SOME GENERALIZED INEQUALITIES INVOLVING THE q-GAMMA FUNCTION

J. K. PRAJAPAT
Department of Mathematics
Central University of Rajasthan
16, Nav Durga Colony, Opposite Hotel Clarks Amer,
J. L. N. Marg, Jaipur-302017, Rajasthan, India
EMail: jkp_0007@rediffmail.com

S. KANT
Department of Mathematics
Government Dungar College
Bikaner-334001,
Rajasthan, India.
EMail: drskant.2007@yahoo.com

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1. Introduction and Preliminary Results

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$, by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \tag{1.1}$$

and the Psi (or digamma) function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \qquad (x > 0). \tag{1.2}$$

The q-psi function is defined for $0 < q < 1$, by

$$\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x), \tag{1.3}$$

where the q-gamma function $\Gamma_q(x)$ is defined by $(0 < q < 1)$

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{i=1}^{\infty} \frac{1-q^i}{1-q^{x+i}}. \tag{1.4}$$

Many properties of the q-gamma function were derived by Askey [2]. The explicit form of the q-psi function $\psi_q(x)$ is

$$\psi_q(x) = -\log(1-q) + \log q \sum_{i=0}^{\infty} \frac{q^{x+i}}{1-q^{x+i}}. \tag{1.5}$$

In particular

$$\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x) \qquad \text{and} \qquad \lim_{q\to 1^-} \psi_q(x) = \psi(x).$$

For the gamma function Alsina and Thomas [1] proved the following double inequality:

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Theorem 1.1. *For all $x \in [0,1]$, and all nonnegative integers n, the following double inequality holds true*

$$\frac{1}{n!} \leq \frac{[\Gamma(1+x)]^n}{\Gamma(1+nx)} \leq 1. \tag{1.6}$$

Sándor [4] and Shabani [5] proved the following generalizations of (1.6) given by Theorem 1.2 and Theorem 1.3 respectively.

Theorem 1.2. *For all $a \geq 1$ and all $x \in [0,1]$, one has*

$$\frac{1}{\Gamma(1+a)} \leq \frac{[\Gamma(1+x)]^a}{\Gamma(1+ax)} \leq 1. \tag{1.7}$$

Theorem 1.3. *Let $a \geq b > 0$, c, d be positive real numbers such that $bc \geq ad > 0$ and $\psi(b+ax) > 0$, where $x \in [0,1]$. Then the following double inequality holds:*

$$\frac{[\Gamma(a)]^c}{[\Gamma(b)]^d} \leq \frac{[\Gamma(a+bx)]^c}{[\Gamma(b+ax)]^d} \leq [\Gamma(a+b)]^{c-d}. \tag{1.8}$$

Recently, Mansour [3] extended above gamma function inequalities to the case of $\Gamma_q(x)$, given by Theorem 1.4, below:

Theorem 1.4. *Let $x \in [0,1]$ and $q \in (0,1)$. If $a \geq b > 0$, c, d are positive real numbers with $bc \geq ad > 0$ and $\psi_q(b+ax) > 0$, then*

$$\frac{[\Gamma_q(a)]^c}{[\Gamma_b(b)]^d} \leq \frac{[\Gamma_q(a+bx)]^c}{[\Gamma_q(b+ax)]^d} \leq [\Gamma_q(a+b)]^{c-d}. \tag{1.9}$$

In our investigation we shall require the following lemmas:

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Lemma 1.5. *Let $q \in (0,1)$, $\alpha > 0$ and a, b be any two positive real numbers such that $a \geq b$. Then*

$$\psi_q(a\alpha + bx) \geq \psi_q(b\alpha + ax) \qquad x \in [0, \alpha], \tag{1.10}$$

and

$$\psi_q(a\alpha + bx) \leq \psi_q(b\alpha + ax) \qquad x \in [\alpha, \infty). \tag{1.11}$$

Proof. By using (1.5), we have

$$\begin{aligned}
\psi_q(a\alpha + bx) - \psi_q(b\alpha + ax) &= \log q \sum_{i=0}^{\infty} \left(\frac{q^{a\alpha+bx+i}}{1 - q^{a\alpha+bx+i}} - \frac{q^{b\alpha+ax+i}}{1 - q^{b\alpha+ax+i}} \right) \\
&= \log q \sum_{i=0}^{\infty} \frac{q^i \left(q^{a\alpha+bx} - q^{b\alpha+ax} \right)}{(1 - q^{a\alpha+bx+i})(1 - q^{b\alpha+ax+i})} \\
&= \log q \sum_{i=0}^{\infty} \frac{q^{b(x+\alpha)+i} \left(q^{(a-b)\alpha} - q^{(a-b)x} \right)}{(1 - q^{a\alpha+bx+i})(1 - q^{b\alpha+ax+i})}.
\end{aligned}$$

Since for $0 < q < 1$, we have $\log q < 0$. In addition, for $a \geq b$, $x \in [0, \alpha]$, we get $(1 - q^{a\alpha+bx+i}) > 0$, $(1 - q^{b\alpha+ax+i}) > 0$ and $q^{(a-b)\alpha} \leq q^{(a-b)x}$. Hence

$$\psi_q(a\alpha + bx) \geq \psi_q(b\alpha + ax) \qquad x \in [0, \alpha].$$

Furthermore, for $a \geq b$ and $x \in [\alpha, \infty)$, we have $(1 - q^{a\alpha+bx+i}) > 0$, $(1 - q^{b\alpha+ax+i}) > 0$ and $q^{(a-b)\alpha} \geq q^{(a-b)x}$. Hence

$$\psi_q(a\alpha + bx) \leq \psi_q(b\alpha + ax) \qquad x \in [\alpha, \infty).$$

which completes the proof. □

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Lemma 1.6. *Let* $x \in [0, \alpha]$, $\alpha > 0$ *and* $q \in (0, 1)$. *If* a, b, c, d *are positive real numbers such that* $a \geq b$ *and* $[bc \geq ad, \psi_q(b\alpha + ax) > 0]$ *or* $[bc \leq ad, \psi_q(a\alpha + bx) < 0]$, *we have*

$$bc\psi_q(a\alpha + bx) - ad\psi_q(b\alpha + ax) \geq 0. \tag{1.12}$$

Proof. Since $bc \geq ad$ and $\psi_q(b\alpha + ax) > 0$, then using (1.10), we obtain

$$\begin{aligned} ad\psi_q(b\alpha + ax) &\leq bc\psi_q(b\alpha + ax) \\ &\leq bc\psi_q(a\alpha + bx). \end{aligned}$$

Similarly, when $bc \leq ad$ and $\psi_q(a\alpha + bx) < 0$, we have

$$bc\psi_q(a\alpha + bx) \geq ad\psi_q(a\alpha + bx) \geq ad\psi_q(b\alpha + ax).$$

This proves Lemma 1.6. □

Similarly, using (1.11) and a similar proof to that above, we have the following lemma:

Lemma 1.7. *Let* $q \in (0, 1)$ *and* $x \in [\alpha, \infty)$, $\alpha > 0$. *If* a, b, c, d *are positive real numbers such that* $a \geq b$ *and* $[bc \geq ad, \psi_q(b\alpha + ax) < 0]$ *or* $[bc \leq ad, \psi_q(a\alpha + bx) < 0]$, *we have*

$$bc\psi_q(a\alpha + bx) - ad\psi_q(b\alpha + ax) \leq 0. \tag{1.13}$$

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2. Main Results

In this section we will establish some generalized double inequalities involving the q- gamma function.

Theorem 2.1. *For all* $q \in (0,1)$, $x \in [0,\alpha]$, $\alpha > 0$ *and positive real numbers* a,b,c,d *such that* $a \geq b$ *and* $[bc \geq ad, \psi_q(b\alpha + ax) > 0]$ *or* $[bc \leq ad, \psi_q(a\alpha + bx) < 0]$, *we have*

$$\frac{[\Gamma_q(a\alpha)]^c}{[\Gamma_q(b\alpha)]^d} \leq \frac{[\Gamma_q(a\alpha + bx)]^c}{[\Gamma_q(b\alpha + ax)]^d} \leq [\Gamma_q\{(a+b)\alpha\}]^{c-d}. \tag{2.1}$$

Proof. Let

$$f(x) = \frac{[\Gamma_q(a\alpha + bx)]^c}{[\Gamma_q(b\alpha + ax)]^d}, \tag{2.2}$$

and assume that $g(x)$ is a function defined by $g(x) = \log f(x)$. Then

$$g(x) = c \log \Gamma_q(a\alpha + bx) - d \log \Gamma_q(b\alpha + ax),$$

so

$$\begin{aligned} g'(x) &= bc\frac{\Gamma_q'(a\alpha + bx)}{\Gamma_q(a\alpha + bx)} - ad\frac{\Gamma_q'(b\alpha + ax)}{\Gamma_q(b\alpha + ax)} \\ &= bc\psi_q(a\alpha + bx) - ad\psi_q(b\alpha + ax). \end{aligned}$$

Thus using Lemma 1.6, we have $g'(x) \geq 0$. This means that $g(x)$ is an increasing function in $[0,\alpha]$, which implies that the function $f(x)$ is also an increasing function in $[0,\alpha]$, so that

$$f(0) \leq f(x) \leq f(\alpha), \qquad x \in [0,\alpha],$$

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and this is equivalent to

$$\frac{[\Gamma_q(a\alpha)]^c}{[\Gamma_q(b\alpha)]^d} \le \frac{[\Gamma_q(a\alpha+bx)]^c}{[\Gamma_q(b\alpha+ax)]^d} \le [\Gamma_q\{(a+b)\alpha\}]^{c-d}.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *For all* $q \in (0,1)$, $x \in [\alpha, \infty)$, $\alpha > 0$ *and positive real numbers* a, b, c, d *such that* $a \ge b$ *and* $[bc \ge ad, \psi_q(b\alpha + ax) < 0]$ *or* $[bc \le ad, \psi_q(a\alpha + bx) > 0]$, *we have*

$$\frac{[\Gamma_q(a\alpha+bx)]^c}{[\Gamma_q(b\alpha+ax)]^d} \le [\Gamma_q(a+b)\alpha]^{c-d} \tag{2.3}$$

and

$$\frac{[\Gamma_q(a\alpha+bx)]^c}{[\Gamma_q(b\alpha+ax)]^d} \le \frac{[\Gamma_q(a\alpha+by)]^c}{[\Gamma_q(b\alpha+ay)]^d}, \qquad \alpha < y < x. \tag{2.4}$$

Proof. Applying Lemma 1.7 and an argument similiar to that of Theorem 2.1, we see that the function $f(x)$ defined by (2.2) is a decreasing function. Therefore we have

$$f(x) \le f(\alpha), \qquad x \in [\alpha, \infty),$$

which gives the desired result. □

Remark 1.

(i) Taking $\alpha = 1$, Theorem 2.1 and Theorem 2.2 yield the results obtained by Mansour [3].

(ii) Taking $\alpha = 1$ and $q \to 1^-$, Theorem 2.1 and Theorem 2.2 yield the results obtained by Shabani [5].

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