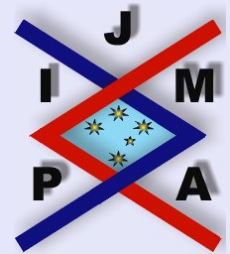
Finally,

$$A_4(n, x) = \int_{(2x, \infty)} g(t) dK_{n,x}(t) - \int_{(2x, \infty)} [g(x) + f(x+)(t-x)] dK_{n,x}(t),$$

and, by the asymptotic assumption on g , Lemma 2.2(b) and Lemma 2.3, we obtain

$$(5.5) \quad |A_4(n, x)| = O_{r,x}(n^{-1}), \quad (n \rightarrow \infty).$$

The conclusion follows from (5.1) – (5.5).



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6. Remarks on Moments

Fix $x > 0$, and let $g(\cdot) := |\cdot - x|^\beta$, with $\beta > 2$. Since

$$\omega_x(g, h) = 2h^\beta, \quad 0 \leq h \leq x,$$

and

$$\sum_{k=1}^n k^{-\beta/2} = O(1), \quad (n \rightarrow \infty),$$

we conclude from Theorem 1.1 that

$$L_n(|\cdot - x|^\beta, x) = O_{r,x}(n^{-1}), \quad (n \rightarrow \infty).$$

In the case that $0 < \beta \leq 2$, we have, by Jensen's inequality (or Hölder's inequality) and Lemma 2.2(a),

$$L_n(|\cdot - x|^\beta, x) = E|Z_{n,x} - x|^\beta \leq (E(Z_{n,x} - x)^2)^{\beta/2} \leq \left(\frac{3x(1+x)^2}{n+2} \right)^{\beta/2},$$

for all $n \geq 1$.



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- [1] G. BLEIMANN, P.L. BUTZER AND L. HAHN, A Bernstein-type operator approximating continuous functions on the semi axis, *Indag. Math.*, **42** (1980), 255–262.
- [2] V. TOTIK, Uniform approximation by Bernstein-type operators, *Indag. Math.*, **46** (1984), 87–93.
- [3] R.A. KHAN, A note on a Bernstein type operator of Bleimann, Butzer and Hahn, *J. Approx. Theory*, **53** (1988), 295–303.
- [4] R.A. KHAN, Some properties of a Bernstein type operator of Bleimann, Butzer and Hahn, In *Progress in Approximation Theory*, (Edited by P. Nevai and A. Pinkus), pp. 497–504, Academic Press, New York (1991).
- [5] J.A. ADELL AND J. DE LA CAL, Preservation of moduli of continuity for Bernstein-type operators, In *Approximation, Probability, and Related Fields*, (Edited by G. Anastassiou and S.T. Rachev), pp. 1–18, Plenum Press, New York (1994).
- [6] J.A. ADELL AND J. DE LA CAL, Bernstein-type operators diminish the ϕ -variation, *Constr. Approx.*, **12** (1996), 489–507.
- [7] J.A. ADELL, F.G. BADÍA AND J. DE LA CAL, On the iterates of some Bernstein-type operators, *J. Math. Anal. Appl.*, **209** (1997), 529–541.
- [8] B. DELLA VECCHIA, Some properties of a rational operator of Bernstein type, In *Progress in Approximation Theory*, (Edited by P. Nevai and A. Pinkus), pp. 177–185, Academic Press, New York (1991).



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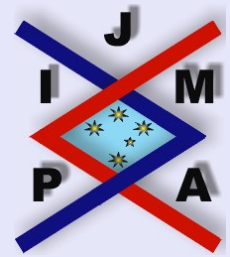
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- [9] U. ABEL AND M. IVAN, Some identities for the operator of Bleimann-Butzer-Hahn involving divided differences, *Calcolo*, **36** (1999), 143–160.
- [10] U. ABEL AND M. IVAN, Best constant for a Bleimann-Butzer-Hahn moment estimation, *East J. Approx.*, **6** (2000), 1–7.
- [11] S. GUO AND M. KHAN, On the rate of convergence of some operators on functions of bounded variation, *J. Approx. Theory*, **58** (1989), 90–101.
- [12] V. GUPTA AND R.P. PANT, Rate of convergence for the modified Szász-Mirakyan operators on functions of bounded variation, *J. Math. Anal. Appl.*, **233** (1999), 476–483.
- [13] X.M. ZENG AND F. CHENG, On the rates of approximation of Bernstein type operators, *J. Approx. Theory*, **109** (2001), 242–256.
- [14] X.M. ZENG AND V. GUPTA, Rate of convergence of Baskakov-Bézier type operators for locally bounded functions, *Computers. Math. Applic.*, **44** (2002), 1445–1453.
- [15] A.N. SHIRYAYEV, *Probability*, Springer, New York (1984).
- [16] N.L. JOHNSON, S. KOTZ AND N. BALAKRISHNAN, *Continuous Univariate Distributions, Vol. 1, 2nd Edition*, Wiley, New York (1994).
- [17] X.M. ZENG, Bounds for Bernstein basis functions and Meyer-König and Zeller basis functions, *J. Math. Anal. Appl.*, **219** (1998), 364–376.



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