A VARIANT OF A GENERAL INEQUALITY OF THE HARDY-KNOPP TYPE

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equalities, Convolution inequalities.

Abstract: In this paper, we prove a variant of a general Hardy-Knopp type inequality. We

also formulate a convolution inequality in the language of topological groups. By our main results we obtain a general form of multidimensional strengthened

Hardy and Pólya-Knopp-type inequalities.

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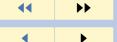
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 1 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

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Contents

l	Introduction	3
2	Main Results	6
3	Multidimensional Hardy and Pólya-Knopp-Type Inequalities	13



Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009



journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction

The well-known Hardy's inequality is stated below (cf. [5, Theorem 327]):

$$(1.1) \qquad \int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx, \quad p > 1, f \ge 0.$$

By replacing f with $f^{\frac{1}{p}}$ in (1.1) and letting $p \to \infty$, we have the Pólya-Knopp inequality (cf. [5, Theorem 335]):

(1.2)
$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x \log f(t)dt\right) dx \le e \int_0^\infty f(x)dx.$$

The constants $(p/(p-1))^p$ and e in (1.1) and (1.2), respectively, are the best possible. On the other hand, the following Hardy-Knopp type inequality (1.3) was proved (cf. [1, Eq.(4.3)] and [7, Theorem 4.1]):

(1.3)
$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \le \int_0^\infty \phi(f(x))\frac{dx}{x},$$

where ϕ is a convex function on $(0,\infty)$. In [7], S. Kaijser et al. also pointed out that (1.1) and (1.2) can be obtained from (1.3). Furthermore, in [2] and [3], Čižmešija and Pečarić proved the so-called strengthened Hardy and Pólya-Knopptype inequalities and their multidimensional forms. In [4, Theorem 1 & Theorem 2], Čižmešija et al. obtained a strengthened Hardy-Knopp type inequality and its dual result. With suitable substitutions, they also showed that the strengthened Hardy and Pólya-Knopp-type inequalities given in the paper [2] are special cases of their results. In the paper [6], Kaijser et al. proved some multidimensional Hardy-type inequalities. They also proved the following generalization of the Hardy and Pólya-



Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents





Page 3 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

Knopp-type inequality:

(1.4)
$$\int_0^b \phi\left(\frac{1}{K(x)}\int_0^x k(x,t)f(t)dt\right)u(x)\frac{dx}{x} \le \int_0^b \phi(f(x))v(x)\frac{dx}{x},$$

where $0 < b \le \infty$, $k(x,t) \ge 0$, $K(x) = \int_0^x k(x,t)dt$, $u(x) \ge 0$, and

$$v(x) = x \int_{x}^{b} \frac{k(z,x)}{K(z)} u(z) \frac{dz}{z}.$$

A dual inequality to (1.4) was also given. Inequality (1.4) can be obtained by using Jensen's inequality and the Fubini theorem. It is elementary but powerful. On the other hand, in the proof of [8, Lemma 3.1], for proving a variant of Schur's lemma, Sinnamon obtained an inequality of the form

(1.5)
$$\left\{ \int_X |T_k f(x)|^q dx \right\}^{\frac{1}{q}} \le \left\{ \int_T |f(t)|^p (Hw(t))^{\frac{p}{q}} w(t)^{1-p} dt \right\}^{\frac{1}{p}},$$

where $1 , X and T are measure spaces, <math>T_k f(x) = \int_T k(x,t) f(t) dt$, w is a positive measurable function on T, and

(1.6)
$$Hw(t) = \int_X k(x,t)^m \left(\int_T k(x,y)^m w(y) dy \right)^{q-\frac{q}{p}} dx, \quad m = \frac{pq}{pq+p-q}.$$

In this paper, let (X, μ) and (T, λ) be two σ -finite measure spaces. Let k be a nonnegative measurable function on $X \times T$ such that

(1.7)
$$\int_T k(x,t)d\lambda(t) = 1 \quad \text{for } \mu\text{-a.e. } x \in X.$$



Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents





Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

For a nonnegative measurable function f on (T, λ) , define

(1.8)
$$T_k f(x) = \int_T k(x, t) f(t) d\lambda(t), \quad x \in X.$$

The purpose of this paper is to establish a modular inequality of the form

$$(1.9) \qquad \left\{ \int_{X} \phi^{q}(T_{k}f(x)) d\mu(x) \right\}^{\frac{1}{q}} \leq \left\{ \int_{T} \phi^{p}(f(t)) (H_{s}w(t))^{\frac{p}{q}} w(t)^{1-sp} d\lambda(t) \right\}^{\frac{1}{p}}$$

for $0 , <math>\phi \in \Phi_s^+(I)$, $s \ge 1/p$, and $H_sw(t)$ is defined by (2.1). As applications, we prove a convolution inequality in the language of integration on a locally compact Abelian group. We also show that by suitable choices of w, we can obtain many forms of strengthened Hardy and Pólya-Knopp-type inequalities. Here $\Phi_s^+(I)$ denotes the class of all nonnegative functions ϕ on $I \subseteq (0,\infty)$ such that $\phi^{1/s}$ is convex on I and ϕ takes its limiting values, finite or infinite, at the ends of I. Note that $\Phi_s^+(I) \subset \Phi_r^+(I)$ for 0 < r < s and we denote $\Phi_\infty^+(I) = \bigcap_{s>0} \Phi_s^+(I)$.

The functions involved in this paper are all measurable on their domains. We work under the convention that $0^0 = \infty^0 = 1$ and $\infty/\infty = 0 \cdot \infty = 0$.



Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents

44 >>

Page 5 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

2. Main Results

The following theorem is based on Jensen's inequality and [8, Lemma 3.1]. For the convenience of readers, we give a complete proof here.

Theorem 2.1. Let $0 , <math>1/p \le s < \infty$, and $\phi \in \Phi_s^+(I)$. Let f be a nonnegative function on (T, λ) and the range of values of f lie in the closure of I. Suppose that w is a positive function on (T, λ) such that the function

(2.1)
$$H_s w(t) = \int_X k(x,t)^m \left(\int_T k(x,y)^m w(y) d\lambda(y) \right)^{sq - \frac{q}{p}} d\mu(x),$$

where m = spq/(spq + p - q), is finite for λ -a.e. $t \in T$. Then we have

(2.2)
$$\left\{ \int_{X} \phi^{q}(T_{k}f(x))d\mu(x) \right\}^{\frac{1}{q}} \leq \left\{ \int_{T} \phi^{p}(f(t))(H_{s}w(t))^{\frac{p}{q}}w(t)^{1-sp}d\lambda(t) \right\}^{\frac{1}{p}}.$$

Proof. Since $\phi^{1/s}$ is convex, $\phi(T_k f(x)) \leq \{T_k(\phi^{1/s}(f))(x)\}^s$ for μ -a.e. $x \in X$ and hence

(2.3)
$$\int_X \phi^q(T_k f(x)) d\mu(x) \le \int_X \left(\int_T k(x,t) \phi^{1/s}(f(t)) d\lambda(t) \right)^{sq} d\mu(x).$$

Let m = spq/(spq + p - q) and w be a positive function on (T, λ) such that $H_sw(t)$ defined by (2.1) is finite for λ -a.e. $t \in T$. By Hölder's inequality with indices sp and $(sp)^*$, we have

(2.4)
$$\int_{T} k(x,t)\phi^{1/s}(f(t))d\lambda(t)$$

$$= \int_{T} k(x,t)^{1-m/(sp)^{*}+m/(sp)^{*}}\phi^{1/s}(f(t))w(t)^{\frac{1}{(sp)^{*}}-\frac{1}{(sp)^{*}}}d\lambda(t)$$



Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents

44 >>

Page 6 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

$$\leq \left(\int_{T} k(x,y)^{m} w(y) d\lambda(y)\right)^{\frac{1}{(sp)^{*}}} \times \left(\int_{T} k(x,t)^{(1-m/(sp)^{*})sp} \phi^{p}(f(t)) w(t)^{-sp/(sp)^{*}} d\lambda(t)\right)^{\frac{1}{(sp)}}$$

and this implies

$$(2.5) \qquad \left\{ \int_{X} \phi^{q}(T_{k}f(x))d\mu(x) \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_{X} \left(\int_{T} k(x,t)^{(1-m/(sp)^{*})sp} \phi^{p}(f(t))w(t)^{-sp/(sp)^{*}} d\lambda(t) \right)^{\frac{q}{p}}$$

$$\times \left(\int_{T} k(x,y)^{m}w(y)d\lambda(y) \right)^{(sp-1)\frac{q}{p}} d\mu(x) \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_{T} \phi^{p}(f(t))(H_{s}w(t))^{\frac{p}{q}}w(t)^{1-sp} d\lambda(t) \right\}^{\frac{1}{p}}.$$

The last inequality is based on the Minkowski's integral inequality with index $\frac{q}{p}$. This completes the proof.

We can apply Theorem 2.1 to obtain some multidimensional strengthened Hardy and Pólya-Knopp-type inequalities. These are discussed in Section 3. In the following corollary, we consider the norm inequality

(2.6)
$$\left\{ \int_X \phi^q(T_k f(x)) d\mu(x) \right\}^{\frac{1}{q}} \le C \left\{ \int_T \phi^p(f(t)) d\lambda(t) \right\}^{\frac{1}{p}}.$$



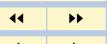
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 7 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

The results of Corollary 2.2 can be obtained by Theorem 2.1 and the fact that $\Phi_s^+(I) \subset \Phi_r^+(I)$ for 0 < r < s.

Corollary 2.2. Let $0 , <math>1/p \le s < \infty$, and $\phi \in \Phi_s^+(I)$. Let f be given as in Theorem 2.1.

(i) If there exists a positive function w on (T, λ) such that the following condition (2.7) holds for some $1/p \le r \le s$ and for some positive constant A_r :

$$(2.7) H_r w(t) \le A_r w(t)^{(r-1/p)q} for \lambda \text{-a.e. } t \in T,$$

then we have (2.6) where the best constant C satisfies

$$(2.8) C \le A_r^{\frac{1}{q}}.$$

(ii) If w satisfies (2.7) for each $1/p \le r \le s$, then we have (2.6) with

$$(2.9) C \le \inf_{1/p \le r \le s} A_r^{\frac{1}{q}}.$$

(iii) If $\phi \in \Phi_{\infty}^+(I)$ and w satisfies (2.7) for each $1/p \le r < \infty$, then we have (2.6) with

$$(2.10) C \le \inf_{1/p \le r < \infty} A_r^{\frac{1}{q}}.$$

In the case $1 and <math>\phi(x) = x$, choose s = r = 1 and then Corollary 2.2 can be reduced to [8, Lemma 3.1].



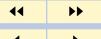
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 8 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

In the following, we consider the particular case X=T=G, where G is a locally compact Abelian group (written multiplicatively), with Haar measure μ . Let h be a nonnegative function on G such that $\int_G h d\mu = 1$. For a nonnegative function f on G, define the convolution operator

(2.11)
$$h * f(x) = \int_{G} h(xt^{-1})f(t)d\mu(t), \quad x \in G.$$

Moreover, if $\int_G h^m d\mu$ is also finite, where m is given in Theorem 2.1, then by (2.1) with $k(x,y) = h(xy^{-1})$ and $w \equiv 1$, we have

(2.12)
$$H_s w(t) = \int_G h(xt^{-1})^m \left(\int_G h(xy^{-1})^m d\mu(y) \right)^{sq - \frac{q}{p}} d\mu(x)$$
$$= \left(\int_G h(x)^m d\mu(x) \right)^{\frac{sq}{m}}.$$

We then obtain the following result:

Corollary 2.3. Let $0 , <math>1/p \le s < \infty$, and $\phi \in \Phi_s^+(I)$. Let h be a nonnegative function on G such that $\int_G h d\mu = 1$ and $\int_G h^m d\mu < \infty$, where m = spq/(spq + p - q). Let f be given as in Theorem 2.1. Then we have

(2.13)
$$\left\{ \int_{G} \phi^{q}(h * f(x)) d\mu(x) \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_{G} h(x)^{m} d\mu(x) \right\}^{\frac{s}{m}} \left\{ \int_{G} \phi^{p}(f(t)) d\mu(t) \right\}^{\frac{1}{p}}.$$



Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 9 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Moreover, if p < q, $\phi \in \Phi_{\infty}^+(I)$ and $\int_G h^r d\mu < \infty$ for some r > 1, then

$$(2.14) \quad \left\{ \int_{G} \phi^{q}(h * f(x)) d\mu(x) \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \exp\left(\int_{G} h(x) \log h(x) d\mu(x) \right) \right\}^{\frac{1}{p} - \frac{1}{q}} \left\{ \int_{G} \phi^{p}(f(t)) d\mu(t) \right\}^{\frac{1}{p}}.$$

Inequality (2.14) can be obtained by letting $s \to \infty$ in (2.13). In the case $\phi(x) = x$ and s = 1 in (2.13), the condition $\int_G h d\mu = 1$ is not necessary and (2.13) can be reduced to Young's inequality:

$$(2.15) \qquad \left\{ \int_{G} (h * f(x))^{q} d\mu(x) \right\}^{\frac{1}{q}} \leq \left\{ \int_{G} h(x)^{m} d\mu(x) \right\}^{\frac{1}{m}} \left\{ \int_{G} f(t)^{p} d\mu(t) \right\}^{\frac{1}{p}},$$

where $1 \le p \le q < \infty$ and m = pq/(pq + p - q). If $\phi(x) = e^x$ and f is replaced by $\log f$ in (2.14), then for 0 ,

$$(2.16) \quad \left\{ \int_{G} \left\{ \exp\left(\int_{G} h(xt^{-1}) \log f(t) d\mu(t) \right) \right\}^{q} d\mu(x) \right\}^{\frac{1}{q}} \\ \leq \left\{ \exp\left(\int_{G} h(x) \log h(x) d\mu(x) \right) \right\}^{\frac{1}{p} - \frac{1}{q}} \left\{ \int_{G} f(t)^{p} d\mu(t) \right\}^{\frac{1}{p}}.$$

Let $G = \mathbb{R}^n$ under addition and μ be the Lebesgue measure. Then (2.15) can be



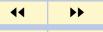
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 10 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

reduced to

(2.17)
$$\left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} h(x-t)f(t)dt \right)^q dx \right\}^{\frac{1}{q}} \\ \leq \left\{ \int_{\mathbb{R}^n} h(x)^m dx \right\}^{\frac{1}{m}} \left\{ \int_{\mathbb{R}^n} f(t)^p dt \right\}^{\frac{1}{p}}.$$

Moreover, if $\int_{\mathbb{R}^n} h(x) dx = 1$ and $\int_{\mathbb{R}^n} h(x)^r dx < \infty$ for some r > 1, then by (2.16),

$$(2.18) \quad \left\{ \int_{\mathbb{R}^n} \left\{ \exp\left(\int_{\mathbb{R}^n} h(x-t) \log f(t) dt \right) \right\}^q dx \right\}^{\frac{1}{q}} \\ \leq \left\{ \exp\left(\int_{\mathbb{R}^n} h(x) \log h(x) dx \right) \right\}^{\frac{1}{p} - \frac{1}{q}} \left\{ \int_{\mathbb{R}^n} f(t)^p dt \right\}^{\frac{1}{p}}.$$

Let $G = (0, \infty)$ under multiplication and $d\mu = x^{-1}dx$. Then by (2.15),

(2.19)
$$\left\{ \int_0^\infty \left(\int_0^\infty h(x/t) f(t) \frac{dt}{t} \right)^q \frac{dx}{x} \right\}^{\frac{1}{q}} \\ \leq \left\{ \int_0^\infty h(x)^m \frac{dx}{x} \right\}^{\frac{1}{m}} \left\{ \int_0^\infty f(t)^p \frac{dt}{t} \right\}^{\frac{1}{p}}.$$

Moreover, if $\int_0^\infty h(x)x^{-1}dx = 1$ and $\int_0^\infty h(x)^r x^{-1}dx < \infty$ for some r > 1,



Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents

44 >>

Page 11 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

then (2.16) can be reduced to

$$(2.20) \quad \left\{ \int_0^\infty \left\{ \exp\left(\int_0^\infty h(x/t) \log f(t) \frac{dt}{t} \right) \right\}^q \frac{dx}{x} \right\}^{\frac{1}{q}} \\ \leq \left\{ \exp\left(\int_0^\infty h(x) \log h(x) \frac{dx}{x} \right) \right\}^{\frac{1}{p} - \frac{1}{q}} \left\{ \int_0^\infty f(t)^p \frac{dt}{t} \right\}^{\frac{1}{p}}.$$

There are multidimensional cases corresponding to (2.19) and (2.20). For example, the 2-dimensional analogue of (2.19) is

$$(2.21) \quad \left\{ \int_0^\infty \int_0^\infty \left(\int_0^\infty \int_0^\infty h\left(\frac{x}{s}, \frac{y}{t}\right) f(s, t) \frac{ds}{s} \frac{dt}{t} \right)^q \frac{dx}{x} \frac{dy}{y} \right\}^{\frac{1}{q}} \\ \leq \left\{ \int_0^\infty \int_0^\infty h(x, y)^m \frac{dx}{x} \frac{dy}{y} \right\}^{\frac{1}{m}} \left\{ \int_0^\infty \int_0^\infty f(s, t)^p \frac{ds}{s} \frac{dt}{t} \right\}^{\frac{1}{p}},$$

and we can also obtain similar results to (2.20).



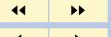
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 12 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

3. Multidimensional Hardy and Pólya-Knopp-Type Inequalities

In this section, we apply our main results to the case $X=T=\mathbb{R}^N$ and obtain some multidimensional forms of the strengthened Hardy and Pólya-Knopp-type inequalities. Let Σ^{N-1} be the unit sphere in \mathbb{R}^N , that is, $\Sigma^{N-1}=\{x\in\mathbb{R}^N:|x|=1\}$, where |x| denotes the Euclidean norm of x. Let A be a Lebesgue measurable subset of Σ^{N-1} , $0< b\leq \infty$, and define

$$E = \{ x \in \mathbb{R}^N : x = s\rho, \ 0 \le s < b, \ \rho \in A \}.$$

For $x \in E$, we define

$$S_x = \{ y \in \mathbb{R}^N : y = s\rho, \ 0 \le s \le |x|, \ \rho \in A \},$$

and denote by $|S_x|$ the Lebesgue measure of S_x . We have the following result:

Theorem 3.1. Let $0 , <math>1/p \le s < \infty$, and $\phi \in \Phi_s^+(I)$. Let g be a nonnegative function on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\int_{S_x} g(x,t) dt = 1$ for almost all $x \in E$ and let f be a nonnegative function on \mathbb{R}^N and the range of values of f lie in the closure of f. Suppose that f is a nonnegative function on f and f is a positive function on f such that the function

(3.1)
$$H_s w(t) = \int_E g(x,t)^m \left(\int_{S_x} g(x,y)^m w(y) dy \right)^{sq - \frac{q}{p}} u(x) \chi_{S_x}(t) dx,$$

where m = spq/(spq + p - q), is finite for almost all $t \in E$. Then we have

(3.2)
$$\left\{ \int_{E} \phi^{q} \left(\int_{S_{x}} g(x,t) f(t) dt \right) u(x) dx \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_{E} \phi^{p} (f(t)) (H_{s} w(t))^{\frac{p}{q}} w(t)^{1-sp} dt \right\}^{\frac{1}{p}}.$$



Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents

44 >>>

Page 13 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Proof. Let $X = T = \mathbb{R}^N$, $d\mu = u(x)\chi_E(x)dx$, $d\lambda = \chi_E(x)dx$, and $k(x,t) = g(x,t)\chi_{S_x}(t)$ in Theorem 2.1. Then $H_s w$ defined by (2.1) can be reduced to (3.1) and we have (3.2) by Theorem 2.1.

In the case p = q = s = 1, then m = 1 and we have

(3.3)
$$\int_{E} \phi \left(\int_{S_{x}} g(x,t) f(t) dt \right) u(x) dx$$

$$\leq \int_{E} \phi(f(t)) \left(\int_{E} g(x,t) u(x) \chi_{S_{x}}(t) dx \right) dt.$$

In particular, if N=1, E=[0,b), $S_x=[0,x)$, and u(x) is replaced by u(x)/x, then (3.3) can be reduced to

$$(3.4) \quad \int_0^b \phi\left(\int_0^x g(x,t)f(t)dt\right)\frac{u(x)}{x}dx \le \int_0^b \phi(f(t))\left(\int_t^b g(x,t)\frac{u(x)}{x}dx\right)dt.$$

Inequality (3.4) was also obtained in [6, Theorem 4.1].

Now we consider (3.2) with $u(x) = |S_x|^a$ and $g(x,t) = |S_x|^{-1}h(|S_t|/|S_x|)$, where $a \in \mathbb{R}$, h is a nonnegative function defined on [0,1) and $\int_0^1 h(x)dx = 1$. By (3.1) with $w(y) = |S_y|^{m(\frac{q}{p}-a-2)/(sq)}$, we have

(3.5)
$$H_s w(t) = \left(\int_0^1 h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi \right)^{sq-\frac{q}{p}} |S_t|^{-1+m(a+2-q/p)/(sq)} \times \int_{(|t|/b)^N}^1 h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi.$$

As a consequence of Theorem 3.1, we have the following result:



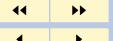
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 14 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

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Corollary 3.2. Let $0 , <math>1/p \le s < \infty$, $\phi \in \Phi_s^+(I)$, and f be given as in Theorem 3.1. Let $a \in \mathbb{R}$, h be given as above, and $\int_0^1 h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi < \infty$, where m = spq/(spq + p - q). Then we have

$$(3.6) \quad \left\{ \int_{E} \phi^{q} \left(\frac{1}{|S_{x}|} \int_{S_{x}} h\left(\frac{|S_{t}|}{|S_{x}|} \right) f(t) dt \right) |S_{x}|^{a} dx \right\}^{\frac{1}{q}} \\ \leq \left(\int_{0}^{1} h(\xi)^{m} \xi^{m(q/p-a-2)/(sq)} d\xi \right)^{s-\frac{1}{p}} \\ \times \left\{ \int_{E} \phi^{p}(f(t)) |S_{t}|^{(a+1)\frac{p}{q}-1} v(t)^{\frac{p}{q}} dt \right\}^{\frac{1}{p}},$$

where

$$v(t) = \int_{(|t|/b)^N}^{1} h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi.$$

By (3.6), we see that

$$(3.7) \qquad \left\{ \int_{E} \phi^{q} \left(\frac{1}{|S_{x}|} \int_{S_{x}} h\left(\frac{|S_{t}|}{|S_{x}|} \right) f(t) dt \right) |S_{x}|^{a} dx \right\}^{\frac{1}{q}}$$

$$\leq C \left\{ \int_{E} \phi^{p}(f(t)) |S_{t}|^{(a+1)\frac{p}{q}-1} dt \right\}^{\frac{1}{p}},$$

where C satisfies

(3.8)
$$C \le \left(\int_0^1 h(\xi)^m \xi^{m(\frac{q}{p} - a - 2)/(sq)} d\xi \right)^{s - \frac{1}{p} + \frac{1}{q}}.$$

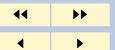


Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page
Contents



Page 15 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

Moreover, if $\phi \in \Phi_{\infty}^+(I)$ and p < q, then the estimation given in (3.8) can be replaced by

(3.9)
$$C \le \left\{ \exp\left(\int_0^1 h(\xi) \log[h(\xi)\xi^{(q-(a+2)p)/(q-p)}] d\xi \right) \right\}^{\frac{1}{p} - \frac{1}{q}}.$$

In the following, we consider the particular case p=q. In this case, m=1 and (3.6) can be reduced to

$$(3.10) \int_{E} \phi^{p} \left(\frac{1}{|S_{x}|} \int_{S_{x}} h\left(\frac{|S_{t}|}{|S_{x}|} \right) f(t) dt \right) |S_{x}|^{a} dx$$

$$\leq \left(\int_{0}^{1} h(\xi) \xi^{(-a-1)/(sp)} d\xi \right)^{sp-1}$$

$$\times \int_{E} \phi^{p}(f(t)) \left(\int_{(|t|/b)^{N}}^{1} h(\xi) \xi^{(-a-1)/(sp)} d\xi \right) |S_{t}|^{a} dt.$$

In the case $\phi \in \Phi_{\infty}^+(I)$, by letting $s \to \infty$ in (3.10), we have

$$(3.11) \int_{E} \phi^{p} \left(\frac{1}{|S_{x}|} \int_{S_{x}} h\left(\frac{|S_{t}|}{|S_{x}|}\right) f(t) dt \right) |S_{x}|^{a} dx$$

$$\leq \left\{ \exp\left(\int_{0}^{1} h(\xi) \log \xi d\xi \right) \right\}^{-a-1} \int_{E} \phi^{p}(f(t)) \left(\int_{(|t|/b)^{N}}^{1} h(\xi) d\xi \right) |S_{t}|^{a} dt.$$

If $h(\xi) = \alpha \xi^{\alpha-1}$, $\alpha > 0$, then we have the following corollary.

Corollary 3.3. Let $0 , <math>1/p \le s < \infty$, $\phi \in \Phi_s^+(I)$, $\alpha > 0$, $a + 1 < \alpha sp$,



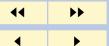
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 16 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

and f be given as in Theorem 3.1. Then we have

$$(3.12) \int_{E} \phi^{p} \left(\frac{\alpha}{|S_{x}|^{\alpha}} \int_{S_{x}} |S_{t}|^{\alpha-1} f(t) dt \right) |S_{x}|^{a} dx$$

$$\leq \left(\frac{\alpha sp}{\alpha sp - a - 1} \right)^{sp} \int_{E} \phi^{p} (f(t)) \left(1 - \left(\frac{|t|}{b} \right)^{N(\alpha sp - a - 1)/(sp)} \right) |S_{t}|^{a} dt.$$

Moreover, if $\phi \in \Phi_{\infty}^+(I)$, then for $a \in \mathbb{R}$, we have

$$(3.13) \quad \int_{E} \phi^{p} \left(\frac{\alpha}{|S_{x}|^{\alpha}} \int_{S_{x}} |S_{t}|^{\alpha-1} f(t) dt \right) |S_{x}|^{a} dx$$

$$\leq e^{(a+1)/\alpha} \int_{E} \phi^{p} (f(t)) \left(1 - \left(\frac{|t|}{b} \right)^{N\alpha} \right) |S_{t}|^{a} dt.$$

Inequality (3.12) was obtained in [3, Theorem 1(i)] for the case $\phi(x) = x, p > 1$, s = 1, a , and <math>E is the ball in \mathbb{R}^N centered at the origin and of radius b. If $\phi(x) = e^x$, p = 1, and f is replaced by $\log f$ in (3.13), then we have [3, Theorem 2(i)]. If $h(\xi) = \alpha(1 - \xi)^{\alpha - 1}$, $\alpha > 0$, then we have the following corollary.

Corollary 3.4. Let $0 , <math>1/p \le s < \infty$, $\phi \in \Phi_s^+(I)$, $\alpha > 0$, a+1 < sp, and f be given as in Theorem 3.1. Then we have

$$(3.14) \quad \int_{E} \phi^{p} \left(\frac{\alpha}{|S_{x}|^{\alpha}} \int_{S_{x}} (|S_{x}| - |S_{t}|)^{\alpha - 1} f(t) dt \right) |S_{x}|^{a} dx$$

$$\leq \left\{ \alpha B \left(\frac{sp - a - 1}{sp}, \alpha \right) \right\}^{sp - 1} \int_{E} \phi^{p} (f(t)) |S_{t}|^{a} v(t) dt,$$



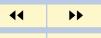
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 17 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

where $B(\delta, \eta)$ is the Beta function and

$$v(t) = \int_{(|t|/b)^N}^{1} \alpha (1-\xi)^{\alpha-1} \xi^{(-a-1)/(sp)} d\xi.$$

Moreover, if $\phi \in \Phi_{\infty}^+(I)$, then for $a \in \mathbb{R}$ we have

$$(3.15) \int_{E} \phi^{p} \left(\frac{\alpha}{|S_{x}|^{\alpha}} \int_{S_{x}} (|S_{x}| - |S_{t}|)^{\alpha - 1} f(t) dt \right) |S_{x}|^{a} dx$$

$$\leq \left\{ \exp \left(\int_{0}^{1} \alpha (1 - \xi)^{\alpha - 1} \log \xi d\xi \right) \right\}^{-a - 1}$$

$$\times \int_{E} \phi^{p} (f(t)) \left(1 - \left(\frac{|t|}{b} \right)^{N} \right)^{\alpha} |S_{t}|^{a} dt.$$

In the following, we consider the dual result of Theorem 3.1. Let $0 \le b < \infty$ and

$$\tilde{E} = \{ x \in \mathbb{R}^N : x = s\rho, b \le s < \infty, \rho \in A \}.$$

For $x \in \tilde{E}$, we define

$$\tilde{S}_x = \{ y \in \mathbb{R}^N : y = s\rho, |x| \le s < \infty, \rho \in A \}.$$

Let u be a nonnegative function on \mathbb{R}^N , $d\mu = u(x)\chi_{\tilde{E}}(x)dx$, $d\lambda = \chi_{\tilde{E}}(t)dt$, and $k(x,t) = g(x,t)\chi_{\tilde{S}_x}(t)$, where g is a nonnegative function on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\int_{\tilde{S}_x} g(x,t)dt = 1$ for almost all $x \in \tilde{E}$. Suppose that w is a positive function on \tilde{E} . Then $H_s w$ defined by (2.1) can be reduced to

(3.16)
$$H_s w(t) = \int_{\tilde{E}} g(x,t)^m \left(\int_{\tilde{S}_x} g(x,y)^m w(y) dy \right)^{sq-\frac{q}{p}} u(x) \chi_{\tilde{S}_x}(t) dx.$$

We have the following theorem.



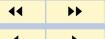
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 18 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Theorem 3.5. Let $0 , <math>1/p \le s < \infty$, $\phi \in \Phi_s^+(I)$, and g, u, w be given as above. Let f be given as in Theorem 3.1. Suppose that $H_sw(t)$ given in (3.16) is finite for almost all $t \in \tilde{E}$. Then we have

$$(3.17) \qquad \left\{ \int_{\tilde{E}} \phi^{q} \left(\int_{\tilde{S}_{x}} g(x,t) f(t) dt \right) u(x) dx \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_{\tilde{E}} \phi^{p} (f(t)) (H_{s} w(t))^{\frac{p}{q}} w(t)^{1-sp} dt \right\}^{\frac{1}{p}}.$$

In the case p = q = s = 1, then m = 1 and we have

(3.18)
$$\int_{\tilde{E}} \phi \left(\int_{\tilde{S}_{x}} g(x,t) f(t) dt \right) u(x) dx$$

$$\leq \int_{\tilde{E}} \phi(f(t)) \left(\int_{\tilde{E}} g(x,t) u(x) \chi_{\tilde{S}_{x}}(t) dx \right) dt.$$

In particular, if N=1, $\tilde{E}=[b,\infty)$, $\tilde{S}_x=[x,\infty)$, and u(x) is replaced by u(x)/x, then by (3.18) we have

(3.19)
$$\int_{b}^{\infty} \phi\left(\int_{x}^{\infty} g(x,t)f(t)dt\right) \frac{u(x)}{x} dx$$

$$\leq \int_{b}^{\infty} \phi(f(t)) \left(\int_{b}^{t} g(x,t) \frac{u(x)}{x} dx\right) dt.$$

Inequality (3.19) was also obtained in [6, Theorem 4.3]. Using a similar method, we can also obtain companion results of (3.6) - (3.15). We omit the details.



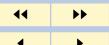
Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents



Page 19 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

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Inequality of Hardy-Knopp Type

Dah-Chin Luor

vol. 10, iss. 3, art. 73, 2009

Title Page

Contents

Page 20 of 20

Go Back

journal of inequalities in pure and applied mathematics

Full Screen

Close

issn: 1443-5756