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A RELATION TO HARDY-HILBERT'S INTEGRAL INEQUALITY AND MULHOLLAND'S INEQUALITY

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ABSTRACT. This paper deals with a relation between Hardy-Hilbert's integral inequality and Mulholland's integral inequality with a best constant factor, by using the Beta function and introducing a parameter λ. As applications, the reverse, the equivalent form and some particular results are considered.

Key words and phrases: Hardy-Hilbert's integral inequality; Mulholland's integral inequality; β function; Hölder's inequality.

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1. INTRODUCTION

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$ satisfy $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then one has two equivalent inequalities as (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}; \tag{1.1}$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \int_0^\infty f^p(x)dx, \tag{1.2}$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ are all the best possible. Inequality (1.1) is called Hardy- Hilbert's integral inequality, which is important in analysis and its applications (cf. Mitrinovic et al. [2]).

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If $0 < \int_1^\infty \frac{1}{x} F^p(x)dx < \infty$ and $0 < \int_1^\infty \frac{1}{y} G^q(y)dy < \infty$, then the Mulholland's integral inequality is as follows (see [1, 3]):

$$\int_1^\infty \int_1^\infty \frac{F(x)F(y)}{xy \ln xy} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_1^\infty \frac{F^p(x)}{x} dx \right\}^{\frac{1}{p}} \left\{ \int_1^\infty \frac{G^q(y)}{y} dy \right\}^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Setting $f(x) = F(x)/x$, and $g(y) = G(y)/y$ in (1.3), by simplification, one has (see [12])

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{\ln xy} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_1^\infty x^{p-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_1^\infty x^{q-1} g^q(x)dx \right\}^{\frac{1}{q}}. \tag{1.4}$$

We still call (1.4) Mulholland's integral inequality.

In 1998, Yang [11] first introduced an independent parameter λ and the β function for given an extension of (1.1) (for $p = q = 2$). Recently, by introducing a parameter λ, Yang [8] and Yang et al. [10] gave some extensions of (1.1) and (1.2) as: If $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$ satisfy $0 < \int_0^\infty x^{1-\lambda} f^p(x)dx < \infty$ and $0 < \int_0^\infty x^{1-\lambda} g^q(x)dx < \infty$, then one has two equivalent inequalities as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < k_\lambda(p) \left\{ \int_0^\infty x^{1-\lambda} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\lambda} g^q(x)dx \right\}^{\frac{1}{q}} \tag{1.5}$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)} \left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \right]^p dy < [k_\lambda(p)]^p \int_0^\infty x^{1-\lambda} f^p(x)dx, \tag{1.6}$$

where the constant factors $k_\lambda(p)$ and $[k_\lambda(p)]^p$ ($k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$, $B(u,v)$ is the β function) are all the best possible. By introducing a parameter α, Kuang [5] gave an extension of (1.1), and Yang [9] gave an improvement of [5] as: If $\alpha > 0, f, g \geq 0$ satisfy $0 < \int_0^\infty x^{(p-1)(1-\alpha)} f^p(x)dx < \infty$ and $0 < \int_0^\infty x^{(q-1)(1-\alpha)} g^q(x)dx < \infty$, then

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\alpha + y^\alpha} dxdy \\ < \frac{\pi}{\alpha \sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty x^{(p-1)(1-\alpha)} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{(q-1)(1-\alpha)} g^q(x)dx \right\}^{\frac{1}{q}}, \end{aligned} \tag{1.7}$$

where the constant $\frac{\pi}{\alpha \sin(\pi/p)}$ is the best possible. Recently, Sulaiman [6] gave some new forms of (1.1) and Hong [14] gave an extension of Hardy-Hilbert's inequality by introducing two parameters λ and α. Yang et al. [13] provided an extensive account of the above results.

The main objective of this paper is to build a relation to (1.1) and (1.4) with a best constant factor, by introducing the β function and a parameter λ, related to the double integral $\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda} dxdy \quad (\lambda > 0)$. As applications, the reversion, the equivalent form and some particular results are considered.

2. SOME LEMMAS

First, we need the formula of the β function as (cf. Wang et al. [7]):

$$B(u,v) := \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt = B(v,u) \qquad (u, v > 0). \tag{2.1}$$

Lemma 2.1 (cf. [4]). *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\omega(\sigma) > 0$, $f, g \geq 0$, $f \in L^p_\omega(E)$ *and* $g \in L^q_\omega(E)$, *then one has the Hölder's inequality with weight as:*

$$\int_E \omega(\sigma) f(\sigma) g(\sigma) d\sigma \leq \left\{ \int_E \omega(\sigma) f^p(\sigma) d\sigma \right\}^{\frac{1}{p}} \left\{ \int_E \omega(\sigma) g^q(\sigma) d\sigma \right\}^{\frac{1}{q}}; \tag{2.2}$$

if $p < 1$ $(p \neq 0)$, *with the above assumption, one has the reverse of (2.2), where the equality (in the above two cases) holds if and only if there exists non-negative real numbers* c_1 *and* c_2, *such that they are not all zero and* $c_1 f^p(\sigma) = c_2 g^q(\sigma)$, *a. e. in* E.

Lemma 2.2. *If* $p \neq 0, 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r = \phi_r(\lambda) > 0\, (r = p, q)$, $\phi_p + \phi_q = \lambda$, *and* $u(t)$ *is a differentiable strict increasing function in* (a, b) $(-\infty \leq a < b \leq \infty)$ *such that* $u(a+) = 0$ *and* $u(b-) = \infty$, *for* $r = p, q$, *define* $\omega_r(x)$ *as*

$$\omega_r(x) := (u(x))^{\lambda - \phi_r} \int_a^b \frac{(u(y))^{\phi_r - 1} u'(y)}{(u(x) + u(y))^\lambda} dy \qquad (x \in (a, b)). \tag{2.3}$$

Then for $x \in (a, b)$, *each* $\omega_r(x)$ *is constant, that is*

$$\omega_r(x) = B(\phi_p, \phi_q) \qquad (r = p, q). \tag{2.4}$$

Proof. For fixed x, setting $v = \frac{u(y)}{u(x)}$ in (2.3), one has

$$\begin{aligned} \omega_r(x) &= (u(x))^{\lambda - \phi_r} \int_a^b \frac{(u(y))^{\phi_r - 1} u'(y)}{(u(x))^\lambda (1 + u(y)/u(x))^\lambda} dy \\ &= (u(x))^{\lambda - \phi_r} \int_0^\infty \frac{(v u(x))^{\phi_r - 1}}{(u(x))^\lambda (1 + v)^\lambda} u(x) dv \\ &= \int_0^\infty \frac{v^{\phi_r - 1}}{(1 + v)^\lambda} dv \qquad (r = p, q). \end{aligned}$$

By (2.1), one has (2.4). The lemma is proved. □

Lemma 2.3. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r > 0$ $(r = p, q)$, *satisfy* $\phi_p + \phi_q = \lambda$, *and* $u(t)$ *is a differentiable strict increasing function in* (a, b) $(-\infty \leq a < b \leq \infty)$ *satisfying* $u(a+) = 0$ *and* $u(b-) = \infty$, *then for* $c = u^{-1}(1)$ *and* $0 < \varepsilon < q\phi_p$,

$$\begin{aligned} I &:= \int_c^b \int_c^b \frac{(u(x))^{\phi_q - \frac{\varepsilon}{p} - 1} u'(x)}{(u(x) + u(y))^\lambda} (u(y))^{\phi_p - \frac{\varepsilon}{q} - 1} u'(y) dx dy \\ &> \frac{1}{\varepsilon} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - O(1); \end{aligned} \tag{2.5}$$

if $0 < p < 1$ *(or* $p < 0$*), with the above assumption and* $0 < \varepsilon < -q\phi_q$ *(or* $0 < \varepsilon < q\phi_p$*), then*

$$I < \frac{1}{\varepsilon} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right). \tag{2.6}$$

Proof. For fixed x, setting $v = \frac{u(y)}{u(x)}$ in I, one has

$$\begin{aligned} I &:= \int_c^b (u(x))^{\phi_q - \frac{\varepsilon}{p} - 1} u'(x) \left[\int_c^b \frac{(u(y))^{\phi_p - \frac{\varepsilon}{q} - 1}}{(u(x)+u(y))^\lambda} u'(y) dy \right] dx \\ &= \int_c^b (u(x))^{-1-\varepsilon} u'(x) \int_{\frac{1}{u(x)}}^\infty \frac{1}{(1+v)^\lambda} v^{\phi_p - \frac{\varepsilon}{q} - 1} dv dx \\ &= \int_c^b \frac{u'(x) dx}{(u(x))^{1+\varepsilon}} \int_0^\infty \frac{v^{\phi_p - \frac{\varepsilon}{q} - 1}}{(1+v)^\lambda} dv - \int_c^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} \int_0^{\frac{1}{u(x)}} \frac{v^{\phi_p - \frac{\varepsilon}{q} - 1}}{(1+v)^\lambda} dv dx \qquad (2.7) \\ &> \frac{1}{\varepsilon} \int_0^\infty \frac{v^{\phi_p - \frac{\varepsilon}{q} - 1}}{(1+v)^\lambda} dv - \int_c^b \frac{u'(x)}{(u(x))} \left[\int_0^{\frac{1}{u(x)}} v^{\phi_p - \frac{\varepsilon}{q} - 1} dv \right] dx \\ &= \frac{1}{\varepsilon} \int_0^\infty \frac{v^{\phi_p - \frac{\varepsilon}{q} - 1}}{(1+v)^\lambda} dv - \left(\phi_p - \frac{\varepsilon}{q} \right)^{-2}. \end{aligned}$$

By (2.1), inequality (2.5) is valid. If $0 < p < 1$ (or $p < 0$), by (2.7), one has

$$I < \int_c^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} dx \int_0^\infty \frac{1}{(1+v)^\lambda} v^{\phi_p - \frac{\varepsilon}{q} - 1} dv,$$

and then by (2.1), inequality (2.6) follows. The lemma is proved. □

3. MAIN RESULTS

Theorem 3.1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r > 0\,(r = p, q)$, $\phi_p + \phi_q = \lambda$, $u(t)$ *is a differentiable strict increasing function in* (a, b) $(-\infty \le a < b \le \infty)$, *such that* $u(a+) = 0$ *and* $u(b-) = \infty$, *and* $f, g \ge 0$ *satisfy* $0 < \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx < \infty$ *and* $0 < \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x) dx < \infty$, *then*

$$\begin{aligned} &\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda} dx dy \qquad (3.1) \\ &\quad < B(\phi_p, \phi_q) \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}}, \end{aligned}$$

where the constant factor $B(\phi_p, \phi_q)$ *is the best possible. If* $p < 1$ $(p \neq 0)$, $\{\lambda; \phi_r > 0\,(r = p, q), \phi_p + \phi_q = \lambda\} \neq \phi$, *with the above assumption, one has the reverse of (3.1), and the constant is still the best possible.*

Proof. By (2.2), one has

$$\begin{aligned} J &:= \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda} dx dy \\ &= \int_a^b \int_a^b \frac{1}{(u(x)+u(y))^\lambda} \left[\frac{(u(x))^{(1-\phi_q)/q}}{(u(y))^{(1-\phi_p)/p}} \frac{(u'(y))^{1/p}}{(u'(x))^{1/q}} f(x) \right] \\ &\qquad \times \left[\frac{(u(y))^{(1-\phi_p)/p}}{(u(x))^{(1-\phi_q)/q}} \frac{(u'(x))^{1/q}}{(u'(y))^{1/p}} g(y) \right] dx dy \end{aligned}$$

$$\leq \left\{ \int_a^b \left[\int_a^b \frac{(u(y))^{\phi_p - 1} u'(y)}{(u(x)+u(y))^\lambda} dy \right] \frac{(u(x))^{(p-1)(1-\phi_q)}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_a^b \left[\int_a^b \frac{(u(x))^{\phi_q - 1} u'(x)}{(u(x)+u(y))^\lambda} dx \right] \frac{(u(y))^{(q-1)(1-\phi_p)}}{(u'(y))^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}}. \tag{3.2}$$

If (3.2) takes the form of equality, then by (2.2), there exist non-negative numbers c_1 and c_2, such that they are not all zero and

$$c_1 \frac{u'(y)(u(x))^{(p-1)(1-\phi_q)}}{(u(y))^{1-\phi_p}(u'(x))^{p-1}} f^p(x) = c_2 \frac{u'(x)(u(y))^{(q-1)(1-\phi_p)}}{(u(x))^{1-\phi_q}(u'(y))^{q-1}} g^q(y),$$
$$\text{a.e. in } (a,b)\times(a,b).$$

It follows that

$$c_1 \frac{(u(x))^{p(1-\phi_q)}}{(u'(x))^p} f^p(x) = c_2 \frac{(u(y))^{q(1-\phi_p)}}{(u'(y))^q} g^q(y) = c_3, \text{ a.e. in } (a,b)\times(a,b),$$

where c_3 is a constant. Without loss of generality, suppose $c_1 \neq 0$. One has

$$\frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) = \frac{c_3 u'(x)}{c_1 u(x)}, \text{ a.e. in } (a,b),$$

which contradicts $0 < \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx < \infty$. Then by (2.3), one has

$$J < \left\{ \int_a^b \omega_p(x) \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega_q(x) \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}}, \tag{3.3}$$

and in view of (2.4), it follows that (3.1) is valid.

For $0 < \varepsilon < q\phi_p$, setting $f_\varepsilon(x) = g_\varepsilon(x) = 0$, $x \in (a,c)$ $(c = u^{-1}(1))$;

$$f_\varepsilon(x) = (u(x))^{\phi_q - \frac{\varepsilon}{p} - 1} u'(x), \quad g_\varepsilon(x) = (u(x))^{\phi_p - \frac{\varepsilon}{q} - 1} u'(x),$$

$x \in [c,b)$, we find

$$\left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g_\varepsilon^q(x) dx \right\}^{\frac{1}{q}} = \frac{1}{\varepsilon}. \tag{3.4}$$

If the constant factor $B(\phi_p, \phi_q)$ in (3.1) is not the best possible, then, there exists a positive constant $k < B(\phi_p, \phi_q)$, such that (3.1) is still valid if one replaces $B(\phi_p, \phi_q)$ by k. In particular, by (2.6) and (3.4), one has

$$\begin{aligned} & B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - \varepsilon O(1) \\ & < \varepsilon \int_a^b \int_a^b \frac{f_\varepsilon(x) g_\varepsilon(y)}{(u(x)+u(y))^\lambda} dx dy \\ & < \varepsilon k \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g_\varepsilon^q(x) dx \right\}^{\frac{1}{q}} = k, \end{aligned}$$

and then $B(\phi_p, \phi_q) \leq k$ $(\varepsilon \to 0^+)$. This contradicts the fact that $k < B(\phi_p, \phi_q)$. Hence the constant factor $B(\phi_p, \phi_q)$ in (3.1) is the best possible.

For $0 < p < 1$ (or $p < 0$), by the reverse of (2.2) and using the same procedures, one can obtain the reverse of (3.1). For $0 < \varepsilon < -q\phi_q$ (or $0 < \varepsilon < q\phi_p$), setting $f_\varepsilon(x)$ and $g_\varepsilon(x)$ as the above, we still have (3.4). If the constant factor $B(\phi_p, \phi_q)$ in the reverse of (3.1) is not the best possible, then, there exists a positive constant $K > B(\phi_p, \phi_q)$, such that the reverse of (3.1) is still valid if one replaces $B(\phi_p, \phi_q)$ by K. In particular, by (2.7) and (3.4), one has

$$\begin{aligned} & B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) \\ & > \varepsilon \int_a^b \int_a^b \frac{f_\varepsilon(x) g_\varepsilon(y)}{(u(x)+u(y))^\lambda} dx dy \\ & > \varepsilon K \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g_\varepsilon^q(x) dx \right\}^{\frac{1}{q}} = K, \end{aligned}$$

and then $B(\phi_p, \phi_q) \geq K$ $(\varepsilon \to 0^+)$. This contradiction concludes that the constant in the reverse of (3.1) is the best possible. The theorem is proved. □

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold.*

(i) *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, *one obtains the equivalent inequality of (3.1) as follows*

$$\int_a^b \frac{u'(y)}{(u(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda} dx \right]^p dy < [B(\phi_p, \phi_q)]^p \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx; \tag{3.5}$$

(ii) *If* $0 < p < 1$, *one obtains the reverse of (3.5) equivalent to the reverse of (3.1);*

(iii) *If* $p < 0$, *one obtains inequality (3.5) equivalent to the reverse of (3.1),*

where the constants in the above inequalities are all the best possible.

Proof. Set

$$g(y) := \frac{u'(y)}{(u(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda} dx \right]^{p-1},$$

and use (3.1) to obtain

$$\begin{aligned} 0 & < \int_a^b \frac{(u(y))^{q(1-\phi_p)-1}}{(u'(y))^{q-1}} g^q(y) dy \\ & = \int_a^b \frac{u'(y)}{(u(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda} dx \right]^p dy \\ & = \int_a^b \int_a^b \frac{f(x) g(y)}{(u(x)+u(y))^\lambda} dx dy \ \leq B(\phi_p, \phi_q) \\ & \quad \times \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{(u(y))^{q(1-\phi_p)-1}}{(u'(y))^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}}; \end{aligned} \tag{3.6}$$

$$\begin{aligned} 0 & < \left\{ \int_a^b \frac{(u(y))^{q(1-\phi_p)-1}}{(u'(y))^{q-1}} g^q(y) dy \right\}^{1-\frac{1}{q}} \\ & = \left\{ \int_a^b \frac{u'(y)}{(u(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda} dx \right]^p dy \right\}^{\frac{1}{p}} \\ & \leq B(\phi_p, \phi_q) \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} < \infty. \end{aligned} \tag{3.7}$$

It follows that (3.6) takes the form of strict inequality by using (3.1); so does (3.7). Hence one can get (3.5). On the other hand, if (3.5) is valid, by (2.2),

$$\begin{aligned}
&\int_a^b\int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda}dxdy\\
&\qquad = \int_a^b \left[\frac{(u'(y))^{\frac{1}{p}}}{(u(y))^{\frac{1}{p}-\phi_p}}\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda}dx\right]\left[\frac{(u(y))^{\frac{1}{p}-\phi_p}}{(u'(y))^{\frac{1}{p}}}g(y)\right]dy\\
&\qquad \le \left\{\int_a^b \frac{u'(y)}{(u(y))^{1-p\phi_p}}\left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda}dx\right]^p dy\right\}^{\frac{1}{p}}\\
&\qquad\qquad \times\left\{\int_a^b \frac{(u(y))^{q(1-\phi_p)-1}}{(u'(y))^{q-1}}g^q(y)dy\right\}^{\frac{1}{q}}. \qquad (3.8)
\end{aligned}$$

Hence by (3.5), (3.1) yields. It follows that (3.1) and (3.5) are equivalent.

If the constant factor in (3.5) is not the best possible, one can get a contradiction that the constant factor in (3.1) is not the best possible by using (3.8). Hence the constant factor in (3.5) is still the best possible.

If $0 < p < 1$ (or $p < 0$), one can get the reverses of (3.6), (3.7) and (3.8), and thus concludes the equivalence. By (3.6), for $0 < p < 1$, one can obtain the reverse of (3.5); for $p < 0$, one can get (3.5). If the constant factor in the reverse of (3.5) (or simply (3.5)) is not the best possible, then one can get a contradiction that the constant factor in the reverse of (3.1) is not the best possible by using the reverse of (3.8). Thus the theorem is proved. □

4. Some Particular Results

We point out that the constant factors in the following particular results of Theorems 3.1 – 3.2 are all the best possible.

4.1. The first reversible form.

Corollary 4.1. *Let the assumptions of Theorems 3.1 – 3.2 hold. For* $\phi_r = (1-\frac{1}{r})(\lambda-2)+1(r=p,q)$, $0 < \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}}f^p(x)dx < \infty$ *and* $0 < \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{q-1}}g^q(x)dx < \infty$, *setting* $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$,

(i) *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda > 2-\min\{p,q\}$, *then we have the following two equivalent inequalities:*

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda}dxdy < k_\lambda(p)\left\{\int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{q-1}}g^q(x)dx\right\}^{\frac{1}{q}} \qquad (4.1)$$

and

$$\int_a^b \frac{u'(y)}{(u(y))^{(p-1)(1-\lambda)}}\left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda}dx\right]^p dy < [k_\lambda(p)]^p\int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}}f^p(x)dx. \qquad (4.2)$$

(ii) *If* $0 < p < 1$ *and* $2-p < \lambda < 2-q$, *one obtains two equivalent reverses of (4.1) and (4.2),*

(iii) *If* $p < 0$ *and* $2-q < \lambda < 2-p$, *we have the reverse of (4.1) and the inequality (4.2), which are equivalent. In particular, by (4.1),*

(a) *setting* $u(x) = x^{\alpha}$ $(\alpha > 0, x \in (0, \infty))$, *one has*

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{(x^{\alpha}+y^{\alpha})^{\lambda}}dxdy < \frac{1}{\alpha}k_{\lambda}(p)\left\{\int_0^{\infty} x^{p-1+\alpha(2-\lambda-p)}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^{\infty} x^{q-1+\alpha(2-\lambda-q)}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.3}$$

(b) *setting* $u(x) = \ln x$, $x \in (1, \infty)$, *one has*

$$\int_1^{\infty}\int_1^{\infty} \frac{f(x)g(y)}{(\ln xy)^{\lambda}}dxdy < k_{\lambda}(p)\left\{\int_1^{\infty} x^{p-1}(\ln x)^{1-\lambda}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_1^{\infty} x^{q-1}(\ln x)^{1-\lambda}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.4}$$

(c) *setting* $u(x) = e^x$, $x \in (-\infty, \infty)$, *one has*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)}{(e^x+e^y)^{\lambda}}dxdy < k_{\lambda}(p)\left\{\int_{-\infty}^{\infty} e^{(2-p-\lambda)x}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_{-\infty}^{\infty} e^{(2-q-\lambda)x}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.5}$$

(d) *setting* $u(x) = \tan x$, $x \in (0, \frac{\pi}{2})$, *one has*

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\tan x+\tan y)^{\lambda}}dxdy < k_{\lambda}(p)\left\{\int_0^{\frac{\pi}{2}} \frac{\tan^{1-\lambda} x}{\sec^{2(p-1)} x}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{2}} \frac{\tan^{1-\lambda} x}{\sec^{2(q-1)} x}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.6}$$

(e) *setting* $u(x) = \sec x - 1$, $x \in (0, \frac{\pi}{2})$, *one has*

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\sec x+\sec y-2)^{\lambda}}dxdy < k_{\lambda}(p)\left\{\int_0^{\frac{\pi}{2}} \frac{(\sec x-1)^{1-\lambda}}{(\sec x\tan x)^{p-1}}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{2}} \frac{(\sec x-1)^{1-\lambda}}{(\sec x\tan x)^{q-1}}g^q(x)dx\right\}^{\frac{1}{q}}. \tag{4.7}$$

4.2. The second reversible form.

Corollary 4.2. *Let the assumptions of Theorems 3.1 – 3.2 hold. For* $\phi_r = \frac{\lambda-1}{2} + \frac{1}{r}$ $(r = p, q)$, $0 < \int_a^b \frac{(u(x))^{p\frac{1-\lambda}{2}}}{(u'(x))^{p-1}}f^p(x)dx < \infty$ *and* $0 < \int_a^b \frac{(u(x))^{q\frac{1-\lambda}{2}}}{(u'(x))^{q-1}}g^q(x)dx < \infty$, *setting* $\widetilde{k_{\lambda}}(p) = B\left(\frac{p\lambda-p+2}{2p}, \frac{q\lambda-q+2}{2q}\right)$,

(i) *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}$, *then one can get two equivalent inequalities as follows:*

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^{\lambda}}dxdy < \widetilde{k_{\lambda}}(p)\left\{\int_a^b \frac{(u(x))^{p\frac{1-\lambda}{2}}}{(u'(x))^{p-1}}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_a^b \frac{(u(x))^{q\frac{1-\lambda}{2}}}{(u'(x))^{q-1}}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.8}$$

(4.9) $$\int_a^b \frac{u'(y)}{(u(y))^{p\frac{1-\lambda}{2}}}\left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda}dx\right]^p dy < \left[\widetilde{k_\lambda}(p)\right]^p \int_a^b \frac{(u(x))^{p\frac{1-\lambda}{2}}}{(u'(x))^{p-1}} f^p(x)dx,$$

(ii) *If* $0 < p < 1$, $1-\frac{2}{p} < \lambda < 1-\frac{2}{q}$, *one can get two equivalent reversions of (4.8) and (4.9),*

(iii) *If* $p < 0$, $1-\frac{2}{q} < \lambda < 1-\frac{2}{p}$, *one can get the reversion of (4.8) and inequality (4.9), which are equivalent. In particular, by (4.8),*

(a) *setting* $u(x) = x^\alpha$ $(\alpha > 0, x \in (0,\infty))$, *one has*

(4.10) $$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda}dxdy < \frac{1}{\alpha}\widetilde{k_\lambda}(p)\left\{\int_0^\infty x^{p-1+\alpha(1-p\frac{1+\lambda}{2})} f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q-1+\alpha(1-q\frac{1+\lambda}{2})} g^q(x)dx\right\}^{\frac{1}{q}};$$

(b) *setting* $u(x) = \ln x$, $x \in (1,\infty)$, *one has*

(4.11) $$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln xy)^\lambda}dxdy < \widetilde{k_\lambda}(p)\left\{\int_1^\infty x^{p-1}(\ln x)^{p\frac{1-\lambda}{2}} f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_1^\infty x^{q-1}(\ln x)^{q\frac{1-\lambda}{2}} g^q(x)dx\right\}^{\frac{1}{q}};$$

(c) *setting* $u(x) = e^x$, $x \in (-\infty,\infty)$, *one has*

(4.12) $$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(e^x + e^y)^\lambda}dxdy < \widetilde{k_\lambda}(p)\left\{\int_{-\infty}^\infty e^{(1-p\frac{1+\lambda}{2})x} f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_{-\infty}^\infty e^{(1-q\frac{1+\lambda}{2})x} g^q(x)dx\right\}^{\frac{1}{q}};$$

(d) *setting* $u(x) = \tan x$, $x \in (0,\frac{\pi}{2})$, *one has*

(4.13) $$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\tan x + \tan y)^\lambda}dxdy < \widetilde{k_\lambda}(p)\left\{\int_0^{\frac{\pi}{2}} \frac{\tan^{p\frac{1-\lambda}{2}} x}{\sec^{2(p-1)} x} f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{2}} \frac{\tan^{q\frac{1-\lambda}{2}} x}{\sec^{2(q-1)} x} g^q(x)dx\right\}^{\frac{1}{q}};$$

(e) *setting* $u(x) = \sec x - 1$, $x \in (0,\frac{\pi}{2})$, *one has*

(4.14) $$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\sec x + \sec y - 2)^\lambda}dxdy < \widetilde{k_\lambda}(p)\left\{\int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{p\frac{1-\lambda}{2}}}{(\sec x \tan x)^{p-1}} f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{q\frac{1-\lambda}{2}}}{(\sec x \tan x)^{q-1}} g^q(x)dx\right\}^{\frac{1}{q}}.$$

4.3. **The form which does not have a reverse.**

Corollary 4.3. *Let the assumptions of Theorems 3.1 – 3.2 hold. For* $\phi_r = \frac{\lambda}{r}(r = p,q)$, *if* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \lambda > 0$, $0 < \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx < \infty$ *and* $0 < \int_a^b \frac{(u(x))^{(q-1)(1-\lambda)}}{(u'(x))^{q-1}} g^q(x)dx <$

∞, *then one can get two equivalent inequalities as:*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_a^b \frac{(u(x))^{(q-1)(1-\lambda)}}{(u'(x))^{q-1}} g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.15}$$

$$\int_a^b \frac{u'(y)}{(u(y))^{1-\lambda}} \left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda} dx\right]^p dy < \left[B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\right]^p \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x)dx. \tag{4.16}$$

In particular, by (4.15),

(a) *setting* $u(x) = x^\alpha (\alpha > 0; x \in (0, \infty))$, *one has*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dxdy < \frac{1}{\alpha} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\int_0^\infty x^{(p-1)(1-\alpha\lambda)} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{(q-1)(1-\alpha\lambda)} g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.17}$$

(b) *setting* $u(x) = \ln x, x \in (1, \infty)$, *one has*

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln xy)^\lambda} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\int_1^\infty x^{p-1} (\ln x)^{(p-1)(1-\lambda)} f^p(x)dx\right\}^{\frac{1}{p}} \times \left\{\int_1^\infty x^{q-1} (\ln x)^{(q-1)(1-\lambda)} g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.18}$$

(c) *setting* $u(x) = e^x, x \in (-\infty, \infty)$, *one has*

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(e^x + e^y)^\lambda} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\int_{-\infty}^\infty e^{(1-p)\lambda x} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_{-\infty}^\infty e^{(1-q)\lambda x} g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.19}$$

(d) *setting* $u(x) = \tan x, x \in (0, \frac{\pi}{2})$, *one has*

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\tan x + \tan y)^\lambda} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\int_0^{\frac{\pi}{2}} \frac{\tan^{(p-1)(1-\lambda)} x}{\sec^{2(p-1)} x} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^{\frac{\pi}{2}} \frac{\tan^{(q-1)(1-\lambda)} x}{\sec^{2(q-1)} x} g^q(x)dx\right\}^{\frac{1}{q}}; \tag{4.20}$$

(e) *setting* $u(x) = \sec x - 1, x \in (0, \frac{\pi}{2})$, *one has*

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\sec x + \sec y - 2)^{\lambda}} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{ \int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{(p-1)(1-\lambda)}}{(\sec x \tan x)^{p-1}} f^p(x)dx \right\}^{\frac{1}{p}} \times \left\{ \int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{(q-1)(1-\lambda)}}{(\sec x \tan x)^{q-1}} g^q(x)dx \right\}^{\frac{1}{q}}. \tag{4.21}$$

Remark 4.4. For $\alpha = 1$, (4.3) reduces to (1.5). For $\lambda = 1$, inequalities (4.3), (4.10) and (4.17) reduce to (1.7), and inequalities (4.4), (4.11) and (4.18) reduce to (1.4). It follows that inequality (3.5) is a relation between (1.4) and (1.7)(ro (1.1)) with a parameter λ. Still for $\lambda = 1$, (4.5), (4.12) and (4.19) reduce to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{e^x + e^y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_{-\infty}^{\infty} e^{(1-p)x} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{\infty} e^{(1-q)x} g^q(x)dx \right\}^{\frac{1}{q}}, \tag{4.22}$$

(4.6), (4.13) and (4.20) reduce to

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{\tan x + \tan y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^{\frac{\pi}{2}} \cos^{2(p-1)} x f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{\pi}{2}} \cos^{2(q-1)} x g^q(x)dx \right\}^{\frac{1}{q}}, \tag{4.23}$$

and (4.7), (4.14) and (4.21) reduce to

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{\sec x + \sec y - 2} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^{\frac{\pi}{2}} \left(\frac{\cos^2 x}{\sin x}\right)^{p-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{\pi}{2}} \left(\frac{\cos^2 x}{\sin x}\right)^{q-1} g^q(x)dx \right\}^{\frac{1}{q}}. \tag{4.24}$$

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