Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 2, Article 75, 2006

AN ITERATIVE METHOD FOR NONCONVEX EQUILIBRIUM PROBLEMS

MESSAOUD BOUNKHEL AND BUSHRA R. AL-SENAN

DEPARTMENT OF MATHEMATICS
KING SAUD UNIVERSITY
RIYADH 11451, SAUDI ARABIA.
bounkhel@ksu.edu.sa

Received 12 October, 2004; accepted 07 April, 2006
Communicated by A.M. Rubinov

ABSTRACT. Using some recent results from nonsmooth analysis, we prove the convergence of a new iterative scheme to a solution of a nonconvex equilibrium problem.

Key words and phrases: Uniform prox-regularity, Uniform regularity over sets, Strong monotonicity.

2000 *Mathematics Subject Classification.* Primary 47B47, 47A30, 47B20; Secondary 47B10.

1. INTRODUCTION

Equilibrium problems theory is an important branch of mathematical sciences which has a wide range of applications in economics, operations research, industrial, physical, and engineering sciences. Many research papers have lately been written, both on the theory and applications of this field (see for instance [8, 10] and the references therein).

One of the typical formulations of equilibrium problems found in the literature is the following:

(EP) $$\text{Find } \bar{x} \in C \quad \text{such that} \quad F(\bar{x}, x) \geq 0 \quad \forall x \in C,$$

where C is a convex subset of a Hilbert space H and $F : H \times H \to \mathbb{R}$ is a given bifunction convex with respect to the second variable and satisfying $F(x, x) = 0$ for all $x \in C$. Recently, more attention has been given to developing efficient and implementable numerical methods to solve (EP), see for example [8] and the references therein. In [8] the author used a modified

ISSN (electronic): 1443-5756

The authors would like to thank the referee for his/her careful reading of the paper and for his important suggestions.

The present paper is a modified version of the same paper which appeared in the same journal JIPAM with the references: Bounkhel, Messaoud; Al-Senan, Bushra R., Generalized proximal method for nonconvex variational inequality, *J. Inequal. Pure Appl. Math.*, **6**(2) (2005), Art. 40. and it has been removed due to issues which have been resolved in the present version.

187-04

proximal method to solve (EP) (see [1]) which generates the sequence $\{x_{k+1}\}$ by solving the subproblem:

$$\text{(SP)} \qquad \begin{cases} \text{Find } x_{k+1} \in C \text{ such that} \\ F(x_{k+1}, x) + \lambda_k^{-1} \langle x_{k+1} - x_k, x - x_{k+1} \rangle \geq 0 \quad \forall x \in C, \end{cases}$$

for a given $\lambda_k > 0$. In this paper, we will study a nonconvex equilibrium problem, by using some recent ideas and techniques from nonsmooth analysis theory to overcome the difficulties arising from the nonconvexity of both C and F. First, we consider the following natural regularization of (EP):

$$\text{(GEP)} \qquad \text{Find } \bar{x} \in C \quad \text{such that} \quad F(\bar{x}, x) + \rho \|x - \bar{x}\|^2 \geq 0 \quad \forall x \in C$$

for a given $\rho \geq 0,$ where C is a closed subset of H and $F : C \times C \to \mathbb{R}$ is a given bifunction satisfying $F(x, x) = 0$ for all $x \in C$. Note that any (EP) can be written in the form of (GEP) with $\rho = 0$.

Problem (GEP) has been denoted in the literature as a uniformly regular equilibrium problem (see e.g. [10]). It is also interesting to point out that the authors in [10] proved (in Section 3) the convergence of some algorithms in the convex case to a solution of (EP) in the finite dimensional setting. It has been commented in Section 4 of [10] that a similar technique used in the convex case can be used for solving the problem (GEP). However, this is just a comment at the end of the paper [10], with no further explanations.

Let us propose the following appropriate reformulation of the subproblem (SP):

$$\text{(GSP)} \qquad \begin{cases} \text{Select } x_{k+1} \in C \text{ such that } x_{k+1} \in x_k + M\lambda_k B \text{ and} \\ \frac{x_k - x_{k+1}}{\lambda_k} \in \partial^p F(x_{k+1}, \cdot)(x_{k+1}) + N_C^p(x_{k+1}), \end{cases}$$

where $M > 0$ is a given positive number. Here ∂^p (resp. N^p) stands for the proximal subdifferential (resp. proximal normal cone). Under natural assumptions, we will prove the convergence of a subsequence of the sequence $\{x_k\}$ generated by (GSP) to a solution of (GEP).

This paper is organized as follows. In Section 2, we recall some definitions and results that will be needed in the paper. In Section 3, we prove the main results of this paper. First, we prove, in Proposition 3.1, that (GSP) is equivalent to (SP) whenever C is a convex subset and $F(x, \cdot)$ is a convex function for all $x \in C$. In Proposition 3.2, we prove under the uniform-prox-regularity of the set C and the uniform-regularity of the bifunction F with respect to the second variable (see Definition 2.2 below), that the sequence $\{x_k\}$ generated by (GSP) satisfies some variational inequality. This result is used to prove, in Theorem 3.3, the convergence of a subsequence of the sequence $\{x_k\}$ to a solution of (GEP), under natural hypotheses and when the set of solutions of (GEP) is assumed to be nonempty.

2. PRELIMINARIES

Throughout the paper H will denote a Hilbert space. We recall some notation and definitions that will be used in the paper. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a function and x any point in H where f is finite. We recall that the proximal subdifferential $\partial^p f(x)$ is the set of all $\xi \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta\mathbb{B}$

$$\langle \xi, x' - x \rangle \leq f(x') - f(x) + \sigma \|x' - x\|^2.$$

Here $\mathbb{B}$ denotes the closed unit ball centered at the origin of H. Recall now that the proximal normal cone of S at x is defined by $N^p(S, x) = \partial^p \psi_S(x)$ where ψ_S denotes the indicator function of S, i.e., $\psi_S(x') = 0$ if $x' \in S$ and $+\infty$ otherwise. Note that (see for instance [11]) for convex functions (resp. convex sets) the proximal subdifferential (resp. proximal normal cone) reduces to the usual subdifferential (resp. usual normal cone) in the sense of convex analysis.

Definition 2.1. For a given $r \in]0, +\infty]$, a subset C is uniformly prox-regular with respect to r (we will say uniformly r-prox-regular) (see [7, 12]) if and only if for all $\bar{x} \in C$ and all $0 \neq \xi \in N^P(C; \bar{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2,$$

for all $x \in C$.

We use the convention $\frac{1}{r} = 0$ for $r = +\infty$. Note that it is not difficult to check that for $r = +\infty$ the uniform r-prox-regularity of C is equivalent to the convexity of C, which makes this class of great importance. Recall that the distance function $d_S(\cdot)$ associated with a closed subset S in H is given by $d_S(x) = \inf\{\|x - y\| : y \in S\}$ with the convention $d_S(x) = +\infty$, when S is empty.

For concrete examples of uniform prox-regular nonconvex sets, we state the following:

(1) The union of two disjoint intervals $[a, b]$ and $[c, d]$ with $c > b$ is nonconvex but uniformly r-prox-regular with any $0 < r < \frac{c-b}{2}$.
(2) The finite union of disjoint intervals is nonconvex but uniformly r-prox-regular and the r depends on the distances between the intervals.
(3) The set

$$\{(x, y) \in \mathbb{R}^2 : \max\{|x - 1|, |y - 2|\} \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : |x - 4| + |y - 2| \leq 1\}$$

is not convex but uniformly r-prox-regular with any $0 < r < \frac{1}{2}$.
(4) More generally, any finite union of disjoint convex subsets in H is nonconvex but uniformly r-prox-regular and the r depends on the distances between the sets. For more examples we refer the reader to [6].

The following proposition recalls an important consequence of the uniform prox-regularity needed in the sequel. For its proof we refer the reader to [6].

Proposition 2.1. *Let C be a nonempty closed subset in H and let $r \in]0, +\infty]$. If the subset C is uniformly r-prox-regular then for any $x \in C$ and any $\xi \in \partial^p d_C(x)$ one has*

$$\langle \xi, x' - x \rangle \leq \frac{2}{r}\|x' - x\|^2 + d_C(x'),$$

for all $x' \in H$ with $d_C(x') < r$.

The following proposition is needed in the proof of our main results in Section 3. It is due to Bounkhel and Thibault [5].

Proposition 2.2. *Let C be a nonempty closed subset in H and let $x \in C$. Then one has*

$$\partial^p d_C(x) = N_C^p(x) \cap \mathbb{B}.$$

Now we recall the following concept of uniform regularity for functions introduced and studied in [2] for solving nonconvex differential inclusions.

Definition 2.2. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function and $O \subset \operatorname{dom} f$ be a nonempty open subset. We will say that f is *uniformly regular* over O with respect to $\beta \geq 0$ (we will also say β-uniformly regular) if for all $\bar{x} \in O$ and for all $\xi \in \partial^p f(\bar{x})$ one has

$$\langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \beta\|x - \bar{x}\|^2 \quad \forall x \in O.$$

We say that f is uniformly regular over a closed set C if there exists an open set O containing C such that f is uniformly regular over O.

A wide family of functions can be proved to be uniformly regular over sets. We state here some examples from [2].

(1) Any l.s.c. proper convex function f is uniformly regular over any nonempty subset of its domain with $\beta = 0$.
(2) Any lower-C^2 function f is uniformly regular over any nonempty *convex compact* subset of its domain. We recall (see [3]) that a function $f : O \to \mathbb{R}$ is said to be lower-C^2 on an open subset O of H if relative to some neighborhood of each point of O there is a representation $f = g - \frac{\rho}{2}\|\cdot\|^2$, in which g is a finite convex function and $\rho \geq 0$. It is very important to point out that this class of nonconvex functions is equivalent (see for instance Theorem 10.33 in [11]) in the finite dimensional setting ($H = \mathbb{R}^n$) to the class of all functions $f : O \to \mathbb{R}$ for which on some neighborhood V of each $\bar{x} \in O$ there exists a representation $f(x) = \max_{t \in T} f_t(x)$ in which f_t are of C^2 on V and the index set T is a compact space and $f_t(x)$ and $\nabla f_t(x)$ depend continuously not just on x but jointly on $(t, x) \in T \times V$. As a particular example of lower-C^2 functions in the finite dimensional setting, one has $f(x) = \max\{f_1(x), \dots, f_m(x),\}$ when f_i is of class C^2.

One could think towards dealing with the class of lower-C^2 instead of the class of uniformly regular functions. The inconvenience of the class of lower-C^2 functions is the need for *convexity* and the *compactness* of the set C to satisfy the inequality in Definition 2.2 which is the exact property needed in our proofs. However, we can find many functions that are uniformly regular over nonconvex noncompact sets. To give an example we need to recall the following result by Bounkhel and Thibault [6].

Theorem 2.3. *Let C be a nonempty closed subset in H and let $r \in]0, +\infty]$. Then C is uniformly r-prox-regular if and only if the following holds for all $x \in H$; with $d_C(x) < r$; and all $\xi \in \partial^p d_C(x)$ one has*

$$\langle \xi, x' - x \rangle \leq \frac{8}{r - d_C(x)}\|x' - x\|^2 + d_C(x') - d_C(x),$$

for all $x' \in H$ with $d_C(x') \leq r$.

From Theorem 2.3 one deduces that for any uniformly r-prox-regular set C (not necessarily convex nor compact) the distance function d_C is uniformly regular over $C + (r - r')B := \{x \in H : d_C(x) \leq r - r'\}$ for every $r' \in]0, r]$.

3. MAIN RESULTS

Now, we are in position to state our first proposition.

Proposition 3.1. *If C is a closed convex set and $F(x, \cdot)$ is a Lipschitz continuous convex function for any $x \in C$, then (GSP) is equivalent to (SP).*

Proof. Let $x_{k+1} \in C$ be generated by (GSP), i.e.,

$$\zeta_{k+1} \in \partial^p F(x_{k+1}, \cdot)(x_{k+1}) + N_C^p(x_{k+1}),$$

with $\zeta_{k+1} = \frac{x_k - x_{k+1}}{\lambda_k}$. Then there exists $\xi_{k+1} \in N_C^p(x_{k+1})$ such that

$$\zeta_{k+1} - \xi_{k+1} \in \partial^p F(x_{k+1}, \cdot)(x_{k+1}).$$

By the convexity of $F(x_{k+1}, \cdot)$ and the definition of the subdifferential for convex functions we have

$$\langle \zeta_{k+1} - \xi_{k+1}, x - x_{k+1} \rangle \leq F(x_{k+1}, x) - F(x_{k+1}, x_{k+1}) \quad \forall x \in C$$

and so

$$\langle \zeta_{k+1}, x - x_{k+1} \rangle \leq F(x_{k+1}, x) + \langle \xi_{k+1}, x - x_{k+1} \rangle \quad \forall x \in C. \tag{3.1}$$

On the other hand, by the convexity of C and by the fact that $\xi_{k+1} \in N_C^p(x_{k+1})$ we get

$$\langle \xi_{k+1}, x - x_{k+1} \rangle \leq 0 \quad \forall x \in C.$$

Combining (3.1) and the last inequality we obtain

$$F(x_{k+1}, x) + \lambda_k^{-1} \langle x_{k+1} - x_k, x - x_{k+1} \rangle \geq 0 \quad \forall x \in C.$$

Conversely, assume that x_{k+1} is generated by (SP), that is,

$$F(x_{k+1}, x) + \lambda_k^{-1} \langle x_{k+1} - x_k, x - x_{k+1} \rangle \geq 0 \quad \forall x \in C.$$

Let $h(x) := F(x_{k+1}, x) + \langle \zeta_{k+1}, x_{k+1} - x \rangle$. Then the last inequality yields

$$h(x) \geq h(x_{k+1}) \quad \forall x \in C.$$

This means that x_{k+1} is a minimum of h over C. Thus

$$0 \in \partial h(x_{k+1}) + N_C(x_{k+1}) = \partial F(x_{k+1}, \cdot)(x_{k+1}) - \zeta_{k+1} + N_C(x_{k+1})$$

and so

$$\zeta_{k+1} \in \partial F(x_{k+1}, \cdot)(x_{k+1}) + N_C(x_{k+1}). \tag{3.2}$$

On the other hand, since $F(x_{k+1}, \cdot)$ is Lipschitz continuous and convex there exists $M > 0$ such that for all x, y one has

$$|F(x_{k+1}, x) - F(x_{k+1}, y)| \leq M \|x - y\|.$$

Let $\epsilon > 0$ be small enough and let $b \in \mathbb{B}$. Then, taking $y = x_{k+1}$ and $x := x_{k+1} + \epsilon b$ in the last inequality yields

$$|F(x_{k+1}, x)| \leq M \|x - x_{k+1}\| = M\epsilon \|b\| \leq M\epsilon.$$

and so

$$\langle \zeta_k, \epsilon b \rangle = \langle \zeta_k, x - x_{k+1} \rangle \leq F(x_{k+1}, x) \leq M\epsilon,$$

and hence $\langle \zeta_k, b \rangle \leq M,$ for all $b \in \mathbb{B}$, which ensures that $\|\zeta_k\| \leq M$.

Thus, this inequality and (3.2) ensure that x_{k+1} is generated by (GSP). □

Proposition 3.2. *If C is uniformly r-prox-regular and if for any $x \in C$, the function $F(x, \cdot)$ is γ-Lipschitz and β-uniformly regular over C, then (GSP) can be written as follows*

$$\lambda_k^{-1} \langle x_k - x_{k+1}, x - x_{k+1} \rangle \leq F(x_{k+1}, x) + \left(\frac{\gamma + M}{2r} + \beta \right) \|x - x_{k+1}\|^2, \quad \forall x \in C.$$

Proof. Let $x_{k+1} \in C$ be generated by (GSP), i.e,

$$\zeta_{k+1} \in \partial^p F(x_{k+1}, \cdot)(x_{k+1}) + N_C^p(x_{k+1}) \text{ and } \|\zeta_{k+1}\| \leq M,$$

with $\zeta_{k+1} = \frac{x_k - x_{k+1}}{\lambda_k}$. Then there exists $\xi_{k+1} \in \partial^p F(x_{k+1}, \cdot)(x_{k+1})$ such that

$$\zeta_{k+1} - \xi_{k+1} \in N_C^p(x_{k+1}).$$

Since $F(x_{k+1}, \cdot)$ is γ-Lipschitz, then (see for instance [11]) $\partial^p F(x_{k+1}, \cdot)(x_{k+1}) \subset \gamma\mathbb{B}$ and so $\|\xi_{k+1}\| \leq \gamma$. Hence $\|\zeta_{k+1} - \xi_{k+1}\| \leq M + \gamma$. By Proposition 2.2 we obtain

$$\zeta_{k+1} - \xi_{k+1} \in N_C^p(x_{k+1}) \cap (\gamma + M)\mathbb{B} = (\gamma + M)\partial^p d_C(x_{k+1}).$$

Then by Proposition 2.1 and by the uniform prox-regularity of C we get

$$\langle \zeta_{k+1} - \xi_{k+1}, x - x_{k+1} \rangle \leq \frac{\gamma + M}{2r} \|x - x_{k+1}\|^2, \quad \forall x \in C. \tag{3.3}$$

On the other hand, by the fact that $\xi_{k+1} \in \partial^p F(x_{k+1}, \cdot)(x_{k+1})$ and $F(x_{k+1}, \cdot)$ is β-uniformly regular over C we have

$$\langle \xi_{k+1}, x - x_{k+1} \rangle \le \beta \|x - x_{k+1}\|^2 + F(x_{k+1}, x) - F(x_{k+1}, x_{k+1}) \quad \forall x \in C.$$

Combining (3.3) and the last inequality we obtain

$$\langle \zeta_{k+1}, x - x_{k+1} \rangle \le F(x_{k+1}, x) + \left(\frac{\gamma + M}{2r} + \beta \right) \|x - x_{k+1}\|^2 \quad \forall x \in C$$

This completes the proof of the proposition. □

Now, we state and prove our main theorem.

Theorem 3.3. *Let C be a closed subset of a Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction. Let $\{x_k\}_k$ be a sequence generated by (GSP). Assume that:*

(1) *C is uniformly r-prox regular;*
(2) *C is ball compact, that is, $C \cap M\mathbb{B}$ is compact for any $M > 0$;*
(3) *The solution set of (GEP) is nonempty;*
(4) *F is σ-strongly monotone, i.e., $F(x, y) + F(y, x) \le -\sigma \|x - y\|^2 \quad \forall x, y \in C$;*
(5) *F is upper semicontinuous with respect to the first variable, i.e.,*

$$\limsup_{x' \to x} F(x', y) \le F(x, y) \quad \forall x, y \in C;$$

(6) *For any $x \in C$, the function $F(x, \cdot)$ is β-uniformly regular over C;*
(7) *For any $x \in C$, the function $F(x, \cdot)$ is γ-Lipschitz;*
(8) *There exists $\lambda > 0$ such that $\lambda_k \ge \lambda$ for all k;*
(9) *The positive number ρ satisfies $\frac{\gamma+M}{2r} + \beta \le \rho \le \frac{\sigma}{2}$;*

Then, there exists $\tilde{x} \in C$ which solves (GEP) such that a subsequence of $\{x_k\}$ converges to $\tilde{x}$.

Proof. Let $\bar{x} \in C$ be a solution of (GEP). By setting $x = x_{k+1}$ in (GEP) and taking into account the strong monotonicity of F and the assumption $\rho \le \frac{\sigma}{2}$, we get

$$F(x_{k+1}, \bar{x}) \le -\rho \|\bar{x} - x_{k+1}\|^2.$$

This combined with Proposition 3.2 gives

$$\langle \zeta_{k+1}, \bar{x} - x_{k+1} \rangle \le \left(-\rho + \frac{\gamma + M}{2r} + \beta \right) \|\bar{x} - x_{k+1}\|^2.$$

So,

$$\langle x_k - x_{k+1}, \bar{x} - x_{k+1} \rangle \le \lambda_k \left(-\rho + \frac{\gamma + M}{2r} + \beta \right) \|\bar{x} - x_{k+1}\|^2. \tag{3.4}$$

Define now the auxiliary real sequence $\phi_k = \frac{1}{2}\|x_k - \bar{x}\|^2$. It is direct to check that

$$\langle x_k - x_{k+1}, \bar{x} - x_{k+1} \rangle = \phi_{k+1} - \phi_k + \frac{1}{2}\|x_{k+1} - x_k\|^2. \tag{3.5}$$

It follows that

$$\phi_{k+1} - \phi_k \le -\frac{1}{2}\|x_{k+1} - x_k\|^2 + \lambda_k \left(-\rho + \frac{\gamma + M}{2r} + \beta \right) \|\bar{x} - x_{k+1}\|^2.$$

Using the assumption $\rho \ge \frac{\gamma+M}{2r} + \beta$ yields

$$\phi_{k+1} \le \phi_k.$$

Therefore, the sequence $\{\phi_k\}$ is a non increasing non negative sequence and so it is convergent to some limit and bounded by some positive number $\alpha > 0$. By (3.4) and (3.5) and by the assumption $\rho \geq \frac{\gamma+M}{2r} + \beta$ we have

$$\frac{1}{2}\|x_{k+1} - x_k\|^2 \leq \phi_k - \phi_{k+1}.$$

Therefore, by the assumption (8)

$$\|\zeta_{k+1}\| = \lambda_k^{-1}\|x_{k+1} - x_k\| \leq \lambda^{-1}\|x_{k+1} - x_k\|,$$

and so $\lim_{k\to\infty} \zeta_{k+1} = 0$. On the other hand, since $\|x_k\| \leq \|\bar{x}\| + \sqrt{2\alpha}$ and C is ball compact there exists a subsequence $\{x_{k_n}\}$ which converges to some limit $\tilde{x} \in C$. Note that this subsequence satisfies

$$(3.6) \quad \langle \zeta_{k_n+1}, x - x_{k_n+1}\rangle \leq F(x_{k_n+1}, x) + \left(\frac{\gamma + M}{2r} + \beta\right)\|x - x_{k_n+1}\|^2, \quad \forall n, \quad \forall x \in C.$$

Thus, by letting $n \to \infty$ in the inequality (3.6) and by taking into account the upper semicontinuity of F with respect to the first variable, we obtain

$$0 \leq F(\tilde{x}, x) + \left(\frac{\gamma + M}{2r} + \beta\right)\|x - \tilde{x}\|^2, \quad \forall x \in C.$$

Therefore, the assumption $\rho \geq \frac{\gamma+M}{2r} + \beta$ concludes

$$F(\tilde{x}, x) + \rho\|x - \tilde{x}\|^2 \geq 0 \quad \forall x \in C,$$

which ensures that the limit $\tilde{x}$ is a solution of (GEP). □

Remark 3.4.

(1) An inspection of our proof of Theorem 3.3 shows that the sequence $\{x_k\}$ generated by (GSP) is bounded, if and only if, there exists at least one solution of (GEP).
(2) Our main Theorem 3.3 extends Theorem 2.1 in [8] from the convex case to the nonconvex case.

REFERENCES

[1] A.S. ANTIPIN AND S. FLAM, Equilibrium programming using proximal-like algorithms, *Math. Program.*, **78**(1) (1997), 29–41.

[2] M. BOUNKHEL, Existence results of nonconvex differential inclusions, *J. Portugalea Mathematica*, **59**(3) (2002), 283–309.

[3] M. BOUNKHEL AND A. JOFRÉ, Subdifferential stability of the distance function and its applications to nonconvex economies and equilibrium, *J. Nonlinear Convex Anal.*, **5**(3) (2004), 331–347.

[4] M. BOUNKHEL, L. TADJ AND A. HAMDI, Iterative schemes to solve nonconvex variational problems, *J. Inequal. Pure and Appl. Math.*, **4**(1) (2003), Art. 14. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=250`].

[5] M. BOUNKHEL AND L. THIBAULT, On various notions of regularity of sets in nonsmooth analysis, *Nonlinear Analysis*, **48**(2) (2002), 223–246.

[6] M. BOUNKHEL AND L. THIBAULT, Nonconvex sweeping process and prox-regularity in Hilbert space, *J. Nonlinear Convex Anal.*, **6**(2) (2005), 359–374.

[7] F.H. CLARKE, R.J. STERN AND P.R. WOLENSKI, Proximal smoothness and the lower C^2 property, *J. Convex Analysis*, **2**(1) (1995), 117–144.

[8] A. MOUDAFI, Second-order differential proximal methods for equilbrium problems, *J. Inequal. Pure and Appl. Math.*, 4(1) (2003), Art. 18. [ONLINE: http://jipam.vu.edu.au/article.php?sid=254].

[9] M.A. NOOR, Iterative schemes for nonconvex variational inequalities, *JOTA*, **121**(2) (2004).

[10] M.A. NOOR AND K.I. NOOR, On equilibrium problems, *Applied Mathematics E-Notes* (AMEN), **4** (2004).

[11] R.T. ROCKAFELLAR AND R. WETS, *Variational Analysis*, Springer Verlag, Berlin, 1998.

[12] R.A. POLIQUIN, R.T. ROCKAFELLAR AND L. THIBAULT, Local differentiability of distance functions, *Trans. Amer. Math. Soc.*, **352**(11) (2000), 5231–5249.