journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 4, Article 114, 29 pp.

STABILITY OF A GENERALIZED MIXED TYPE ADDITIVE, QUADRATIC, CUBIC AND QUARTIC FUNCTIONAL EQUATION

K. RAVI, J.M. RASSIAS, M. ARUNKUMAR, AND R. KODANDAN

DEPARTMENT OF MATHEMATICS
SACRED HEART COLLEGE
TIRUPATTUR - 635 601
TAMILNADU, INDIA.
shckravi@yahoo.co.in

PEDAGOGICAL DEPARTMENT E.E.
SECTION OF MATHEMATICS AND INFORMATICS
NATIONAL AND CAPODISTRIAN UNIVERSITY OF ATHENS
4, AGAMEMNONOS STR. AGHIA PARASKEVI
ATHENS 15342, GREECE.
jrassias@primedu.uoa.gr
URL: http://www.primedu.uoa.gr/~jrassias/

DEPARTMENT OF MATHEMATICS
SACRED HEART COLLEGE
TIRUPATTUR - 635 601
TAMILNADU, INDIA.
annarun2002@yahoo.co.in

DEPARTMENT OF MATHEMATICS
SRINIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES
CHITTOOR - 517 127, ANDHRA PRADESH.
Rkodandan1979@rediff.co.in

Received 06 July, 2009; accepted 06 November, 2009
Communicated by S.S. Dragomir

ABSTRACT. In this paper, we obtain the general solution and the generalized Hyers-Ulam-Rassias stability of the generalized mixed type of functional equation

$$f(x+ay)+f(x-ay)=a^2[f(x+y)+f(x-y)]+2\left(1-a^2\right)f(x) + \frac{\left(a^4-a^2\right)}{12}[f(2y)+f(-2y)-4f(y)-4f(-y)].$$

for fixed integers a with $a \neq 0, \pm 1$.

Key words and phrases: Additive function, Quadratic function, Cubic function, Quartic function, Generalized Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability, J.M. Rassias stability.

2000 *Mathematics Subject Classification.* 39B52, 39B82.

183-09

1. INTRODUCTION

S.M. Ulam [31] is the pioneer of the stability problem in functional equations. In 1940, while he was delivering a talk before the Mathematics Club of the University of Wisconsin, he discussed a number of unsolved problems. Among them was the following question concerning the stability of homomorphisms:

"Let G be group and H be a metric group with metric $d(\cdot,\cdot)$. Given $\epsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies

$$d\left(f(xy), f(x)f(y)\right) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with

$$d\left(f(x), a(x)\right) < \varepsilon$$

for all $x \in G$."

In 1941, D.H. Hyers [12] gave the first affirmative partial answer to the question of Ulam for Banach spaces. He proved the following celebrated theorem.

Theorem 1.1 ([12])**.** *Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \tag{1.1}$$

for all $x, y \in X$. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in X$ and $a : X \to Y$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \le \varepsilon \tag{1.3}$$

for all $x \in X$.

In 1950, Aoki [2] generalized the Hyers theorem for additive mappings. In 1978, Th.M. Rassias [26] provided a generalized version of the Hyers theorem which permitted the Cauchy difference to become unbounded. He proved the following:

Theorem 1.2 ([26])**.** *Let X be a normed vector space and Y be a Banach space. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\left(||x||^p + ||y||^p\right) \tag{1.4}$$

for all $x, y \in X$, where θ and p are constants with $\theta > 0$ and $p < 1$, then the limit

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.5}$$

exists for all $x \in X$ and $T : X \to Y$ is the unique additive mapping which satisfies

$$\|f(x) - T(x)\| \le \frac{2\theta}{2-2^p}||x||^p \tag{1.6}$$

for all $x \in X$. If $p < 0$, then inequality (1.4) holds for $x, y \neq 0$ and (1.6) for $x \neq 0$. Also if for each $x \in X$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then T is linear.

It was shown by Z. Gajda [9], as well as Th.M. Rassias and P. Semrl [27] that one cannot prove a Th.M. Rassias type theorem when $p = 1$. The counter examples of Z. Gajda, as well as of Th.M. Rassias and P. Semrl [27] have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings; P. Gavruta [10] and S.M. Jung [17] among others have studied the Hyers-Ulam-Rassias stability of functional equations. The inequality (1.4) that was introduced by Th.M. Rassias [26] provided much influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as the Hyers-Ulam-Rassias stability of functions.

In 1982, J.M. Rassias [24] following the spirit of the approach of Th.M. Rassias [26] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

Theorem 1.3 ([24])**.** *Let X be a real normed linear space and Y be a real completed normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exists constants $\theta > 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q \tag{1.7}$$

for all $x, y \in X$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.8}$$

exists for all $x \in X$ and $L : X \to Y$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2 - 2^r|} \|x\|^r \tag{1.9}$$

for all $x \in X$. If, in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}-$ linear mapping.

However, the case $r = 1$ in inequality (1.9) is singular. A counter example has been given by P. Gavruta [11]. The above-mentioned stability involving a product of different powers of norms was called Ulam-Gavruta-Rassias stability by M.A. Sibaha et al., [30], as well as by K. Ravi and M. Arunkumar [28]. This stability result was also called the Hyers-Ulam-Rassias stability involving a product of different powers of norms by Park [23].

In 1994, a generalization of Th.M. Rassias' theorem and J.M. Rassias' theorem was obtained by P. Gavruta [10], who replaced the factors $\theta(||x||^p + ||y||^p)$ and $\theta(||x||^p||y||^p)$ by a general control function $\varphi(x, y)$. In the past few years several mathematicians have published various generalizations and applications of Hyers- Ulam- Rassias stability to a number of functional equations and mappings (see [4, 5, 13, 18, 19]). Very recently, J.M. Rassias [29] in the inequality (1.7) replaced the bound by a mixed one involving the product and sum of powers of norms, that is, $\theta\{||x||^p||y||^p + (||x||^{2p} + ||y||^{2p})\}$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.10}$$

is said to be a *quadratic functional equation* because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.10). A quadratic functional equation was used to characterize inner product spaces [1, 20]. It is well known that a function f is a solution of (1.10) if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x (see [20]). The biadditive function B is given by

$$B(x,y) = \frac{1}{4}[f(x+y) + f(x-y)]. \tag{1.11}$$

The functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.12}$$

is called a *cubic functional equation*, because the cubic function $f(x) = cx^3$ is a solution of the equation (1.12). The general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.12) was discussed by K.W. Jun and H.M. Kim [14]. They proved that a function f between real vector spaces X and Y is a solution of (1.12) if and only if there exists a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables.

The *quartic functional equation*

$$f(x+2y) + f(x-2y) - 6f(x) = 4[f(x+y) + f(x-y)] + 24f(y) \tag{1.13}$$

was introduced by J.M. Rassias [25]. Later S.H. Lee et al., [21] remodified J.M. Rassias's equation and obtained a new quartic functional equation of the form

$$f(2x+y) + f(2x-y) = 4[f(x+y) + f(x-y)] + 24f(x) - 6f(y) \tag{1.14}$$

and discussed its general solution. In fact S.H. Lee et al., [21] proved that a function f between vector spaces X and Y is a solution of (1.14) if and only if there exists a unique symmetric multi - additive function $Q : X \times X \times X \times X \to Y$ such that $f(x) = Q(x, x, x, x)$ for all $x \in X$. It is easy to show that the function $f(x) = kx^4$ is the solution of (1.13) and (1.14).

A function

$$f(x) = Q(x_1, x_2, x_3, x_4) \tag{1.15}$$

is called symmetric multi additive if Q is additive with respect to each variable $x_i,\ i = 1, 2, 3, 4$ in (1.15).

A function f is defined as

$$f(x) = \frac{\beta(x) - \alpha(x)}{12}$$

where $\alpha(x) = f(2x) - 16f(x), \beta(x) = f(2x) - 4f(x)$, further, f satisfies $f(2x) = 4f(x)$ and $f(2x) = 16f(x)$ is said to be a *quadratic - quartic function*.

K.W. Jun and H.M. Kim [16] introduced the following generalized *quadratic and additive type functional equation*

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1 \le i < j \le n} f(x_i + x_j) \tag{1.16}$$

in the class of functions between real vector spaces. For $n = 3$, Pl. Kannappan proved that a function f satisfies the functional equation (1.16) if and only if there exists a symmetric bi-additive function A and an additive function B such that $f(x) = B(x, x) + A(x)$ for all x (see [20]). The Hyers-Ulam stability for the equation (1.16) when $n = 3$ was proved by S.M. Jung [18]. The Hyers-Ulam-Rassias stability for the equation (1.16) when $n = 4$ was also investigated by I.S. Chang et al., [3].

The general solution and the generalized Hyers-Ulam stability for the *quadratic and additive type functional equation*

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y) \tag{1.17}$$

for any positive integer a with $a \neq -1, 0, 1$ was discussed by K.W. Jun and H.M. Kim [15]. Recently A. Najati and M.B. Moghimi [22] investigated the generalized Hyers-Ulam-Rassias stability for a *quadratic and additive type functional equation* of the form

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 4f(x) \tag{1.18}$$

Very recently, the authors [6, 7] investigated a mixed type functional equation of cubic and quartic type and obtained its general solution. The stability of generalized mixed type functional equations of the form

$$f(x+ky) + f(x-ky) = k^2[f(x+y) + f(x-y)] + 2\left(1-k^2\right)f(x) \tag{1.19}$$

for fixed integers k with $k \neq 0, \pm 1$ in quasi -Banach spaces was investigated by M. Eshaghi Gordji and H. Khodaie [8]. The mixed type functional equation (1.19) is additive, quadratic and cubic.

In this paper, the authors introduce a mixed type functional equation of the form

$$f(x+ay)+f(x-ay)=a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(x) + \frac{a^4-a^2}{12}\left[f(2y)+f(-2y)-4f(y)-4f(-y)\right] \tag{1.20}$$

which is additive, quadratic, cubic and quartic and obtain its general solution and generalized Hyers-Ulam-Rassias stability for fixed integers a with $a \neq 0, \pm 1$.

2. GENERAL SOLUTION

In this section, we present the general solution of the functional equation (1.20). Throughout this section let E_1 and E_2 be real vector spaces.

Theorem 2.1. *Let $f : E_1 \to E_2$ be a function satisfying (1.20) for all $x, y \in E_1$. If f is even then f is quadratic - quartic.*

Proof. Let f be an even function, i.e., $f(-x) = f(x)$. Then equation (1.20) becomes

$$f(x+ay)+f(x-ay)=a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(x) + \frac{a^4-a^2}{6}\left[f(2y)-4f(y)\right] \tag{2.1}$$

for all $x, y \in E_1$. Interchanging x and y in (2.1) and using the evenness of f, we get

$$f(ax+y)+f(ax-y)=a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(y) + \frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right] \tag{2.2}$$

for all $x, y \in E_1$. Setting (x, y) as $(0, 0)$ in (2.2), we obtain $f(0) = 0$. Replacing y by $x + y$ in (2.2) and using the evenness of f, we have

$$f((a+1)x+y)+f((a-1)x-y) = a^2\left[f(2x+y)+f(y)\right]+2\left(1-a^2\right)f(x+y)+\frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right] \tag{2.3}$$

for all $x, y \in E_1$. Replacing y by $x - y$ in (2.2), we obtain

$$f((a+1)x-y)+f((a-1)x+y) = a^2\left[f(2x-y)+f(y)\right]+2\left(1-a^2\right)f(x-y)+\frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right] \tag{2.4}$$

for all $x, y \in E_1$. Adding (2.3) and (2.4), we get

$$f((a+1)x+y)+f((a-1)x-y)+f((a+1)x-y)+f((a-1)x+y) = a^2\left[f(2x+y)+f(2x-y)+2f(y)\right]+2\left(1-a^2\right)\left[f(x+y)+f(x-y)\right] + \frac{a^4-a^2}{6}\left[2f(2x)-8f(x)\right] \tag{2.5}$$

for all $x, y \in E_1$. Replacing y by $ax + y$ in (2.2), we obtain

$$f(2ax+y) + f(y) = a^2 [f((a+1)x+y) + f((1-a)x-y)] + 2(1-a^2) f(ax+y) + \frac{a^4-a^2}{6}[f(2x) - 4f(x)] \tag{2.6}$$

for all $x, y \in E_1$. Replacing y by $ax - y$ in (2.3), we get

$$f(2ax-y) + f(y) = a^2 [f((a+1)x-y) + f((1-a)x+y)] + 2(1-a^2) f(ax-y) + \frac{a^4-a^2}{6}[f(2x) - 4f(x)] \tag{2.7}$$

for all $x, y \in E_1$. Adding (2.6) and (2.7), we obtain

$$\begin{aligned} &f(2ax+y) + f(2ax-y) + 2f(y) \\ &= a^2 [f((a+1)x+y) + f((a+1)x-y) + f((a-1)x+y) + f((a-1)x-y)] \\ &\quad + 2(1-a^2)[f(ax+y) + f(ax-y)] + \frac{a^4-a^2}{3}[f(2x) - 4f(x)] \end{aligned} \tag{2.8}$$

for all $x, y \in E_1$. Using (2.5) in (2.8), we arrive at

$$\begin{aligned} &f(2ax+y) + f(2ax-y) + 2f(y) \\ &= a^4 [f(2x+y) + f(2x-y)] + 2a^4 f(y) + 2a^2(1-a^2)[f(x+y) + f(x-y)] \\ &\quad + \frac{a^2(a^4-a^2)}{3}[f(2x) - 4f(x)] + 2(1-a^2)[f(ax+y) + f(ax-y)] \\ &\quad + \frac{a^4-a^2}{3}[f(2x) - f(x)] \end{aligned} \tag{2.9}$$

for all $x, y \in E_1$. Replacing x by $2x$ in (2.2), we get

$$\begin{aligned} &f(2ax+y) + f(2ax-y) \\ &= a^2 [f(2x+y) + f(2x-y)] + 2(1-a^2) f(y) + \frac{a^4-a^2}{6}[f(4x) - 4f(2x)] \end{aligned} \tag{2.10}$$

for all $x, y \in E_1$. Using (2.10) in (2.9), we obtain

$$\begin{aligned} &a^2 [f(2x+y) + f(2x-y)] + 2(1-a^2) f(y) + \frac{a^4-a^2}{6}[f(4x) - 4f(2x)] + 2f(y) \\ &= a^4 [f(2x+y) + f(2x-y)] + 2a^2(1-a^2)[f(x+y) + f(x-y)] \\ &\quad + \frac{a^2(a^4-a^2)}{3}[f(2x) - 4f(x)] + 2(1-a^2)[f(ax+y) + f(ax-y)] \\ &\quad + \frac{a^4-a^2}{3}[f(2x) - 4f(x)] + 2a^4 f(y) \end{aligned} \tag{2.11}$$

for all $x, y \in E_1$. Using (2.2) in (2.11), we get

$$\begin{aligned}
(2.12)\quad & a^2\left[f(2x+y)+f(2x-y)\right] \\
&\qquad + 2\left(1-a^2\right) f(y) + \frac{a^4-a^2}{6}\left[f(4x)-4f(2x)\right]+2f(y) \\
&= a^4\left[f(2x+y)+f(2x-y)\right] + 2a^2\left(1-a^2\right)\left[f(x+y)+f(x-y)\right] \\
&\quad + \frac{a^2\left(a^4-a^2\right)}{3}\left[f(2x)-4f(x)\right] + \frac{a^4-a^2}{3}\left[f(2x)-4f(x)\right] + 2a^4 f(y) \\
&\quad + 2\left(1-a^2\right)\left[a^2\left(f(x+y)+f(x-y)\right) + 2\left(1-a^2\right) f(y) + \frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]\right]
\end{aligned}$$

for all $x, y \in E_1$. Letting $y = 0$ in (2.2), we obtain

$$(2.13)\qquad 2f(ax) = 2a^2 f(x) + \frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]$$

for all $x, y \in E_1$. Replacing y by x in (2.2), we get

$$(2.14)\quad f((a+1)x)+f((a-1)x) = a^2 f(2x)+2\left(1-a^2\right) f(x)+\frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]$$

for all $x \in E_1$. Replacing y by ax in (2.2), we obtain

$$\begin{aligned}
(2.15)\quad f(2ax) &= a^2\left[f((1+a)x)+f((1-a)x)\right] \\
&\qquad + 2\left(1-a^2\right) f(ax) + \frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]
\end{aligned}$$

for all $x \in E_1$. Letting $y = 0$ in (2.10), we get

$$(2.16)\qquad f(2ax) = a^2 f(2x) + \frac{a^4-a^2}{12}\left[f(4x)-4f(2x)\right]$$

for all $x \in E_1$. From (2.15) and (2.16), we arrive at

$$\begin{aligned}
(2.17)\quad a^2 f(2x) + \frac{a^4-a^2}{12}\left[f(4x)-4f(2x)\right] &= a^2\left[f((1+a)x)+f((1-a)x)\right] \\
&\qquad + 2\left(1-a^2\right) f(ax) + \frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]
\end{aligned}$$

for all $x \in E_1$. Using (2.13) and (2.14) in (2.17), we obtain

$$\begin{aligned}
(2.18)\quad & a^2 f(2x) + \frac{a^4-a^2}{12}\left[f(4x)-4f(2x)\right] \\
&= a^2\left[a^2 f(2x) + 2\left(1-a^2\right) f(x) + \frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]\right] \\
&\quad + \left(1-a^2\right)\left[2a^2 f(x) + \frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]\right] \\
&\quad + \frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]
\end{aligned}$$

for all $x \in E_1$. Comparing (2.12) and (2.18), we arrive at

$$(2.19)\quad f(2x+y)+f(2x-y) = 4\left[f(x+y)+f(x-y)\right] - 8f(x) + 2f(2x) - 6f(y)$$

for all $x, y \in E_1$. Replacing y by $2y$ in (2.19), we get

$$(2.20)\quad f(2x+2y)+f(2x-2y) = 4\left[f(x+2y)+f(x-2y)\right]-8f(x)+2f(2x)-6f(2y)$$

for all $x, y \in E_1$. Interchanging x and y in (2.19) and using the evenness of f, we obtain

$$f(x+2y)+f(x-2y)=4[f(x+y)+f(x-y)]-8f(y)+2f(2y)-6f(x) \tag{2.21}$$

for all $x, y \in E_1$. Using (2.21) in (2.20), we get

$$\begin{aligned} &f(2x+2y)+f(2x-2y) \\ &\quad = 16[f(x+y)+f(x-y)]+2f(2y)-32f(y)+2f(2x)-32f(x) \end{aligned} \tag{2.22}$$

for all $x, y \in E_1$. Rearranging (2.22), we have

$$\begin{aligned} &\{f(2x+2y)-16f(x+y)\}+\{f(2x-2y)-16f(x-y)\} \\ &\quad = 2\{f(2x)-16f(x)\}+2\{f(2y)-16f(y)\} \end{aligned} \tag{2.23}$$

for all $x, y \in E_1$. Let $\alpha : E_1 \to E_2$ defined by

$$\alpha(x)=f(2x)-16f(x), \qquad \forall x \in E_1. \tag{2.24}$$

Applying (2.24) in (2.23), we arrive at

$$\alpha(x+y)+\alpha(x-y)=2\alpha(x)+2\alpha(y) \qquad \forall x \in E_1. \tag{2.25}$$

Hence $\alpha : E_1 \to E_2$ is quadratic mapping.

Since α is quadratic, we have $\alpha(2x)=4\alpha(x)$ for all $x \in E_1$. Then

$$f(4x)=20f(2x)-64f(x) \tag{2.26}$$

for all $x \in E_1$. Replacing (x, y) by $(2x, 2y)$ in (2.19), we get

$$\begin{aligned} &f(2(2x+y))+f(2(2x-y)) \\ &\quad = 4[f(2(x+y))+f(2(x-y))]-8f(2x)+2f(4x)-6f(2y) \end{aligned} \tag{2.27}$$

for all $x, y \in E_1$. Using (2.26) in (2.27), we obtain

$$\begin{aligned} &f(2(2x+y))+f(2(2x-y)) \\ &\quad = 4[f(2(x+y))+f(2(x-y))]+32\{f(2x)-4f(x)\}-6f(2y) \end{aligned} \tag{2.28}$$

for all $x, y \in E_1$. Multiplying (2.19) by 4, we arrive at

$$\begin{aligned} &4f(2x+y)+4f(2x-y) \\ &\quad = 16[f(x+y)+f(x-y)]+8\{f(2x)-4f(x)\}-24f(y) \end{aligned} \tag{2.29}$$

for all $x, y \in E_1$. Subtracting (2.29) from (2.28), we get

$$\begin{aligned} &\{f(2(2x+y))-4f(2x+y)\}+\{f(2(2x-y))-4f(2x-y)\} \\ &\quad = 4\{f(2(x+y))-4f(x+y)\}+4\{f(2(x-y))-4f(x-y)\} \\ &\qquad + 24\{f(2x)-4f(x)\}-6\{f(2y)-4f(y)\} \end{aligned} \tag{2.30}$$

for all $x, y \in E_1$. Let $\beta : E_1 \to E_2$ be defined by

$$\beta(x)=f(2x)-4f(x), \forall x \in E_1. \tag{2.31}$$

Applying (2.30) in (2.31), we arrive at

$$\beta(2x+y)+\beta(2x-y)=4[\beta(x+y)+\beta(x-y)]+24\beta(x)-6\beta(y) \tag{2.32}$$

for all $x, y \in E_1$. Hence $\beta : E_1 \to E_2$ is quartic mapping.

On the other hand, we have

$$f(x)=\frac{\beta(x)-\alpha(x)}{12} \qquad \forall x \in E_1. \tag{2.33}$$

This means that f is quadratic-quartic function. This completes the proof of the theorem. □

Theorem 2.2. *Let $f : E_1 \to E_2$ be a function satisfying (1.20) for all $x, y \in E_1$. If f is odd then f is additive - cubic.*

Proof. Let f be an odd function (i.e., $f(-x) = -f(x)$). Then equation (1.20) becomes

$$f(x + ay) + f(x - ay) = a^2 [f(x + y) + f(x - y)] + 2\left(1 - a^2\right) f(x) \tag{2.34}$$

for all $x, y \in E_1$. By Lemma 2.2 of [13], f is additive-cubic. □

Theorem 2.3. *Let $f : E_1 \to E_2$ be a function satisfying (1.20) for all $x, y \in E_1$ if and only if there exists functions $A : E_1 \to E_2$, $B : E_1 \times E_1 \to E_2$, $C : E_1 \times E_1 \times E_1 \to E_2$ and $D : E_1 \times E_1 \times E_1 \times E_1 \to E_2$ such that*

$$f(x) = A(x) + B(x, x) + C(x, x, x) + D(x, x, x, x) \tag{2.35}$$

for all $x \in E_1$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is symmetric multi-additive.

Proof. Let $f : E_1 \to E_2$ be a function satisfying (1.20). We decompose f into even and odd parts by setting

$$f_e(x) = \frac{1}{2}\{f(x) + f(-x)\}, \qquad f_o(x) = \frac{1}{2}\{f(x) - f(-x)\}$$

for all $x \in E_1$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in E_1$. It is easy to show that the functions f_e and f_o satisfy (1.20). Hence by Theorem 2.1 and 2.2, we see that the function f_e is quadratic-quartic and f_o is additive-cubic, respectively. Thus there exist a symmetric bi-additive function $B : E_1 \times E_1 \to E_2$ and a symmetric multi-additive function $D : E_1 \times E_1 \times E_1 \times E_1 \to E_2$ such that $f_e(x) = B(x, x) + D(x, x, x, x)$ for all $x \in E_1$, and the function $A : E_1 \to E_2$ is additive and $C : E_1 \times E_1 \times E_1 \to E_2$ such that $f_o(x) = A(x) + C(x, x, x)$, where C is symmetric for each fixed one variable and is additive for fixed two variables. Hence we get (2.35) for all $x \in E_1$.

Conversely let $f(x) = A(x) + B(x, x) + C(x, x, x) + D(x, x, x, x)$ for all $x \in E_1$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is symmetric multi-additive. Then it is easy to show that f satisfies (1.20). □

3. STABILITY OF THE FUNCTIONAL EQUATION (1.20)

In this section, we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.20). Throughout this section, let E_1 be a real normed space and E_2 be a Banach space. Define a difference operator $Df : E_1 \times E_1 \to E_2$ by

$$\begin{aligned} Df(x, y) = f(x + ay) + f(x - ay) - a^2 [f(x + y) + f(x - y)] - 2\left(1 - a^2\right) f(x) \\ - \frac{a^4 - a^2}{12} [f(2y) + f(-2y) - 4f(y) - 4f(-y)] \end{aligned}$$

for all $x, y \in E_1$.

Theorem 3.1. *Let $\phi_b : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi_b(2^n x, 2^n y)}{4^n} \quad \text{converges and} \quad \lim_{n\to\infty} \frac{\phi_b(2^n x, 2^n y)}{4^n} = 0 \tag{3.1}$$

for all $x, y \in E_1$ and let $f : E_1 \to E_2$ be an even function which satisfies the inequality

$$\|Df(x, y)\| \le \phi_b(x, y) \tag{3.2}$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \to E_2$ such that

$$\|f(2x) - 16f(x) - B(x)\| \le \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k} \tag{3.3}$$

for all $x \in E_1$, where the mapping $B(x)$ and $\Phi_b(2^k x)$ are defined by

$$B(x) = \lim_{n\to\infty} \frac{1}{4^n} \left\{ f\left(2^{n+1}x\right) - 16f(2^n x) \right\} \tag{3.4}$$

$$\Phi_b\left(2^k x\right) = \frac{1}{a^4 - a^2}\Big[12\left(1-a^2\right)\phi_b\left(0, 2^k x\right) + 12a^2\phi_b\left(2^k x, 2^k x\right) + 6\phi_b\left(0, 2^{k+1}x\right) + 12\phi_b\left(2^k ax, 2^k x\right)\Big] \tag{3.5}$$

for all $x \in E_1$.

Proof. Using the evenness of f, from (3.2) we get

$$\Big\| f(x+ay) + f(x-ay) - a^2[f(x+y) + f(x-y)] - 2\left(1-a^2\right)f(x) - \frac{(a^4-a^2)}{12}[2f(2y) - 8f(y)] \Big\| \le \phi_b(x,y) \tag{3.6}$$

for all $x, y \in E_1$. Interchanging x and y in (3.6), we obtain

$$\Big\| f(ax+y) + f(ax-y) - a^2[f(x+y) + f(x-y)] - 2\left(1-a^2\right)f(y) - \frac{(a^4-a^2)}{12}[2f(2x) - 8f(x)] \Big\| \le \phi_b(y,x) \tag{3.7}$$

for all $x, y \in E_1$. Letting $y = 0$ in (3.7), we get

$$\Big\| 2f(ax) - 2a^2 f(x) - \frac{(a^4-a^2)}{12}[2f(2x) - 8f(x)] \Big\| \le \phi_b(0,x) \tag{3.8}$$

for all $x \in E_1$. Putting $y = x$ in (3.7), we obtain

$$\Big\| f((a+1)x) + f((a-1)x) - a^2 f(2x) - 2\left(1-a^2\right)f(x) - \frac{(a^4-a^2)}{12}[2f(2x) - 8f(x)] \Big\| \le \phi_b(x,x) \tag{3.9}$$

for all $x \in E_1$. Replacing x by $2x$ in (3.8), we get

$$\Big\| 2f(2ax) - 2a^2 f(2x) - \frac{(a^4-a^2)}{12}[2f(4x) - 8f(2x)] \Big\| \le \phi_b(0,2x) \tag{3.10}$$

for all $x \in E_1$. Setting y by ax in (3.7), we obtain

$$\Big\| f(2ax) - a^2[f((1+a)x) + f((1-a)x)] - 2\left(1-a^2\right)f(ax) - \frac{(a^4-a^2)}{12}[2f(2x) - 8f(x)] \Big\| \le \phi_b(ax,x) \tag{3.11}$$

for all $x \in E_1$. Multiplying (3.8), (3.9), (3.10) and (3.11) by $12(1-a^2)$, $12a^2$, 6 and 12 respectively, we have

$$\begin{aligned}
&\left(a^4-a^2\right)\|f(4x)-20f(2x)+64f(x)\| \\
&=\Bigg\|\Big\{24\left(1-a^2\right)f(ax)-24a^2\left(1-a^2\right)f(x) \\
&\qquad -\frac{12\left(1-a^2\right)\left(a^4-a^2\right)}{12}[2f(2x)-8f(x)]\Big\} \\
&\qquad +\Big\{12a^2f((a+1)x)+12a^2f((a-1)x)-12a^4f(2x) \\
&\qquad -24a^2\left(1-a^2\right)f(x)-\frac{12a^2\left(a^4-a^2\right)}{12}[2f(2x)-8f(x)]\Big\} \\
&\qquad +\Big\{-12f(2ax)+12a^2f(2x)+\frac{6\left(a^4-a^2\right)}{12}[2f(4x)-8f(2x)]\Big\} \\
&\qquad +\Big\{12f(2ax)-12a^2\left[f((1+a)x)+f((1-a)x)\right] \\
&\qquad -24\left(1-a^2\right)f(ax)-\frac{12\left(a^4-a^2\right)}{12}[2f(2x)-8f(x)]\Big\}\Bigg\| \\
&\le 12\left(1-a^2\right)\phi_b(0,x)+12a^2\phi_b(x,x)+6\phi_b(0,2x)+12\phi_b(ax,x)
\end{aligned}$$

for all $x \in E_1$. Hence from the above inequality, we get

$$\begin{aligned}
&\|f(4x)-20f(2x)+64f(x)\| \\
&\quad\le \frac{1}{\left(a^4-a^2\right)}\left[12\left(1-a^2\right)\phi_b(0,x)+12a^2\phi_b(x,x)+6\phi_b(0,2x)+12\phi_b(ax,x)\right]
\end{aligned}\tag{3.12}$$

for all $x \in E_1$. From (3.12), we arrive at

$$\|f(4x)-20f(2x)+64f(x)\| \le \Phi_b(x), \tag{3.13}$$

where

$$\Phi_b(x)=\frac{1}{a^4-a^2}\left[12\left(1-a^2\right)\phi_b(0,x)+12a^2\phi_b(x,x)+6\phi_b(0,2x)+12\phi_b(ax,x)\right]$$

for all $x \in E_1$. It is easy to see from (3.13) that

$$\|f(4x)-16f(2x)-4\{f(2x)-16f(x)\}\| \le \Phi_b(x) \tag{3.14}$$

for all $x \in E_1$. Using (2.24) in (3.14), we obtain

$$\|\alpha(2x)-4\alpha(x)\| \le \Phi_b(x) \tag{3.15}$$

for all $x \in E_1$. From (3.15), we have

$$\left\|\frac{\alpha(2x)}{4}-\alpha(x)\right\| \le \frac{\Phi_b(x)}{4} \tag{3.16}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 4 in (3.16), we obtain

$$\left\|\frac{\alpha(2^2x)}{4^2}-\frac{\alpha(2x)}{4}\right\| \le \frac{\Phi_b(2x)}{4^2} \tag{3.17}$$

for all $x \in E_1$. From (3.16) and (3.17), we arrive at

$$\left\|\frac{\alpha\left(2^2 x\right)}{4^2}-\alpha(x)\right\| \leq\left\|\frac{\alpha\left(2^2 x\right)}{4^2}-\frac{\alpha(2 x)}{4}\right\|+\left\|\frac{\alpha(2 x)}{4}-\alpha(x)\right\| \leq \frac{1}{4}\left[\Phi_b(x)+\frac{\Phi_b(2 x)}{4}\right] \tag{3.18}$$

for all $x \in E_1$. In general for any positive integer n, we get

$$\left\|\frac{\alpha\left(2^n x\right)}{4^n}-\alpha(x)\right\| \leq \frac{1}{4} \sum_{k=0}^{n-1} \frac{\Phi_b\left(2^k x\right)}{4^k} \leq \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b\left(2^k x\right)}{4^k} \tag{3.19}$$

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{\frac{\alpha(2^n x)}{4^n}\right\}$, replace x by $2^m x$ and divide by 4^m in (3.19). For any $m, n>0$, we have

$$\begin{aligned}
\left\|\frac{\alpha\left(2^{n+m} x\right)}{4^{n+m}}-\frac{\alpha\left(2^m x\right)}{4^m}\right\| &= \frac{1}{4^m}\left\|\frac{\alpha\left(2^n 2^m x\right)}{4^n}-\alpha\left(2^m x\right)\right\| \\
&\leq \frac{1}{4} \sum_{k=0}^{n-1} \frac{\Phi_b\left(2^{k+m} x\right)}{4^{k+m}} \\
&\leq \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b\left(2^{k+m} x\right)}{4^{k+m}} \\
&\rightarrow 0 \quad \text{as} \quad m \rightarrow \infty
\end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{\frac{\alpha(2^n x)}{4^n}\right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a quadratic mapping $B: E_1 \rightarrow E_2$ such that

$$B(x)=\lim_{n \rightarrow \infty} \frac{\alpha\left(2^n x\right)}{4^n} \qquad \forall x \in E_1 .$$

Letting $n \rightarrow \infty$ in (3.19) and using (2.24), we see that (3.3) holds for all $x \in E_1$. To prove that B satisfies (1.20), replace (x, y) by $(2^n x, 2^n y)$ and divide by 4^n in (3.2). We obtain

$$\begin{aligned}
\frac{1}{4^n}\Big\| & f\left(2^n(x+a y)\right)+f\left(2^n(x-a y)\right)-a^2\left[f\left(2^n(x+y)\right)+f\left(2^n(x-y)\right)\right] \\
& -2\left(1-a^2\right) f\left(2^n x\right)-\frac{\left(a^4-a^2\right)}{12}\left[f\left(2^n(2 y)\right)+f\left(2^n(-2 y)\right)\right] \\
& -\frac{\left(a^4-a^2\right)}{12}\left[-4 f\left(2^n y\right)-4 f\left(2^n(-y)\right)\right]\Big\| \leq \frac{\phi\left(2^n x, 2^n y\right)}{4^n}
\end{aligned}$$

for all $x, y \in E_1$. Letting $n \rightarrow \infty$ in the above inequality, we see that

$$\begin{aligned}
\Big\| & B(x+a y)+B(x-a y)-a^2[B(x+y)+B(x-y)]-2\left(1-a^2\right) B(x) \\
& -\frac{\left(a^4-a^2\right)}{12}[B(2 y)+B(-2 y)-4 B(y)-4 B f(-y)]\Big\| \leq 0,
\end{aligned}$$

which gives

$$B(x+ay)+B(x-ay)=a^2\left[B(x+y)+B(x-y)\right]+2\left(1-a^2\right)B(x)$$
$$+\frac{\left(a^4-a^2\right)}{12}\left[B(2y)+B(-2y)-4B(y)-4Bf(-y)\right]$$

for all $x,y\in E_1$. Hence B satisfies (1.20). To prove that B is unique, let B' be another quadratic function satisfying (1.20) and (3.3). We have

$$\begin{aligned}\|B(x)-B'(x)\|&=\frac{1}{4^n}\|B(2^n x)-B'(2^n x)\|\\&\leq\frac{1}{4^n}\{\|B(2^n x)-\alpha(2^n x)\|+\|\alpha(2^n x)-B'(2^n x)\|\}\\&\leq\frac{1}{4^n}\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k}\\&\to 0\qquad\text{as}\qquad n\to\infty\end{aligned}$$

for all $x\in E_1$. Hence B is unique. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.20).

Corollary 3.2. *Let ε,p be nonnegative real numbers. Suppose that an even function $f:E_1\to E_2$ satisfies the inequality*

$$\|Df(x,y)\|\leq\begin{cases}\varepsilon\left(\|x\|^p+\|y\|^p\right), & 0\leq p<2;\\ \varepsilon, & 0\leq p<1;\\ \varepsilon\|x\|^p\|y\|^p, & 0\leq p<1;\\ \varepsilon\left(\|x\|^p\|y\|^p+\left\{\|x\|^{2p}+\|y\|^{2p}\right\}\right), & \end{cases}\tag{3.20}$$

for all $x,y\in E_1$. Then there exists a unique quadratic function $B:E_1\to E_2$ such that

$$\|f(2x)-16f(x)-B(x)\|\leq\begin{cases}\frac{\lambda_1\|x\|^p}{4-2^p},\\ 10\lambda_2,\\ \frac{\lambda_3\|x\|^{2p}}{4-2^{2p}},\\ \frac{\lambda_4\|x\|^{2p}}{4-2^{2p}},\end{cases}\tag{3.21}$$

where

$$\lambda_1=\frac{\varepsilon\left\{24+12a^2+12\left(a^p\right)+6\left(2^p\right)\right\}}{a^4-a^2},\qquad\lambda_2=\frac{\varepsilon}{a^4-a^2},$$
$$\lambda_3=\frac{12\varepsilon\left\{a^2+a^p\right\}}{a^4-a^2}\qquad\text{and}\qquad\lambda_4=\frac{\varepsilon\left\{24+24a^2+12\left(a^p\right)+12\left(a^{2p}\right)+6\left(2^{2p}\right)\right\}}{a^4-a^2}$$

for all $x\in E_1$.

Theorem 3.3. *Let $\phi_d:E_1\times E_1\to[0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty}\frac{\phi_d\left(2^n x,2^n y\right)}{16^n}\quad\text{converges and}\quad\lim_{n\to\infty}\frac{\phi_d\left(2^n x,2^n y\right)}{16^n}=0\tag{3.22}$$

for all $x,y\in E_1$ and let $f:E_1\to E_2$ be an even function which satisfies the inequality

$$\|Df(x,y)\|\leq\phi_d(x,y)\tag{3.23}$$

for all $x, y \in E_1$. Then there exists a unique quartic function $D : E_1 \to E_2$ such that

$$\|f(2x) - 4f(x) - D(x)\| \leq \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d\left(2^k x\right)}{16^k} \tag{3.24}$$

for all $x \in E_1$, where the mapping $D(x)$ and $\Phi_d(2^k x)$ are defined by

$$D(x) = \lim_{n\to\infty} \frac{1}{16^n} \left\{ f\left(2^{n+1}x\right) - 4f\left(2^n x\right) \right\}, \tag{3.25}$$

$$\begin{aligned} \Phi_d\left(2^k x\right) = \frac{1}{a^4 - a^2} \Big[& 12\left(1 - a^2\right) \phi_d\left(0, 2^k x\right) + 12a^2 \phi_d\left(2^k x, 2^k x\right) \\ & + 6\phi_d\left(0, 2^{k+1}x\right) + 12\phi_d\left(2^k ax, 2^k x\right) \Big] \end{aligned} \tag{3.26}$$

for all $x \in E_1$.

Proof. Along similar lines to those in the proof of Theorem 3.1, we have

$$\|f(4x) - 20f(2x) + 64f(x)\| \leq \Phi_d(x), \tag{3.27}$$

where

$$\Phi_d(x) = \frac{1}{a^4 - a^2} \left[12\left(1 - a^2\right) \phi_d(0, x) + 12a^2 \phi_d(x, x) + 6\phi_d(0, 2x) + 12\phi_d(ax, x) \right]$$

for all $x \in E_1$. It is easy to see from (3.27) that

$$\|f(4x) - 4f(2x) - 16\{f(2x) - 4f(x)\}\| \leq \Phi_d(x) \tag{3.28}$$

for all $x \in E_1$. Using (2.31) in (3.28), we obtain

$$\|\beta(2x) - 16\beta(x)\| \leq \Phi_d(x) \tag{3.29}$$

for all $x \in E_1$. From (3.29), we have

$$\left\| \frac{\beta(2x)}{16} - \beta(x) \right\| \leq \frac{\Phi_d(x)}{16} \tag{3.30}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 16 in (3.30), we obtain

$$\left\| \frac{\beta(2^2 x)}{16^2} - \frac{\beta(2x)}{16} \right\| \leq \frac{\Phi_d(2x)}{16^2} \tag{3.31}$$

for all $x \in E_1$. From (3.30) and (3.31), we arrive at

$$\begin{aligned} \left\| \frac{\beta(2^2 x)}{16^2} - \beta(x) \right\| &\leq \left\| \frac{\beta(2^2 x)}{16^2} - \frac{\beta(2x)}{16} \right\| + \left\| \frac{\beta(2x)}{16} - \beta(x) \right\| \\ &\leq \frac{1}{16} \left[\Phi_d(x) + \frac{\Phi_d(2x)}{16} \right] \end{aligned} \tag{3.32}$$

for all $x \in E_1$. In general for any positive integer n, we get

$$\begin{aligned} \left\| \frac{\beta(2^n x)}{16^n} - \beta(x) \right\| &\leq \frac{1}{16} \sum_{k=0}^{n-1} \frac{\Phi_d\left(2^k x\right)}{16^k} \\ &\leq \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d\left(2^k x\right)}{16^k} \end{aligned} \tag{3.33}$$

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{\frac{\beta(2^n x)}{16^n}\right\}$, replace x by $2^m x$ and divide by 16^m in (3.33). For any $m, n > 0$, we then have

$$\begin{aligned} \left\| \frac{\beta\left(2^{n+m}x\right)}{16^{n+m}} - \frac{\beta\left(2^m x\right)}{16^m} \right\| &= \frac{1}{16^m} \left\| \frac{\beta\left(2^n 2^m x\right)}{16^n} - \beta\left(2^m x\right) \right\| \\ &\leq \frac{1}{16} \sum_{k=0}^{n-1} \frac{\Phi_d\left(2^{k+m}x\right)}{16^{k+m}} \\ &\leq \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d\left(2^{k+m}x\right)}{16^{k+m}} \\ &\to 0 \quad \text{as} \quad m \to \infty \end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{\frac{\beta(2^n x)}{16^n}\right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a quartic mapping $D : E_1 \to E_2$ such that

$$D\left(x\right) = \lim_{n\to\infty} \frac{\beta\left(2^n x\right)}{16^n} \qquad \forall x \in E_1.$$

Letting $n \to \infty$ in (3.33) and using (2.31) we see that (3.24) holds for all $x \in E_1$. The proof that D satisfies (1.20) and is unique is similar to that for Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.20).

Corollary 3.4. *Let ε, p be nonnegative real numbers. Suppose that an even function $f : E_1 \to E_2$ satisfies the inequality*

$$\|Df\left(x,y\right)\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 4; \\ \varepsilon, & \\ \varepsilon\|x\|^p\|y\|^p, & 0 \leq p < 2; \\ \varepsilon\left(\|x\|^p\|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < 2 \end{cases} \tag{3.34}$$

for all $x, y \in E_1$. Then there exists a unique quartic function $D : E_1 \to E_2$ such that

$$\|f\left(2x\right) - 4f\left(x\right) - D\left(x\right)\| \leq \begin{cases} \frac{\lambda_1\|x\|^p}{16-2^p}, \\ 2\lambda_2, \\ \frac{\lambda_3\|x\|^{2p}}{16-2^{2p}}, \\ \frac{\lambda_4\|x\|^{2p}}{16-2^{2p}}, \end{cases} \tag{3.35}$$

for all $x \in E_1$, where λ_i $(i = 1, 2, 3, 4)$ are given in Corollary 3.2.

Theorem 3.5. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi_b\left(2^n x, 2^n y\right)}{4^n}, \quad \sum_{n=0}^{\infty} \frac{\phi_d\left(2^n x, 2^n y\right)}{16^n} \qquad \text{converges} \tag{3.36}$$

and

$$\lim_{n\to\infty} \frac{\phi_b\left(2^n x, 2^n y\right)}{4^n} = 0 = \lim_{n\to\infty} \frac{\phi_d\left(2^n x, 2^n y\right)}{16^n} \tag{3.37}$$

for all $x, y \in E_1$. Suppose that an even function $f : E_1 \to E_2$ satisfies the inequalities (3.2) and (3.23) for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that

$$\|f(x) - B(x) - D(x)\| \leq \frac{1}{12}\left\{\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^k x\right)}{16^k}\right\} \tag{3.38}$$

for all $x \in E_1$, where $\Phi_b\left(2^k x\right)$ and $\Phi_d\left(2^k x\right)$ are defined in (3.5) and (3.26), respectively for all $x \in E_1$.

Proof. By Theorems 3.1 and 3.3, there exists a unique quadratic function $B_1 : E_1 \to E_2$ and a unique quartic function $D_1 : E_1 \to E_2$ such that

$$\|f(2x) - 16f(x) - B_1(x)\| \leq \frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k} \tag{3.39}$$

and

$$\|f(2x) - 4f(x) - D_1(x)\| \leq \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^k x\right)}{16^k} \tag{3.40}$$

for all $x \in E_1$. Now from (3.39) and (3.40), one can see that

$$\begin{aligned}
&\left\|f(x) + \frac{1}{12}B_1(x) - \frac{1}{12}D_1(x)\right\| \\
&\qquad = \left\|\left\{-\frac{f(2x)}{12} + \frac{16f(x)}{12} + \frac{B_1(x)}{12}\right\} + \left\{\frac{f(2x)}{12} - \frac{4f(x)}{12} - \frac{D_1(x)}{12}\right\}\right\| \\
&\qquad \leq \frac{1}{12}\left\{\|f(2x) - 16f(x) - B_1(x)\| + \|f(2x) - 4f(x) - D_1(x)\|\right\} \\
&\qquad \leq \frac{1}{12}\left\{\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^k x\right)}{16^k}\right\}
\end{aligned}$$

for all $x \in E_1$. Thus we obtain (3.38) by defining $B(x) = \frac{-1}{12}B_1(x)$ and $D(x) = \frac{1}{12}D_1(x)$, where $\Phi_b\left(2^k x\right)$ and $\Phi_d\left(2^k x\right)$ are defined in (3.5) and (3.26), respectively for all $X \in E_1$. □

The following corollary is the immediate consequence of Theorem 3.5 concerning the stability of (1.20).

Corollary 3.6. *Let ϵ, p be nonnegative real numbers. Suppose an even function $f : E_1 \to E_2$ satisfies the inequality*

$$\|Df(x,y)\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 2; \\ \varepsilon, & \\ \varepsilon\|x\|^p\|y\|^p, & 0 \leq p < 1; \\ \varepsilon\left(\|x\|^p\|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < 1 \end{cases} \tag{3.41}$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that

$$\|f(x) - B(x) - D(x)\| \leq \begin{cases} \frac{\lambda_1 \|x\|^p}{12}\left\{\frac{1}{4-2^p} + \frac{1}{16-2^p}\right\}, \\ \lambda_2 \\ \frac{\lambda_3 \|x\|^{2p}}{12}\left\{\frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}}\right\}, \\ \frac{\lambda_4 \|x\|^{p}}{12}\left\{\frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}}\right\}, \end{cases} \tag{3.42}$$

for all $x \in E_1$, where λ_i $(i = 1, 2, 3, 4)$ are given in Corollary 3.2.

Theorem 3.7. *Let $\phi_a : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} \quad \textit{converges and} \quad \lim_{n\to\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} = 0 \tag{3.43}$$

for all $x, y \in E_1$ and let $f : E_1 \to E_2$ be an odd function with $f(0) = 0$ which satisfies the inequality

$$\|Df(x, y)\| \leq \phi_a(x, y) \tag{3.44}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Phi_a\left(2^k x\right)}{2^k} \tag{3.45}$$

for all $x \in E_1$, where the mapping $A(x)$ and $\Phi_a(2^k x)$ are defined by

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n}\left\{f\left(2^{n+1}x\right) - 8f(2^n x)\right\} \tag{3.46}$$

$$\begin{aligned} \Phi_a\left(2^k x\right) &= \frac{1}{a^4 - a^2}\left[\left(5 - 4a^2\right)\phi_a\left(2^k x, 2^k x\right) + a^2\phi_a\left(2^{k+1}x, 2^{k+1}x\right) + 2a^2\phi_a\left(2^{k+1}x, 2^k x\right)\right. \\ &\quad + \left(4 - 2a^2\right)\phi_a\left(2^k x, 2^{k+1}x\right) + \phi_a\left(2^k x, 2^k 3x\right) + 2\phi_a\left(2^k(1+a)x, 2^k x\right) \\ &\quad \left. + 2\phi_a\left(2^k(1-a)x, 2^k x\right) + \phi_a\left(2^k(1+2a)x, 2^k x\right) + \phi_a\left(2^k(1-2a)x, 2^k x\right)\right] \end{aligned} \tag{3.47}$$

for all $x \in E_1$.

Proof. Using the oddness of f and from (3.44), we get

$$\left\|f(x + ay) + f(x - ay) - a^2[f(x + y) + f(x - y)] - 2\left(1 - a^2\right)f(x)\right\| \leq \phi_a(x, y) \tag{3.48}$$

for all $x \in E_1$. Replacing y by x in (3.48), we obtain

$$\left\|f((1 + a)x) + f((1 - a)x) - a^2 f(2x) - 2\left(1 - a^2\right)f(x)\right\| \leq \phi_a(x, x) \tag{3.49}$$

for all $x \in E_1$. Replacing x by $2x$ in (3.49), we get

$$\left\|f(2(1 + a)x) + f(2(1 - a)x) - a^2 f(4x) - 2\left(1 - a^2\right)f(2x)\right\| \leq \phi_a(2x, 2x) \tag{3.50}$$

for all $x \in E_1$. Again replacing (x, y) by $(2x, x)$ in (3.48), we obtain

$$\left\|f((2 + a)x) + f((2 - a)x) - a^2 f(3x) - a^2 f(x) - 2\left(1 - a^2\right)f(2x)\right\| \leq \phi_a(2x, x) \tag{3.51}$$

for all $x \in E_1$. Replacing y by $2x$ in (3.48), we get

$$\left\|f\left(\left(1+2a\right)x\right)+f\left(\left(1-2a\right)x\right)-a^2 f\left(3x\right)+a^2 f\left(x\right)-2\left(1-a^2\right)f\left(x\right)\right\| \leq \phi_a\left(x,2x\right) \tag{3.52}$$

for all $x \in E_1$. Replacing y by $3x$ in (3.48), we obtain

$$\left\|f\left(\left(1+3a\right)x\right)+f\left(\left(1-3a\right)x\right)-a^2 f\left(4x\right)+a^2 f\left(2x\right)-2\left(1-a^2\right)f\left(x\right)\right\| \leq \phi_a\left(x,3x\right) \tag{3.53}$$

for all $x \in E_1$. Replacing (x, y) by $((1 + a)x, x)$ in (3.48), we get

$$\left\|f\left(\left(1+2a\right)x\right)+f\left(x\right)-a^2 f\left(\left(2+a\right)x\right)-a^2 f\left(ax\right)-2\left(1-a^2\right)f\left(\left(1+a\right)x\right)\right\| \leq \phi_a\left(\left(1+a\right)x,x\right) \tag{3.54}$$

for all $x \in E_1$. Again replacing (x, y) by $((1 - a)x, x)$ in (3.48), we obtain

$$\left\|f\left(\left(1-2a\right)x\right)+f\left(x\right)-a^2 f\left(\left(2-a\right)x\right)+a^2 f\left(ax\right)-2\left(1-a^2\right)f\left(\left(1-a\right)x\right)\right\| \leq \phi_a\left(\left(1-a\right)x,x\right) \tag{3.55}$$

for all $x \in E_1$. Adding (3.54) and (3.55), we arrive at

$$\begin{aligned}\|f\left(\left(1+2a\right)x\right)+f\left(\left(1-2a\right)x\right)+2f\left(x\right)-a^2 f\left(\left(2+a\right)x\right)-a^2 f\left(\left(2-a\right)x\right)\\ -2\left(1-a^2\right)f\left(\left(1+a\right)x\right)-2\left(1-a^2\right)f\left(\left(1-a\right)x\right)\big\|\\ \leq \phi_a\left(\left(1+a\right)x,x\right)+\phi_a\left(\left(1-a\right)x,x\right)\end{aligned} \tag{3.56}$$

for all $x \in E_1$. Replacing (x, y) by $((1 + 2a)x, x)$ in (3.48), we get

$$\begin{aligned}\big\|f\left(\left(1+3a\right)x\right)+f\left(\left(1+a\right)x\right)-a^2 f\left(2\left(1+a\right)x\right)-a^2 f\left(2ax\right)\\ -2\left(1-a^2\right)f\left(\left(1+2a\right)x\right)\big\| \leq \phi_a\left(\left(1+2a\right)x,x\right)\end{aligned} \tag{3.57}$$

for all $x \in E_1$. Again replacing (x, y) by $((1 - 2a)x, x)$ in (3.48), we obtain

$$\begin{aligned}\big\|f\left(\left(1-3a\right)x\right)+f\left(\left(1-a\right)x\right)-a^2 f\left(2\left(1-a\right)x\right)+a^2 f\left(2ax\right)\\ -2\left(1-a^2\right)f\left(\left(1-2a\right)x\right)\big\| \leq \phi_a\left(\left(1-2a\right)x,x\right)\end{aligned} \tag{3.58}$$

for all $x \in E_1$. Adding (3.57) and (3.58), we arrive at

$$\begin{aligned}\|f\left(\left(1+3a\right)x\right)+f\left(\left(1-3a\right)x\right)+f\left(\left(1+a\right)x\right)+f\left(\left(1-a\right)x\right)-a^2 f\left(2\left(1+a\right)x\right)\\ -a^2 f\left(2\left(1-a\right)x\right)-2\left(1-a^2\right)f\left(\left(1+2a\right)x\right)-2\left(1-a^2\right)f\left(\left(1-2a\right)x\right)\big\|\\ \leq \phi_a\left(\left(1+2a\right)x,x\right)+\phi_a\left(\left(1-2a\right)x,x\right)\end{aligned} \tag{3.59}$$

for all $x \in E_1$. Now multiplying (3.49) by $2(1-a^2)$, (3.51) by a^2 and adding (3.52) and (3.56), we have

$$\begin{aligned}
&\left(a^4-a^2\right)\|f(3x)-4f(2x)+5f(x)\| \\
&=\Big\|\Big\{2\left(1-a^2\right)f((1+a)x)+2\left(1-a^2\right)f((1-a)x)-2a^2\left(1-a^2\right)f(2x) \\
&\qquad -4\left(1-a^2\right)^2 f(x)\Big\}+\left\{a^2 f((2+a)x)+a^2 f((2-a)x)-a^4 f(3x)\right. \\
&\qquad \left.-a^4 f(x)-2a^2\left(1-a^2\right)f(2x)\right\}+\{-f((1+2a)x) \\
&\qquad -f((1-2a)x)+a^2 f(3x)-a^2 f(x)+2\left(1-a^2\right)f(x)\} \\
&\qquad +\{f((1+2a)x)+f((1-2a)x)+2f(x)-a^2 f((2+a)x) \\
&\qquad -a^2 f((2-a)x)-2\left(1-a^2\right)f((1+a)x)-2\left(1-a^2\right)f((1-a)x)\}\Big\| \\
&\le 2\left(1-a^2\right)\phi_a(x,x)+a^2\phi_a(2x,x)+\phi_a(x,2x) \\
&\qquad +\phi_a((1+a)x,x)+\phi_a((1-a)x,x)
\end{aligned}$$

for all $x \in E_1$. Hence from the above inequality, we get

$$\begin{aligned}
\|f(3x)-4f(2x)+5f(x)\| \le \frac{1}{(a^4-a^2)}&\left[2\left(1-a^2\right)\phi_a(x,x)+a^2\phi_a(2x,x)\right. \\
&\left.+\phi_a(x,2x)+\phi_a((1+a)x,x)+\phi_a((1-a)x,x)\right]
\end{aligned} \tag{3.60}$$

for all $x \in E_1$. Now multiplying (3.50) by a^2, (3.52) by $2(1-a^2)$ and adding (3.49), (3.53) and (3.59), we have

$$\begin{aligned}
&\left(a^4-a^2\right)\|f(4x)-2f(3x)-2f(2x)+6f(x)\| \\
&=\Big\|\{-f((1+a)x)-f((1-a)x)+a^2 f(2x)+2\left(1-a^2\right)f(x)\} \\
&\qquad +\left\{a^2 f(2(1+a)x)+a^2 f(2(1-a)x)-a^4 f(4x)-2a^2\left(1-a^2\right)f(2x)\right\} \\
&\qquad +\Big\{2\left(1-a^2\right)f((1+2a)x)+2\left(1-a^2\right)f((1-2a)x)-2a^2\left(1-a^2\right)f(3x) \\
&\qquad +2a^2\left(1-a^2\right)f(x)-4\left(1-a^2\right)^2 f(x)\Big\}+\{-f((1+3a)x) \\
&\qquad -f((1-3a)x)+a^2 f(4x)-a^2 f(2x)+2\left(1-a^2\right)f(x)\}+\{f((1+3a)x) \\
&\qquad +f((1-3a)x)+f((1+a)x)+f((1-a)x)-a^2 f(2(1+a)x) \\
&\qquad -a^2 f(2(1-a)x)-2\left(1-a^2\right)f((1+2a)x)-2\left(1-a^2\right)f((1-2a)x)\}\Big\| \\
&\le \phi_a(x,x)+a^2\phi_a(2x,2x)+2\left(1-a^2\right)\phi_a(x,2x) \\
&\qquad +\phi_a(x,3x)+\phi_a((1+2a)x,x)+\phi_a((1-2a)x,x)
\end{aligned}$$

for all $x \in E_1$. Hence from the above inequality, we get

$$\begin{aligned}
&\|f(4x)-2f(3x)-2f(2x)+6f(x)\| \\
&\quad \le \frac{1}{(a^4-a^2)}\left[\phi_a(x,x)+a^2\phi_a(2x,2x)+2\left(1-a^2\right)\phi_a(x,2x)+\phi_a(x,3x)\right. \\
&\qquad\qquad \left.+\phi_a((1+2a)x,x)+\phi_a((1-2a)x,x)\right]
\end{aligned} \tag{3.61}$$

for all $x \in E_1$. From (3.60) and (3.61), we arrive at

$$\begin{aligned}
&\|f(4x) - 10f(2x) + 16f(x)\| \\
&= \|2f(3x) - 8f(2x) + 10f(x) + f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\
&\leq 2\|f(3x) - 4f(2x) + 5f(x)\| + \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\
&\leq \frac{1}{(a^4 - a^2)} \left[\left(5 - 4a^2\right) \phi_a(x, x) + a^2 \phi_a(2x, 2x) + 2a^2 \phi_a(2x, x) \right. \\
&\qquad + \left(4 - 2a^2\right) \phi_a(x, 2x) + \phi_a(x, 3x) + 2\phi_a((1+a)x, x) \\
&\qquad \left. + 2\phi_a((1-a)x, x) + \phi_a((1+2a)x, x) + \phi_a((1-2a)x, x) \right]
\end{aligned} \tag{3.62}$$

for all $x \in E_1$. From (3.62), we have

$$\|f(4x) - 10f(2x) + 16f(x)\| \leq \Phi_a(x), \tag{3.63}$$

where

$$\begin{aligned}
\Phi_a(x) = \frac{1}{(a^4 - a^2)} &\left[\left(5 - 4a^2\right) \phi_a(x, x) + a^2 \phi_a(2x, 2x) + 2a^2 \phi_a(2x, x) \right. \\
&+ \left(4 - 2a^2\right) \phi_a(x, 2x) + \phi_a(x, 3x) + 2\phi_a((1+a)x, x) \\
&\left. + 2\phi_a((1-a)x, x) + \phi_a((1+2a)x, x) + \phi_a((1-2a)x, x) \right]
\end{aligned}$$

for all $x \in E$. It is easy to see from (3.63)

$$\|f(4x) - 8f(2x) - 2\{f(2x) - 8f(x)\}\| \leq \Phi_a(x) \tag{3.64}$$

for all $x \in E_1$. Define a mapping $\gamma : E_1 \to E_2$ by

$$\gamma(x) = f(2x) - 8f(x) \tag{3.65}$$

for all $x \in E_1$. Using (3.65) in (3.64), we obtain

$$\|\gamma(2x) - 2\gamma(x)\| \leq \Phi_a(x) \tag{3.66}$$

for all $x \in E_1$. From (3.66), we have

$$\left\| \frac{\gamma(2x)}{2} - \gamma(x) \right\| \leq \frac{\Phi_a(x)}{2} \tag{3.67}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 2 in (3.67), we obtain

$$\left\| \frac{\gamma(2^2 x)}{2^2} - \frac{\gamma(2x)}{2} \right\| \leq \frac{\Phi_a(2x)}{2^2} \tag{3.68}$$

for all $x \in E_1$. From (3.67) and (3.68), we arrive at

$$\begin{aligned}
\left\| \frac{\gamma(2^2 x)}{2^2} - \gamma(x) \right\| &\leq \left\| \frac{\gamma(2^2 x)}{2^2} - \frac{\gamma(2x)}{2} \right\| + \left\| \frac{\gamma(2x)}{2} - \gamma(x) \right\| \\
&\leq \frac{1}{2} \left[\Phi_a(x) + \frac{\Phi_a(2x)}{2} \right]
\end{aligned} \tag{3.69}$$

for all $x \in E_1$. In general for any positive integer n, we get

$$\begin{aligned}
\left\| \frac{\gamma(2^n x)}{2^n} - \gamma(x) \right\| &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\Phi_a\left(2^k x\right)}{2^k} \\
&\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a\left(2^k x\right)}{2^k}
\end{aligned} \tag{3.70}$$

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{\frac{\gamma(2^n x)}{2^n}\right\}$, replace x by $2^m x$ and divide by 2^m in (3.70). Then for any $m, n > 0$, we have

$$\begin{aligned}\left\|\frac{\gamma\left(2^{n+m}x\right)}{2^{n+m}} - \frac{\gamma\left(2^m x\right)}{2^m}\right\| &= \frac{1}{2^m}\left\|\frac{\gamma\left(2^n 2^m x\right)}{2^n} - \gamma\left(2^m x\right)\right\| \\ &\leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{\Phi_a\left(2^{k+m}x\right)}{2^{k+m}} \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a\left(2^{k+m}x\right)}{2^{k+m}} \\ &\to 0 \quad \text{as} \quad m \to \infty\end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{\frac{\gamma(2^n x)}{2^n}\right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a additive mapping $A : E_1 \to E_2$ such that

$$A\left(x\right) = \lim_{n\to\infty}\frac{\gamma\left(2^n x\right)}{2^n} \qquad \forall x \in E_1.$$

Letting $n \to \infty$ in (3.70) and using (3.65) we see that (3.45) holds for all $x \in E_1$. The proof that A satisfies (1.20) and is unique is similar to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 3.7 concerning the stability of (1.20).

Corollary 3.8. *Let ε, p be nonnegative real numbers. Suppose that an odd function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality*

$$(3.71) \qquad \left\|Df\left(x,y\right)\right\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 2; \\ \varepsilon, & \\ \varepsilon\left\|x\right\|^p\left\|y\right\|^p, & 0 \leq p < \frac{1}{2}; \\ \varepsilon\left(\|x\|^p\|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < \frac{1}{2}\end{cases}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ such that

$$(3.72) \qquad \left\|f\left(2x\right) - 8f\left(x\right) - A\left(x\right)\right\| \leq \begin{cases} \frac{\lambda_5\|x\|^p}{2-2^p}, \\ \lambda_6, \\ \frac{\lambda_7\|x\|^{2p}}{2-2^{2p}}, \\ \frac{\lambda_8\|x\|^{2p}}{2-2^{2p}}, \end{cases}$$

where

$$\begin{aligned}\lambda_5 &= \frac{\varepsilon}{a^4 - a^2}\big\{21 - 8a^2 + 2^p\left(2a^2 + 4\right) + 3^p + 2\left(1+a\right)^p \\ &\qquad + 2\left(1-a\right)^p + \left(1+2a\right)^p + \left(1-2a\right)^p\big\}, \\ \lambda_6 &= \frac{\varepsilon\left(16 - 3a^2\right)}{a^4 - a^2}, \\ \lambda_7 &= \frac{\varepsilon}{a^4 - a^2}\big\{5 - 4a^2 + 2^{2p}a^2 + 4\left(2^p\right) + 3^p + 2\left(1+a\right)^p \\ &\qquad + 2\left(1-a\right)^p + \left(1+2a\right)^p + \left(1-2a\right)^p\big\}\end{aligned}$$

and

$$\lambda_8 = \frac{\varepsilon}{a^4 - a^2} \left\{26 - 12a^2 + 2^{2p}\left(3a^2+4\right) + 3^{2p} + 2(1+a)^{2p} + 2(1-a)^{2p} + (1+2a)^{2p} + (1-2a)^{2p} + 4(2^p) + 3^p + 2(1+a)^p + 2(1-a)^p + (1+2a)^p + (1-2a)^p\right\}$$

for all $x \in E_1$.

Theorem 3.9. *Let* $\phi_c : E_1 \times E_1 \to [0, \infty)$ *be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} \quad \text{converges and} \quad \lim_{n\to\infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} = 0 \tag{3.73}$$

for all $x, y \in E_1$ *and let* $f : E_1 \to E_2$ *be an odd function with* $f(0) = 0$ *that satisfies the inequality*

$$\|Df(x,y)\| \le \phi_c(x,y) \tag{3.74}$$

for all $x, y \in E_1$. *Then there exists a unique cubic function* $C : E_1 \to E_2$ *such that*

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{1}{8}\sum_{k=0}^{\infty} \frac{\Phi_c\left(2^k x\right)}{8^k} \tag{3.75}$$

for all $x \in E_1$, *where the mapping* $C(x)$ *and* $\Phi_c(2^k x)$ *are defined by*

$$C(x) = \lim_{n\to\infty} \frac{1}{8^n}\left\{f\left(2^{n+1}x\right) - 2f(2^n x)\right\} \tag{3.76}$$

$$\Phi_c\left(2^k x\right) = \frac{1}{a^4 - a^2}\Big[\left(5 - 4a^2\right)\phi_c\left(2^k x, 2^k x\right) + a^2\phi_c\left(2^{k+1}x, 2^{k+1}x\right) + 2a^2\phi_c\left(2^{k+1}x, 2^k x\right) + \left(4 - 2a^2\right)\phi_c\left(2^k x, 2^{k+1}x\right) + \phi_c\left(2^k x, 2^k 3x\right) + 2\phi_c\left(2^k(1+a)x, 2^k x\right) + 2\phi_c\left(2^k(1-a)x, 2^k x\right) + \phi_c\left(2^k(1+2a)x, 2^k x\right) + \phi_c\left(2^k(1-2a)x, 2^k x\right)\Big] \tag{3.77}$$

for all $x \in E_1$.

Proof. Following along similar lines to those in the proof of Theorem 3.7, we have

$$\|f(4x) - 10f(2x) + 16f(x)\| \le \Phi_c(x), \tag{3.78}$$

where

$$\Phi_c(x) = \frac{1}{(a^4 - a^2)}\Big[\left(5 - 4a^2\right)\phi_c(x,x) + a^2\phi_c(2x,2x) + 2a^2\phi_c(2x,x) + \left(4 - 2a^2\right)\phi_c(x,2x) + \phi_c(x,3x) + 2\phi_c((1+a)x,x) + 2\phi_c((1-a)x,x) + \phi_c((1+2a)x,x) + \phi_c((1-2a)x,x)\Big]$$

for all $x \in E_1$. It is easy to see from (3.78) that

$$\|f(4x) - 2f(2x) - 8\{f(2x) - 2f(x)\}\| \le \Phi_c(x) \tag{3.79}$$

for all $x \in E_1$. Define a mapping $\delta : E_1 \to E_2$ by

$$\delta(x) = f(2x) - 2f(x) \tag{3.80}$$

for all $x \in E_1$. Using (3.80) in (3.79), we obtain

$$\|\delta(2x) - 8\delta(x)\| \le \Phi_c(x) \tag{3.81}$$

for all $x \in E_1$. From (3.81), we have

$$\left\| \frac{\delta(2x)}{8} - \delta(x) \right\| \leq \frac{\Phi_c(x)}{8} \tag{3.82}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 8 in (3.82), we obtain

$$\left\| \frac{\delta(2^2x)}{8^2} - \frac{\delta(2x)}{8} \right\| \leq \frac{\Phi_c(2x)}{8^2} \tag{3.83}$$

for all $x \in E_1$. From (3.82) and (3.83), we arrive at

$$\begin{aligned} \left\| \frac{\delta(2^2x)}{8^2} - \delta(x) \right\| &\leq \left\| \frac{\delta(2^2x)}{8^2} - \frac{\delta(2x)}{8} \right\| + \left\| \frac{\delta(2x)}{8} - \delta(x) \right\| \\ &\leq \frac{1}{8}\left[\Phi_c(x) + \frac{\Phi_c(2x)}{8} \right] \end{aligned} \tag{3.84}$$

for all $x \in E_1$. In general for any positive integer n, we get

$$\begin{aligned} \left\| \frac{\delta(2^nx)}{8^n} - \delta(x) \right\| &\leq \frac{1}{8}\sum_{k=0}^{n-1} \frac{\Phi_c\left(2^kx\right)}{8^k} \\ &\leq \frac{1}{8}\sum_{k=0}^{\infty} \frac{\Phi_c\left(2^kx\right)}{8^k} \end{aligned} \tag{3.85}$$

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{\frac{\delta(2^nx)}{8^n}\right\}$, replace x by 2^mx and divide by 8^m in (3.85). Then for any $m, n > 0$, we have

$$\begin{aligned} \left\| \frac{\delta(2^{n+m}x)}{8^{n+m}} - \frac{\delta(2^mx)}{8^m} \right\| &= \frac{1}{8^m} \left\| \frac{\delta(2^n2^mx)}{8^n} - \delta(2^mx) \right\| \\ &\leq \frac{1}{8}\sum_{k=0}^{n-1} \frac{\Phi_c\left(2^{k+m}x\right)}{8^{k+m}} \\ &\leq \frac{1}{8}\sum_{k=0}^{\infty} \frac{\Phi_c\left(2^{k+m}x\right)}{8^{k+m}} \\ &\to 0 \quad \text{as} \quad m \to \infty \end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{\frac{\delta(2^nx)}{8^n}\right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a cubic mapping $C : E_1 \to E_2$ such that

$$C(x) = \lim_{n\to\infty} \frac{\delta(2^nx)}{8^n} \quad \forall x \in E_1.$$

Letting $n \to \infty$ in (3.84) and using (3.80) we see that (3.75) holds for all $x \in E_1$. The rest of the proof, which proves that C satisfies (1.20) and is unique, is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.9 concerning the stability of (1.20).

Corollary 3.10. *Let ε, p be nonnegative real numbers. Suppose that an odd function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y)\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 3; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < \frac{3}{2}; \\ \varepsilon\left(\|x\|^p \|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < \frac{3}{2} \end{cases} \tag{3.86}$$

for all $x, y \in E_1$. Then there exists a unique cubic function $C : E_1 \to E_2$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \begin{cases} \frac{\lambda_5 \|x\|^p}{8-2^p}, \\ \frac{\lambda_6}{7}, \\ \frac{\lambda_7 \|x\|^{2p}}{8-2^{2p}}, \\ \frac{\lambda_8 \|x\|^{2p}}{8-2^{2p}}, \end{cases} \tag{3.87}$$

for all $x \in E_1$, where λ_i $(i = 5, 6, 7, 8)$ are given in Corollary 3.8.

Theorem 3.11. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n}, \quad \sum_{n=0}^{\infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} \quad \text{converges} \tag{3.88}$$

and

$$\lim_{n\to\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} = 0 = \lim_{n\to\infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} \tag{3.89}$$

for all $x, y \in E_1$. Suppose that an odd function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequalities (3.44) and (3.74) for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ and a unique cubic function $C : E_1 \to E_2$ such that

$$\|f(x) - A(x) - C(x)\| \leq \frac{1}{6}\left\{\frac{1}{2}\sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k} + \frac{1}{8}\sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k}\right\} \tag{3.90}$$

for all $x \in E_1$, where $\Phi_a(2^k x)$ and $\Phi_c(2^k x)$ are defined by (3.47) and (3.77), respectively for all $x \in E_1$.

Proof. By Theorems 3.7 and 3.9, there exists a unique additive function $A_1 : E_1 \to E_2$ and a unique cubic function $C_1 : E_1 \to E_2$ such that

$$\|f(2x) - 8f(x) - A_1(x)\| \leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k} \tag{3.91}$$

and

$$\|f(2x) - 2f(x) - C_1(x)\| \leq \frac{1}{8}\sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k} \tag{3.92}$$

for all $x \in E_1$. Now from (3.91) and (3.92), one can see that

$$\begin{aligned}
&\left\| f(x) + \frac{1}{6}A_1(x) - \frac{1}{6}C_1(x) \right\| \\
&\quad = \left\| \left\{ -\frac{f(2x)}{6} + \frac{8f(x)}{6} + \frac{A_1(x)}{6} \right\} + \left\{ \frac{f(2x)}{6} - \frac{2f(x)}{6} - \frac{C_1(x)}{6} \right\} \right\| \\
&\quad \leq \frac{1}{6}\left\{ \| f(2x) - 8f(x) - A_1(x) \| + \| f(2x) - 2f(x) - C_1(x) \| \right\} \\
&\quad \leq \frac{1}{6}\left\{ \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Phi_a\left(2^k x\right)}{2^k} + \frac{1}{8}\sum_{k=0}^{\infty} \frac{\Phi_c\left(2^k x\right)}{8^k} \right\}
\end{aligned}$$

for all $x \in E_1$. Thus we obtain (3.90) by defining $A(x) = \frac{-1}{6}A_1(x)$ and $C(x) = \frac{1}{6}C_1(x)$, where $\Phi_a\left(2^k x\right)$ and $\Phi_c\left(2^k x\right)$ are defined in (3.47) and (3.77), respectively for all $x \in E_1$. □

The following corollary is an immediate consequence of Theorem 3.11 concerning the stability of (1.20).

Corollary 3.12. *Let ε, p be nonnegative real numbers. Suppose that an odd function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y)\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 1; \\ \varepsilon, & \\ \varepsilon\|x\|^p\|y\|^p, & 0 \leq p < \frac{1}{2}; \\ \varepsilon\left(\|x\|^p\|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < \frac{1}{2} \end{cases} \tag{3.93}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ and a unique cubic function $C : E_1 \to E_2$ such that

$$\|f(x) - A(x) - C(x)\| \leq \begin{cases} \frac{\lambda_5\|x\|^p}{6}\left\{\frac{1}{2-2^p} + \frac{1}{8-2^p}\right\}, \\ \frac{4\lambda_6}{21}, \\ \frac{\lambda_7\|x\|^{2p}}{6}\left\{\frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}}\right\}, \\ \frac{\lambda_8\|x\|^p}{6}\left\{\frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}}\right\}, \end{cases} \tag{3.94}$$

for all $x \in E_1$, where $\lambda_i \ (i = 5, 6, 7, 8)$ are given in Corollary 3.8.

Theorem 3.13. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function that satisfies (3.36), (3.37), (3.88) and (3.89) for all $x, y \in E_1$. Suppose that a function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequalities (3.2), (3.23), (3.44) and (3.74) for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$, a unique quadratic function $B : E_1 \to E_2$, a unique cubic function $C : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that*

$$\|f(x) - A(x) - B(x) - C(x) - D(x)\| \leq \frac{1}{2}\left\{\widetilde{\Phi}_a(x) + \widetilde{\Phi}_b(x) + \widetilde{\Phi}_c(x) + \widetilde{\Phi}_d(x)\right\} \tag{3.95}$$

for all $x \in E_1$, where $\widetilde{\Phi}_a(x), \widetilde{\Phi}_b(x), \widetilde{\Phi}_c(x)$ and $\widetilde{\Phi}_d(x)$ are defined by

$$\widetilde{\Phi}_a(x) = \frac{1}{6}\left\{\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a\left(2^k x\right)}{2^k} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a\left(-2^k x\right)}{2^k}\right\}, \tag{3.96}$$

$$\widetilde{\Phi}_b(x) = \frac{1}{12}\left\{\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k} + \frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(-2^k x\right)}{4^k}\right\}, \tag{3.97}$$

$$\widetilde{\Phi}_c(x) = \frac{1}{6}\left\{\frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c\left(2^k x\right)}{8^k} + \frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c\left(-2^k x\right)}{8^k}\right\}, \tag{3.98}$$

$$\widetilde{\Phi}_d(x) = \frac{1}{12}\left\{\frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^k x\right)}{16^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(-2^k x\right)}{16^k}\right\}, \tag{3.99}$$

respectively for all $x \in E_1$.

Proof. Let $f_e(x) = \frac{1}{2}\{f(x) + f(-x)\}$ for all $x \in E_1$. Then $f_e(0) = 0$, $f_e(x) = f_e(-x)$. Hence

$$\begin{aligned}
\|Df_e(x,y)\| &= \frac{1}{2}\{\|Df(x,y) + Df(-x,-y)\|\} \\
&\leq \frac{1}{2}\{\|Df(x,y)\| + \|Df(-x,-y)\|\} \\
&\leq \frac{1}{2}\{\phi(x,y) + \phi(-x,-y)\}
\end{aligned}$$

for all $x \in E_1$. Hence from Theorem 3.5, there exists a unique quadratic function $B : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that

$$\begin{aligned}
&\|f(x) - B(x) - D(x)\| \\
&\leq \frac{1}{2}\left\{\frac{1}{12}\left[\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^k x\right)}{16^k}\right]\right. \\
&\qquad \left. + \frac{1}{12}\left[\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(-2^k x\right)}{4^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(-2^k x\right)}{16^k}\right]\right\} \\
&\leq \frac{1}{2}\left\{\widetilde{\Phi}_b(x) + \widetilde{\Phi}_d(x)\right\},
\end{aligned} \tag{3.100}$$

where $\widetilde{\Phi}_b(x)$ and $\widetilde{\Phi}_d(x)$ are given in (3.97) and (3.99) for all $x \in E_1$. Again $f_o(x) = \frac{1}{2}\{f(x) - f(-x)\}$ for all $x \in E_1$. Then $f_o(0) = 0$, $f_o(x) = -f_o(-x)$. Hence

$$\begin{aligned}
\|Df_o(x,y)\| &= \frac{1}{2}\{\|Df(x,y) - Df(-x,-y)\|\} \\
&\leq \frac{1}{2}\{\|Df(x,y)\| + \|Df(-x,-y)\|\} \\
&\leq \frac{1}{2}\{\phi(x,y) + \phi(-x,-y)\}
\end{aligned}$$

for all $x \in E_1$. Hence from Theorem 3.11, there exists a unique additive function $A : E_1 \to E_2$ and a unique cubic function $C : E_1 \to E_2$ such that

$$\begin{aligned} \|f(x) - A(x) - C(x)\| &\leq \frac{1}{2}\left\{\frac{1}{6}\left[\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a(2^k x)}{2^k} + \frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c(2^k x)}{8^k}\right]\right. \\ &\quad \left. + \frac{1}{6}\left[\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a(-2^k x)}{2^k} + \frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c(-2^k x)}{8^k}\right]\right\} \\ &\leq \frac{1}{2}\left\{\widetilde{\Phi}_a(x) + \widetilde{\Phi}_c(x)\right\}, \end{aligned} \tag{3.101}$$

where $\widetilde{\Phi}_a(x)$ and $\widetilde{\Phi}_c(x)$ are given in (3.96) and (3.98) for all $x \in E_1$. Since $f(x) = f_e(x) + f_o(x)$, then it follows from (3.100) and (3.101) that

$$\begin{aligned} &\|f(x) - A(x) - B(x) - C(x) - D(x)\| \\ &\quad = \|\{f_e(x) - B(x) - D(x)\} + \{f_o(x) - C(x) - D(x)\}\| \\ &\quad \leq \|f_e(x) - B(x) - D(x)\| + \|f_o(x) - C(x) - D(x)\| \\ &\quad \leq \frac{1}{2}\left\{\widetilde{\Phi}_a(x) + \widetilde{\Phi}_b(x) + \widetilde{\Phi}_c(x) + \widetilde{\Phi}_d(x)\right\} \end{aligned}$$

for all $x \in E_1$. Hence the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 3.13 concerning the stability of (1.20).

Corollary 3.14. *Let ε, p be nonnegative real numbers. Suppose a function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x,y)\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 1; \\ \varepsilon, & \\ \varepsilon\|x\|^p\|y\|^p, & 0 \leq p < \frac{1}{2}; \\ \varepsilon\left(\|x\|^p\|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < \frac{1}{2} \end{cases} \tag{3.102}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$, a unique quadratic function $B : E_1 \to E_2$, a unique cubic function $C : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that

$$\begin{aligned} &\|f(x) - A(x) - B(x) - C(x) - D(x)\| \\ &\leq \begin{cases} \frac{1}{2}\left\{\frac{\lambda_1}{6}\left\{\frac{1}{4-2^p} + \frac{1}{16-2^p}\right\} + \frac{\lambda_5}{3}\left\{\frac{1}{2-2^p} + \frac{1}{8-2^p}\right\}\right\}\|x\|^p; \\ \frac{1}{2}\left\{\lambda_2 + \frac{4\lambda_6}{21}\right\}; \\ \frac{1}{2}\left\{\frac{\lambda_3}{6}\left\{\frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}}\right\} + \frac{\lambda_7}{3}\left\{\frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}}\right\}\right\}\|x\|^{2p}; \\ \frac{1}{2}\left\{\frac{\lambda_4}{6}\left\{\frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}}\right\} + \frac{\lambda_8}{3}\left\{\frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}}\right\}\right\}\|x\|^{2p} \end{cases} \end{aligned} \tag{3.103}$$

for all $x \in E_1$, where $\lambda_i \ (i = 1, \ldots, 8)$ are respectively, given in Corollaries 3.6 and 3.12.

REFERENCES

[1] J. ACZEL AND J. DHOMBRES, *Functional Equations in Several Variables*, Cambridge University, Press, Cambridge, 1989.

[2] T. AOKI, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2** (1951), 64–66.

[3] I.S. CHANG, E.H. LEE AND H.M. KIM, On the Hyers-Ulam-Rassias stability of a quadratic functional equations, *Math. Ineq. Appl.*, **6**(1) (2003), 87–95.

[4] S. CZERWIK, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, 2002.

[5] S. CZERWIK, *Stability of Functional Equations of Ulam-Hyers Rassias Type*, Hadronic Press, Plam Harbor, Florida, 2003.

[6] M. ESHAGHI GORDJI, A. EBADIAN AND S. ZOLFAGHRI, Stability of a functional equation deriving from cubic and quartic functions, *Abstract and Applied Analysis*, (submitted).

[7] M. ESHAGHI GORDJI, S. KABOLI AND S. ZOLFAGHRI, Stability of a mixed type quadratic, cubic and quartic functional equations, `arxiv: 0812.2939v1 Math FA`, 15 Dec 2008.

[8] M. ESHAGHI GORDJI AND H. KHODAIE, Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces, `arxiv: 0812. 2939v1 Math FA`, 15 Dec 2008.

[9] Z. GAJDA, On the stability of additive mappings, *Inter. J. Math. Math. Sci.*, **14** (1991), 431–434.

[10] P. GAVURUTA, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431–436.

[11] P. GAVURUTA, An answer to a question of J.M.Rassias concerning the stability of Cauchy functional equation, *Advances in Equations and Inequalities*, Hadronic Math. Ser., (1999), 67–71.

[12] D.H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci., U.S.A.*, **27** (1941), 222–224.

[13] D.H. HYERS, G. ISAC AND Th.M. RASSIAS, *Stability of Functional Equations in Several Variables*, Birkhauser Basel, 1998.

[14] K.W. JUN AND H.M. KIM, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, *J. Math. Anal. Appl.*, **274** (2002), 867–878.

[15] K.W. JUN AND H.M. KIM, On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive type functional equation, *Bull. Korean Math. Soc.*, **42**(1) (2005), 133–148.

[16] K.W. JUN AND H.M. KIM, On the stability of an n-dimensional quadratic and additive type functional equation, *Math. Ineq. Appl.*, **9**(1) (2006), 153–165.

[17] S.M. JUNG, On the Hyers–Ulam–Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **204** (1996), 221–226.

[18] S.M. JUNG, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.*, **222** (1998), 126–137.

[19] S.M. JUNG, On the Hyers-Ulam-Rassias stability of a quadratic functional equation, *J. Math. Anal. Appl.*, **232** (1999), 384–393.

[20] Pl. KANNAPPAN, Quadratic functional equation inner product spaces, *Results Math.*, **27**(3-4) (1995), 368–372.

[21] S.H. LEE, S.M. IM AND I.S. HWANG, Quartic functional equations, *J. Math. Anal. Appl.*, **307** (2005), 387–394.

[22] A. NAJATI AND M.B. MOGHIMI, On the stability of a quadratic and additive functional equation, *J. Math. Anal. Appl.*, **337** (2008), 399–415.

[23] C. PARK, Homomorphisms and derivations in C^*-algebras, Hindawi Publ. Co., *Abstract and Applied Analysis*, Volume 2007, Article ID 80630, doi:10.1155/2007/80630, 1–12.

[24] J.M. RASSIAS, On approximately of approximately linear mappings by linear mappings, *J. Funct. Anal. USA*, **46** (1982), 126–130.

[25] J.M. RASSIAS, Solution of the Ulam stability problem for the quartic mapping, *Glasnik Mathematica*, **34**(54) (1999), 243–252.

[26] Th.M. RASSIAS, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300.

[27] Th.M. RASSIAS AND P. SEMRL, On the Hyers-Ulam stability of linear mappings, *J. Math. Anal. Appl.*, **173** (1993), 325–338.

[28] K. RAVI AND M. ARUNKUMAR, On the Ulam-Gavruta-Rassias stability of the orthogonally Euler-Lagrange type functional equation, *Internat. J. Appl. Math. Stat.*, **7** (2007), 143–156.

[29] K. RAVI, M. ARUNKUMAR AND J.M. RASSIAS, Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *Int. J. Math. Stat.*, **3** (2008), A08, 36–46.

[30] M.A. SIBAHA, B. BOUIKHALENE AND E. ELQORACHI, Ulam-Gavruta-Rassias stability for a linear functional equation, *Internat. J. Appl. Math. Stat.*, **7** (2007), 157–168.

[31] S.M. ULAM, *Problems in Modern Mathematics*, Rend. Chap.VI, Wiley, New York, 1960.