



## MATRIX TRACE INEQUALITIES ON THE TSALLIS ENTROPIES

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*Received 28 June, 2007; accepted 29 January, 2008*

*Communicated by F. Zhang*

*Dedicated to Professor Kunio Oshima on his 60th birthday.*

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**ABSTRACT.** Maximum entropy principles in nonextensive statistical physics are revisited as an application of the Tsallis relative entropy defined for non-negative matrices in the framework of matrix analysis. In addition, some matrix trace inequalities related to the Tsallis relative entropy are studied.

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*Key words and phrases:* Matrix trace inequality, Tsallis entropy, Tsallis relative entropy and maximum entropy principle.

*2000 Mathematics Subject Classification.* 47A63, 94A17, 15A39.

### 1. INTRODUCTION

In 1988, Tsallis introduced the one-parameter extended entropy for the analysis of a physical model in statistical physics [10]. In our previous papers, we studied the properties of the Tsallis relative entropy [5, 4] and the Tsallis relative operator entropy [17, 6]. The problems on the maximum entropy principle in Tsallis statistics have been studied for classical systems and quantum systems [9, 11, 2, 1]. Such problems were solved by the use of the Lagrange multipliers formalism. We give a new approach to such problems, that is, we solve them by applying the non-negativity of the Tsallis relative entropy without using the Lagrange multipliers formalism. In addition, we show further results on the Tsallis relative entropy.

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The author would like to thank the reviewer for providing valuable comments to improve the manuscript. The author would like to thank Professor K. Yanagi and Professor K. Kuriyama for providing valuable comments and constant encouragement. This work was supported by the Japanese Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Encouragement of Young Scientists (B), 17740068. This work was also partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (B), 18300003.

In the present paper, the set of  $n \times n$  complex matrices is denoted by  $M_n(\mathbb{C})$ . That is, we deal with  $n \times n$  matrices because of Lemma 2.2 in Section 2. However some results derived in the present paper also hold for the infinite dimensional case. In the sequel, the set of all density matrices (quantum states) is represented by

$$D_n(\mathbb{C}) \equiv \{X \in M_n(\mathbb{C}) : X \geq 0, \text{Tr}[X] = 1\}.$$

$X \in M_n(\mathbb{C})$  is called by a non-negative matrix and denoted by  $X \geq 0$ , if we have  $\langle Xx, x \rangle \geq 0$  for all  $x \in \mathbb{C}^n$ . That is, for a Hermitian matrix  $X$ ,  $X \geq 0$  means that all eigenvalues of  $X$  are non-negative. In addition,  $X \geq Y$  is defined by  $X - Y \geq 0$ . For  $-I \leq X \leq I$  and  $\lambda \in (-1, 0) \cup (0, 1)$ , we denote the generalized exponential function by  $\exp_\lambda(X) \equiv (I + \lambda X)^{1/\lambda}$ . As the inverse function of  $\exp_\lambda(\cdot)$ , for  $X \geq 0$  and  $\lambda \in (-1, 0) \cup (0, 1)$ , we denote the generalized logarithmic function by  $\ln_\lambda X \equiv \frac{X^\lambda - I}{\lambda}$ . Then the Tsallis relative entropy and the Tsallis entropy for non-negative matrices  $X$  and  $Y$  are defined by

$$D_\lambda(X|Y) \equiv \text{Tr} [X^{1-\lambda} (\ln_\lambda X - \ln_\lambda Y)], \quad S_\lambda(X) \equiv -D_\lambda(X|I).$$

These entropies are generalizations of the von Neumann entropy [16] and of the Umegaki relative entropy [14] in the sense that

$$\lim_{\lambda \rightarrow 0} S_\lambda(X) = S_0(X) \equiv -\text{Tr}[X \log X]$$

and

$$\lim_{\lambda \rightarrow 0} D_\lambda(X|Y) = D_0(X|Y) \equiv \text{Tr}[X(\log X - \log Y)].$$

## 2. MAXIMUM ENTROPY PRINCIPLE IN NONEXTENSIVE STATISTICAL PHYSICS

In this section, we study the maximization problem of the Tsallis entropy with a constraint on the  $\lambda$ -expectation value. In quantum systems, the expectation value of an observable (a Hermitian matrix)  $H$  in a quantum state (a density matrix)  $X \in D_n(\mathbb{C})$  is written as  $\text{Tr}[XH]$ . Here, we consider the  $\lambda$ -expectation value  $\text{Tr}[X^{1-\lambda}H]$  as a generalization of the usual expectation value. Firstly, we impose the following constraint on the maximization problem of the Tsallis entropy:

$$\widetilde{C}_\lambda \equiv \{X \in D_n(\mathbb{C}) : \text{Tr}[X^{1-\lambda}H] = 0\},$$

for a given  $n \times n$  Hermitian matrix  $H$ . We denote a usual matrix norm by  $\|\cdot\|$ , namely for  $A \in M_n(\mathbb{C})$  and  $x \in \mathbb{C}^n$ ,

$$\|A\| \equiv \max_{\|x\|=1} \|Ax\|.$$

Then we have the following theorem.

**Theorem 2.1.** *Let  $Y = Z_\lambda^{-1} \exp_\lambda(-H/\|H\|)$ , where  $Z_\lambda \equiv \text{Tr}[\exp_\lambda(-H/\|H\|)]$ , for an  $n \times n$  Hermitian matrix  $H$  and  $\lambda \in (-1, 0) \cup (0, 1)$ . If  $X \in \widetilde{C}_\lambda$ , then  $S_\lambda(X) \leq -c_\lambda \ln_\lambda Z_\lambda^{-1}$ , where  $c_\lambda \equiv \text{Tr}[X^{1-\lambda}]$ .*

*Proof.* Since  $Z_\lambda \geq 0$  and we have  $\ln_\lambda(x^{-1}Y) = \ln_\lambda Y + (\ln_\lambda x^{-1})Y^\lambda$  for a non-negative matrix  $Y$  and scalar  $x$ , we calculate

$$\begin{aligned} \text{Tr}[X^{1-\lambda} \ln_\lambda Y] &= \text{Tr}[X^{1-\lambda} \ln_\lambda \{Z_\lambda^{-1} \exp_\lambda(-H/\|H\|)\}] \\ &= \text{Tr}[X^{1-\lambda} \{-H/\|H\| + \ln_\lambda Z_\lambda^{-1} (I - \lambda H/\|H\|)\}] \\ &= \text{Tr}[X^{1-\lambda} \{\ln_\lambda Z_\lambda^{-1} I - Z_\lambda^{-\lambda} H/\|H\|\}] = c_\lambda \ln_\lambda Z_\lambda^{-1}, \end{aligned}$$

since  $\ln_\lambda Z_\lambda^{-1} = \frac{Z_\lambda^{-\lambda}-1}{\lambda}$  by the definition of the generalized logarithmic function  $\ln_\lambda(\cdot)$ . By the non-negativity of the Tsallis relative entropy:

$$(2.1) \quad \text{Tr}[X^{1-\lambda} \ln_\lambda Y] \leq \text{Tr}[X^{1-\lambda} \ln_\lambda X],$$

we have

$$S_\lambda(X) = -\text{Tr}[X^{1-\lambda} \ln_\lambda X] \leq -\text{Tr}[X^{1-\lambda} \ln_\lambda Y] = -c_\lambda \ln_\lambda Z_\lambda^{-1}.$$

□

Next, we consider the slightly changed constraint:

$$C_\lambda \equiv \{X \in D_n(\mathbb{C}) : \text{Tr}[X^{1-\lambda} H] \leq \text{Tr}[Y^{1-\lambda} H] \quad \text{and} \quad \text{Tr}[X^{1-\lambda}] \leq \text{Tr}[Y^{1-\lambda}]\}$$

for a given  $n \times n$  Hermitian matrix  $H$ , as the maximization problem for the Tsallis entropy. To this end, we prepare the following lemma.

**Lemma 2.2.** *For a given  $n \times n$  Hermitian matrix  $H$ , if  $n$  is a sufficiently large integer, then we have  $Z_\lambda \geq 1$ .*

*Proof.*

(i) For a fixed  $0 < \lambda < 1$  and a sufficiently large  $n$ , we have

$$(2.2) \quad (1/n)^\lambda \leq 1 - \lambda.$$

From the inequalities  $-\|H\| I \leq H \leq \|H\| I$ , we have

$$(2.3) \quad (1 - \lambda)^{\frac{1}{\lambda}} I \leq \exp_\lambda(-H/\|H\|) \leq (1 + \lambda)^{\frac{1}{\lambda}} I.$$

By inequality (2.2), we have

$$\frac{1}{n} I \leq (1 - \lambda)^{\frac{1}{\lambda}} I \leq \exp_\lambda(-H/\|H\|),$$

which implies  $Z_\lambda \geq 1$ .

(ii) For a fixed  $-1 < \lambda < 0$  and a sufficiently large  $n$ , we have

$$(2.4) \quad (1/n)^\lambda \geq 1 - \lambda.$$

Analogously to (i), we have inequalities (2.3) for  $-1 < \lambda < 0$ . By inequality (2.4), we have

$$\frac{1}{n} I \leq (1 - \lambda)^{\frac{1}{\lambda}} I \leq \exp_\lambda(-H/\|H\|),$$

which implies  $Z_\lambda \geq 1$ .

□

Then we have the following theorem by the use of Lemma 2.2.

**Theorem 2.3.** *Let  $Y = Z_\lambda^{-1} \exp_\lambda(-H/\|H\|)$ , where  $Z_\lambda \equiv \text{Tr}[\exp_\lambda(-H/\|H\|)]$ , for  $\lambda \in (-1, 0) \cup (0, 1)$  and an  $n \times n$  Hermitian matrix  $H$ . If  $X \in C_\lambda$  and  $n$  is sufficiently large, then  $S_\lambda(X) \leq S_\lambda(Y)$ .*

*Proof.* Due to Lemma 2.2, we have  $\ln_\lambda Z_\lambda^{-1} \leq 0$  for a sufficiently large  $n$ . Thus we have  $\ln_\lambda Z_\lambda^{-1} \text{Tr}[X^{1-\lambda}] \geq \ln_\lambda Z_\lambda^{-1} \text{Tr}[Y^{1-\lambda}]$  for  $X \in C_\lambda$ . Similarly to the proof of Theorem 2.1, we have

$$\begin{aligned} \text{Tr}[X^{1-\lambda} \ln_\lambda Y] &= \text{Tr}[X^{1-\lambda} \ln_\lambda \{Z_\lambda^{-1} \exp_\lambda(-H/\|H\|)\}] \\ &= \text{Tr}[X^{1-\lambda} \{-H/\|H\| + \ln_\lambda Z_\lambda^{-1} (I - \lambda H/\|H\|)\}] \\ &= \text{Tr}[X^{1-\lambda} \{\ln_\lambda Z_\lambda^{-1} I - Z_\lambda^{-\lambda} H/\|H\|\}] \\ &\geq \text{Tr}[Y^{1-\lambda} \{\ln_\lambda Z_\lambda^{-1} I - Z_\lambda^{-\lambda} H/\|H\|\}] \\ &= \text{Tr}[Y^{1-\lambda} \{-H/\|H\| + \ln_\lambda Z_\lambda^{-1} (I - \lambda H/\|H\|)\}] \\ &= \text{Tr}[Y^{1-\lambda} \ln_\lambda \{Z_\lambda^{-1} \exp_\lambda(-H/\|H\|)\}] \\ &= \text{Tr}[Y^{1-\lambda} \ln_\lambda Y]. \end{aligned}$$

By Eq.(2.1) we have

$$S_\lambda(X) = -\text{Tr}[X^{1-\lambda} \ln_\lambda X] \leq -\text{Tr}[X^{1-\lambda} \ln_\lambda Y] \leq -\text{Tr}[Y^{1-\lambda} \ln_\lambda Y] = S_\lambda(Y).$$

□

**Remark 2.4.** Since  $-x^{1-\lambda} \ln_\lambda x$  is a strictly concave function,  $S_\lambda$  is a strictly concave function on the set  $C_\lambda$ . This means that the maximizing  $Y$  is uniquely determined so that we may regard  $Y$  as a generalized Gibbs state, since an original Gibbs state  $e^{-\beta H} / \text{Tr}[e^{-\beta H}]$ , where  $\beta \equiv 1/T$  and  $T$  represents a physical temperature, gives the maximum value of the von Neumann entropy. Thus, we may define a generalized Helmholtz free energy by

$$F_\lambda(X, H) \equiv \text{Tr}[X^{1-\lambda} H] - \|H\| S_\lambda(X).$$

This can be also represented by the Tsallis relative entropy such as

$$F_\lambda(X, H) = \|H\| D_\lambda(X|Y) + \ln_\lambda Z_\lambda^{-1} \text{Tr}[X^{1-\lambda} (\|H\| - \lambda H)].$$

The following corollary easily follows by taking the limit as  $\lambda \rightarrow 0$ .

**Corollary 2.5** ([12, 15]). *Let  $Y = Z_0^{-1} \exp(-H/\|H\|)$ , where  $Z_0 \equiv \text{Tr}[\exp(-H/\|H\|)]$ , for an  $n \times n$  Hermitian matrix  $H$ .*

- (i) *If  $X \in \widetilde{C}_0$ , then  $S_0(X) \leq \log Z_0$ .*
- (ii) *If  $X \in C_0$ , then  $S_0(X) \leq S_0(Y)$ .*

### 3. ON SOME TRACE INEQUALITIES RELATED TO THE TSALLIS RELATIVE ENTROPY

In this section, we consider an extension of the following inequality [8]:

$$(3.1) \quad \text{Tr}[X(\log X + \log Y)] \leq \frac{1}{p} \text{Tr}[X \log X^{p/2} Y^p X^{p/2}]$$

for non-negative matrices  $X$  and  $Y$ , and  $p > 0$ .

For the proof of the following Theorem 3.3, we use the following famous inequalities.

**Lemma 3.1** ([8]). *For any Hermitian matrices  $A$  and  $B$ ,  $0 \leq \lambda \leq 1$  and  $p > 0$ , we have the inequality:*

$$\text{Tr} \left[ \left( e^{pA} \#_{\lambda} e^{pB} \right)^{1/p} \right] \leq \text{Tr} \left[ e^{(1-\lambda)A + \lambda B} \right],$$

where the  $\lambda$ -geometric mean for positive matrices  $A$  and  $B$  is defined by

$$A \#_{\lambda} B \equiv A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^{\lambda} A^{1/2}.$$

**Lemma 3.2** ([7, 13]). *For any Hermitian matrices  $G$  and  $H$ , we have the Golden-Thompson inequality:*

$$\text{Tr} \left[ e^{G+H} \right] \leq \text{Tr} \left[ e^G e^H \right].$$

**Theorem 3.3.** *For positive matrices  $X$  and  $Y$ ,  $p \geq 1$  and  $0 < \lambda \leq 1$ , we have*

$$(3.2) \quad D_{\lambda}(X|Y) \leq -\text{Tr}[X \ln_{\lambda}(X^{-p/2} Y^p X^{-p/2})^{1/p}].$$

*Proof.* First of all, we note that we have the following inequality [3]

$$(3.3) \quad \text{Tr}[(Y^{1/2} X Y^{1/2})^{rp}] \geq \text{Tr}[(Y^{r/2} X^r Y^{r/2})^p]$$

for non-negative matrices  $X$  and  $Y$ , and  $0 \leq r \leq 1$ ,  $p > 0$ . Similar to the proof of Theorem 2.2 in [5], inequality (3.2) easily follows by setting  $A = \log X$  and  $B = \log Y$  in Lemma 3.1 such that

$$(3.4) \quad \begin{aligned} \text{Tr}[(X^p \#_{\lambda} Y^p)^{1/p}] &\leq \text{Tr}[e^{\log X^{1-\lambda} + \log Y^{\lambda}}] \\ &\leq \text{Tr}[e^{\log X^{1-\lambda}} e^{\log Y^{\lambda}}] \\ &= \text{Tr}[X^{1-\lambda} Y^{\lambda}], \end{aligned}$$

by Lemma 3.2. In addition, we have

$$(3.5) \quad \text{Tr}[X^r Y^r] \leq \text{Tr}[(Y^{1/2} X Y^{1/2})^r], \quad (0 \leq r \leq 1),$$

on taking  $p = 1$  of inequality (3.3). By (3.4) and (3.5) we obtain:

$$\begin{aligned} \text{Tr}[(X^p \#_{\lambda} Y^p)^{1/p}] &= \text{Tr} \left[ \left\{ X^{p/2} (X^{-p/2} Y^p X^{-p/2})^{\lambda} X^{p/2} \right\}^{1/p} \right] \\ &\geq \text{Tr}[X (X^{-p/2} Y^p X^{-p/2})^{\lambda/p}]. \end{aligned}$$

Thus we have,

$$\begin{aligned}
 D_\lambda(X|Y) &= \frac{\text{Tr}[X - X^{1-\lambda}Y^\lambda]}{\lambda} \\
 &\leq \frac{\text{Tr}[X - X(X^{-p/2}Y^pX^{-p/2})^{\lambda/p}]}{\lambda} \\
 &= -\frac{\text{Tr}[X \{((X^{-p/2}Y^pX^{-p/2})^{1/p})^\lambda - I\}]}{\lambda} \\
 &= -\text{Tr}[X \ln_\lambda(X^{-p/2}Y^pX^{-p/2})^{1/p}].
 \end{aligned}$$

□

**Remark 3.4.** For positive matrices  $X$  and  $Y$ ,  $0 < p < 1$  and  $0 < \lambda \leq 1$ , the following inequality does not hold in general:

$$(3.6) \quad D_\lambda(X|Y) \leq -\text{Tr}[X \ln_\lambda(X^{-p/2}Y^pX^{-p/2})^{1/p}].$$

Indeed, the inequality (3.6) is equivalent to

$$(3.7) \quad \text{Tr}[X(X^{-p/2}Y^pX^{-p/2})^{\lambda/p}] \leq \text{Tr}[X^{1-\lambda}Y^\lambda].$$

Then we have many counter-examples. If we set  $p = 0.3$ ,  $\lambda = 0.9$  and  $X = \begin{pmatrix} 10 & 3 \\ 3 & 9 \end{pmatrix}$ ,  $Y = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ , then inequality (3.7) fails. (R.H.S. minus L.H.S. of (3.7) approximately becomes -0.00309808.) Thus, inequality (3.6) is not true in general.

### Corollary 3.5.

(i) For positive matrices  $X$  and  $Y$ , the trace inequality

$$D_\lambda(X|Y) \leq -\text{Tr}[X \ln_\lambda(X^{-1/2}YX^{-1/2})]$$

holds.

(ii) For positive matrices  $X$  and  $Y$ , and  $p \geq 1$ , we have inequality (3.1).

*Proof.*

(i) Put  $p = 1$  in (1) of Theorem 3.3.

(ii) Take the limit as  $\lambda \rightarrow 0$ .

□

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